Appendix II: High School Mathematics Competitions in Alberta

A. Brief History

Although there are older mathematics competitions in Canada for high school students, Alberta was the first to have a province-wide contest. It started in 1957 as the "Alberta Matriculation Prize and Scholarship Examination," sponsored by the Nickle Family Foundation (Calgary), the Canadian Mathematical Congress (now Society), the Mathematics Council of The Alberta Teachers' Association, the Department of Mathematics and Statistics at the University of Calgary and the Department of Mathematics at the University of Alberta.

From the very beginning, the Alberta contest emphasized problem solving questions. Multiple choice questions were introduced in 1967 to accommodate the rising number of participants. That year the contest was renamed the "Alberta High School Mathematics Prize Examination."

In 1969, the Canadian Mathematics Olympiad (C.M.O.) came into being. The Alberta contest acquired an additional role, to be the qualifying round for the national contest. The latter was introduced principally in anticipation of Canada's participation in the International Mathematical Olympiad (I.M.O.), which had been initiated by Romania in 1959. However, it was not until 1981, when the United States hosted the event, that Canada entered a team. It included two members from Alberta, Arthur Baragar and John Bowman.

In 1983, the Alberta High School Mathematics Prize Examination Board was formed to administer the Alberta contest. One of the most pleasant duties of the new board was to welcome two new sponsors, the Peter H. Denham Memorial Fund in Mathematics (Edmonton) and the publishing house of W. H. Freeman (New York).

That same year, 1983, the two parts of the Alberta contest were separated. The muliple choice part retained the title, "Alberta High School Mathematics Prize Examination." Competition among school teams was introduced and the contest was written in the fall. It served as well as the qualifying round for the problem solving part of the contest, which was written in the following spring and renamed the "Alberta High School Mathematics Scholarship Examination."

In 1988, Dover Publications Incorporated (New York) joined the list of sponsors. The board was renamed the "Alberta High School Mathematics Competition Board" (A.H.S.M.C. Board), to emphasize that the contest is not an examination. The two former "examinations" now became the first and second rounds of the "Alberta High School Mathematics Competition."

The late Leo Moser is acknowledged as the father of the Alberta contest. Other individuals have devoted considerable time and effort to this endeavor. The list from the University of Calgary includes Allan Gibbs, Tony Holland, Harold Lampkin, Bill Sands, Jonathan Schaer and Bob Woodrow. (Unfortunately, this part of the record is sadly inadequate and we apologize to those who should be mentioned but are not.) The list from the University of Alberta includes Ken Andersen, Alvin Baragar, Bill Bruce, Graham Chambers, Jim Fisher, Herb Freedman, Murray Klamkin, Ted Lewis, Andy Liu, Jack Macki, Jim Muldowney, Arturo Pianzola, Jim Pounder, Roy Sinclair, Sudarshan Sehgal and Jim Timourian. The late Geoffrey Butler served as the chairman of the Alberta board and of the Canadian Mathematics Olympiad Committee, as well as the leader of the Canadian teams in the International Mathematical Olympiads of 1981 to 1984.

B. Current Information

Contests

The first round of the Alberta High School Mathematics Competition is held in November and the second round in the following February. The top 22 students in the latter competition are nominated to write the Canadian Mathematics Olympiad, which is administered by the Canadian Mathematical Society. All three contests are written in the students' own schools. Top performers in the C.M.O. may earn a spot on the Canadian National Team which competes in the International Mathematical Olympiad, usually held in Europe.

Eligibility

All students enrolled in a high school program in Alberta or the Northwest Territories are eligible to take part in the first round. Those students who qualify through the first round may participate in the second round. A limited number of special applications may be accepted.

Date

The first round is scheduled each year for the morning of the third Tuesday in November. The duration of the competition is one hour and it must start between 8:30 and 9:30. The second round is scheduled from 9 to 12 in the morning of the second Tuesday in February.

Format

The first round is a 60-minute paper which consists of 16 multiple choice questions, to be graded on the computer. Soft pencils (regular pencils will do) *must* be used. Five points are given for each correct answer, two points for each question not attempted and zero points for each incorrect (including multiple) answer. The scores range from 20 to 100 points.

The second round is a three-hour paper consisting of five essay-type questions. Twenty points are given for each complete solution. Partial credits are given for significant progress. The scores range from zero to 100 points.

Pencil, eraser, graph paper, scratch paper, ruler and compass are allowed. Calculators are *not* allowed

All examination materials are the property of the Alberta High School Mathematics Competition Board.

Applications

Applications to write the first round are made through a teacher designated as the contest manager in each school and must *arrive* at the official address at least three weeks before the contest is to be written. Each school entering at least three students is considered to have a team. Team membership is determined after the results are known. The top three students will constitute the team and the team score is the total score of these three members.

The top 50 students will be invited to write the second round. Invitation will also be extended to the top ten students in Grade 11 and the top five students in Grade 10. In exceptional circumstances, contest managers may nominate a number of additional candidates from their schools to write the second round. Special applications must be submitted to the Board and must *arrive* at the official address at least three weeks before the contest is to be written. The number of such candidates from each school is limited and is determined by the Board. It is roughly proportional to the number of participants in the first round from the school.

Fees

The application fee for the first round is $1 per student. There is no application fee for the second round, except for the special applications for which it is $5 per student.

All fees are payable to the University of Alberta and must be submitted with the application.

Hotline

Quick consultation with the A.H.S.M.C. Board is possible via telephone. The current contact is Professor Alvin Baragar, 492-3398.

School Prizes

In the first round, the Peter H. Denham Memorial Plaque goes to the first place school.

Book prizes of appropriate values are awarded to: (1) the top three schools; (2) the top school in each zone which does not qualify under (1); (3) the top school which has not won any book prizes from the A.H.S.M.C. Board.

The prize in (3) was created in memory of the former board chairman, Geoffrey Butler.

In addition, certificates go to each school which entered a team and its contest manager.

Individual Prizes

In the first round, the title of W. H. Freeman Scholar goes to the first place student.

Book prizes of appropriate values are awarded to: (1) the top three students; (2) the top student in each of Grade 10 and Grade 11 if not a recipient under (1); (3) the top two students in each zone if not already recipients under (1) or (2).

In addition, a certificate is awarded to each student invited to write the second round, as well as to the top student from each school which entered a team, if none of these was previously a recipient under previous prize categories.

In the second round, five fellowships are awarded on the basis of performance: (1) The Nickle Family Foundation Fellowship (\$500); (2) The Peter H. Denham Memorial Fellowship (\$250); (3) The Canadian Mathematical Society Fellowship (\$150); (4) The Alberta Teachers' Association Grade 11 Fellowship (\$50); (5) The Alberta Teachers' Association Grade 10 Fellowship (\$50).

Winners of these fellowships must be Canadian citizens or landed immigrants of Canada. The Nickle Family Foundation Fellowship is awarded on the condition that the winner will attend an Alberta university and is to be credited towards the winner's tuition fees at such an institution.

In normal circumstances, these fellowships are awarded in descending order of scores. In case of ineligibility, they may be withheld, subdivided or offered to candidates with lower scores. No student may win more than one of these fellowships in the competition of any one year. All fellowship winners (plus students who would have been winners but who are ineligible for fellowships) receive a certificate.

The decisions of the Alberta High School Mathematics Competition Board are final.

Geographical Zones

For the purpose of assuring some regional distribution of prizes, Alberta and the Northwest Territories are divided into four zones as follows:
Zone 1: The City of Calgary.
Zone 2: Southern Alberta (north to and including the City of Red Deer and excluding the City of Calgary)
Zone 3: The City of Edmonton.
Zone 4: Northern Alberta (excluding the City of Edmonton) and the Northwest Territories.

C. Further Information

Each of Calgary and Edmonton cities has its own contest for junior high students. Each contest is administered by a group of dedicated teachers. The contests are written in the spring and consist of some multiple choice questions and some problem solving questions.

Alberta students can write two sequences of contests administered from outside the province.

One is the Canadian Mathematics Competition sponsored by the University of Waterloo. It began in 1963 as the Junior Mathematics Contest. It now consists of two "Gauss" contests, one for Grade 7 and one for Grade 8, a "Pascal" contest for Grade 9, a "Cayley" contest for Grade 10, a "Fermat" contest for Grade 11, a "Euclid" contest for Grade 12, and a "Descartes" contest which is open to all students but is primarily for Ontario Grade 13 students. These contests are written in the spring. The papers for the earlier grades consist entirely of multiple choice questions. The regional coordinator is Professor Bob Woodrow of the University of Calgary.

The other sequence is run by the Mathematical Association of America. It consists of an American Junior High School Mathematics Examination for Grades 7 and 8, which is scheduled in December, and an American High School Mathematics Examination for Grades 9, 10, 11 and 12, scheduled in the spring. The latter serves as a qualifying round for the American Invitational Mathematics Examination, which also occurs in the spring and which, in turn, serves as a qualifying round for the U.S.A. Mathematics Olympiad.

The junior high and high school contests consist entirely of multiple choice questions. The questions in the Invitational ask for integral answers between 0 and 999, so that they are really multiple choice with a thousand alternatives. The U.S.A. Mathematics Olympiad consists of five problem solving questions. The regional coordinator is Professor Bill Sands of the University of Calgary.

D. Sample Papers

The Alberta High School Mathematics Competition Board publishes a newsletter, *Postulate,* which contains contest papers of recent years. This section reproduces questions from the papers of the first ten contests, 1957 to 1966. Some questions have been slightly edited to correct typographical errors and remove ambiguities.

Year 1957

Problem 1

Express $(1/(1+x))/(1-1/(1+x)) + (1/(1+x))/(x/(1-x)) + (1/(1-x))/(x/(1+x))$ as a simple fraction.

Problem 2

Solve the equation
$\sqrt{16x+1} - 2(\sqrt[4]{16x+1}) = 3.$

Problem 3

In the centre of the flat rectangular top of a building which is 21 metres long and 16 metres wide, a flagpole is to be erected, 8 metres high. To support the pole, four cables are needed. The cables start from the same point, 2 metres below the top of the pole, and end at the four corners of the top of the building. How long is each of the cables?

Problem 4

Solve the equation $x + 2 = 0$.

Problem 5

Solve the equation $x^2 + x + 1 = 0$.

Problem 6

Solve the equation $x^3 + x = 10$.

Problem 7

Given that $ax^2 + bx + c = 0$ has the roots m and n, prove that
(a) $m + n = -b/a$;
(b) $mn = c/a$.

Problem 8

From a 12 by 18 sheet of tin, we wish to make a box by cutting a square from each corner and turning up the sides. Draw a graph showing how the volume of the box obtained varies with the size of the squares cut out. For what size of squares would the largest box be obtained?

Problem 9

Solve the system of equations $x + y + z = 3$, $x + 2y + 3z = 8$ and $x + 3y + 4z = 11$.

Problem 10

Solve the system of equations $x + y + z = 1$, $2x + 3y + 4z = 2$ and $3x + 5y + 7z = 4$.

Problem 11

A certain sample of radium is decreasing according to the equation $A = 3(2^{-t/1800})$, where t is in years and A is in milligrams.
(a) How much radium was in the sample at $t = 0$?
(b) How much will there be 900 years later?

Problem 12

Two circles, each of radius 1, are such that the centre of each lies on the circumference of the other. Find the area common to both circles.

Problem 13

A man takes a trip from A to B at an average speed of 40 kph and returns at an average speed of 60 kph. What is his average speed for the entire trip?

Problem 14

Four men dined at a hotel. They checked their hats in the cloakroom. Each of the four came away wearing a hat belonging to one of the other three. In how many different ways could this have happened?

Problem 15

Given a cylinder of height 6 cm and base radius 2 cm, prove that a spider can go from any point on the surface to any other point on the surface along a path of total length less than 9 cm.

Problem 16

Prove that the difference between the sum of n terms of $n/n + (n - 1)/n + (n - 2)/n + \cdots$ and the sum of the infinite series $n/(n + 1) + n/(n + 1)^2 + n/(n + 1)^3 + \cdots$ is equal to $(n - 1)/2$.

Problem 17

If two sides of a quadrilateral are parallel to each other, prove that the straight line joining their midpoints passes through the point of intersection of the diagonals.

Problem 18

In a certain school all students study Mathematics, Physics and French. Forty percent prefer both Mathematics and Physics to French. Fifty percent prefer Mathematics to French and sixty percent prefer Physics to French. If all students have definite preferences between subjects, what percentage prefer French to both Mathematics and Physics?

Year 1958

Problem 1

Evaluate $1/7 + 2/3 + 5/8$.

Problem 2

Express $c/(a + (b/c)) + (a + c)/(a - (b/c))$ as a simple fraction.

Problem 3

Evaluate $9!(1/8!1! + 1/7!2! + 1/6!3! + 1/5!4!)$.

Problem 4

Solve the equation $(x - 2)(x - 3) = 1$.

Problem 5

Solve the equation $1/(x - 2) + 1/(x - 3) = 1$.

Problem 6

Solve the system of equations $3x + 2y + z = 1$, $4x - y = 2$ and $x - y + 2z = 3$.

Problem 7

You are travelling along a road at 4 kph parallel to a double track railroad. Two trains meet you, each being of the same length. The first takes 40 seconds to pass you, the second 30 seconds. Prove that it takes 4 minutes for the second train to completely pass the first.

Problem 8

Find the greatest common divisor of 3910, 8551 and 11475.

Problem 9

A quartet is chosen by lot from all the high school students in a certain school. There are 40 students in Grade X, 30 in Grade XI and 20 in Grade XII. In how many ways can the quartet consist of
(a) students from Grade X alone;
(b) at least three students from Grade XII?

Problem 10

Let $f(n + 1) = 5f(n) - 6f(n - 1)$ for a function f defined for all positive integers n. Prove that $f(n) = 3^n - 2^n$ is a solution of this equation.

Problem 11

For $f(n) = 3^n - 2^n$, find the sum $f(1) + f(2) + \cdots + f(9)$.

Problem 12

Use Newton's Binomial Theorem to compute the cube root of 26 to three decimal places.

Problem 13

State the Remainder Theorem (for polynomials).

Problem 14

Solve the equation $4x^3 - 8x^2 - 3x + 9 = 0$.

Problem 15

Let P, R and S be any three points on a circle such that RS extended meets the tangent at P at the point T. Prove that $PS/PR = ST/PT = PT/RT$.

Problem 16

(a) Prove that the sum of the first n positive integers is $n(n + 1)/2$.
(b) Prove that the sum of the cubes of the first n positive integers is $n^2(n + 1)^2/4$.

Problem 17

Sketch the locus corresponding to the equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$ and also the locus corresponding to $y^2 - x^2 - 8xy/3 = 0$.

Problem 18

Solve the system of equations
$x^2 + 4y^2 - 6x - 16y + 21 = 0$ and
$y^2 - x^2 - 8xy/3 = 0$.

Problem 19

At $r\%$ interest compounded semi-annually, how much would p dollars deposited on each of the dates: January 1, 1940, 1941, 1942, 1943 and 1944 be worth altogether on January 1, 1945?

Problem 20

Prove the Sine Law (for triangles).

Problem 21

Prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

Problem 22

From the top of a tower, the angle of depression of a point A, in the same horizontal plane as the base of the tower, is x. On the top of the tower is a flagstaff whose length is equal to one-quarter of the tower. Prove that the tangent of the angle which the flagstaff subtends at A is equal to $\sin x \cos x/(4 + \sin^2 x)$.

Year 1959

Problem 1

Solve the equation $2x - 6 + \sqrt{3x - 2} = 0$.

Problem 2

Solve the equation $x^3 + 3x^2 - 4x - 12 = 0$.

Problem 3

Solve the system of equations $6/x + 15/y = 4$, $18/y - 16/z = 1$ and $14/x + 12/z = 5$.

Problem 4

Given that the roots of $ax^2 + bx + c = 0$ are m and n, find the condition on a, b and c so that
(a) $m = 1/n$;
(b) $m = -n$.

Problem 5

Prove that $n!/(n - 2)!2! + (n + 1)!/(n - 1)!2! = n^2$.

Problem 6

$ABCD$ is a kite-shaped figure with $AB = AD$ and $CB = CD$. Find the point P the sum of whose distances from the four vertices is as small as possible.

Problem 7

Six papers are set in an examination, two of them in mathematics. In how many different orders can the papers be given, provided only that the two mathematics papers are not consecutive?

Problem 8

Prove that $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$.

Problem 9

A vertical flagpole stands on a hillside which makes an angle A with the horizontal. At a distance k down the slope from the pole, it subtends an angle B. Prove that the height of the pole is given by $k \sin B/\cos (A + B)$.

Problem 10

If 1 is added to the product of four consecutive positive integers, is the sum always the square of an integer?

Problem 11

If 41 is added to the sum of two consecutive positive integers, is the sum always a prime number?

Problem 12

What is the coefficient of r^2 in the expansion $(r^2 - 2r \cos \theta + 1)^{-1/2}$ in powers of r?

Problem 13

The first three terms of an arithmetic progression are m, $4m - 1$ and $5m + 3$. What is the sum of the first $4m$ terms?

Problem 14

Explain why and how it is possible to attach definite meanings to zero, negative and fractional powers of a positive number.

Problem 15

A golfer can score 2, 3, 4 or 5 strokes per hole. How many different score sequences yield a score of 25?

Problem 16

The lines OAB and OC intersect at O, with $OA = a$ and $OB = b$. A point P moves along OC, and P_1 and P_2 are two positions of P such that the angles AP_1B and AP_2B are equal.
(a) Prove that $OP_1 \cdot OP_2 = ab$.
(b) At what distance of P from O is the angle APB a maximum?

Problem 17

What is wrong with the following argument? [see Figure 1]

$$
\begin{aligned}
(x + 1)^2 &= x^2 + 2x + 1 \\
(x + 1)^2 - (2x + 1) &= x^2 \\
(x + 1)^2 - (2x + 1) - x(2x + 1) &= x^2 - x(2x + 1) \\
(x + 1)^2 - (x + 1)(2x + 1) + (2x + 1)^2/4 &= x^2 - x(2x + 1) + (2x + 1)^2/4 \\
((x + 1) - (2x + 1)/2)^2 &= (x - (2x + 1)/2)^2 \\
x + 1 - (2x + 1)/2 &= x - (2x + 1)/2 \\
1 &= 0
\end{aligned}
$$

Figure 1

Problem 18

PQ is a chord passing through the focus $(a,0)$ of a parabola with vertex at the origin. The slope of PQ is m.
(a) Prove that the coordinates of its midpoint are $a(1 + 2/m^2)$ and $2a/m$.
(b) Prove that the locus of the midpoints of all chords passing through the focus is again a parabola.
(c) Find the focus and directrix of the parabola in (b).

Problem 19

Given m vertical and n horizontal lines, prove that the number of rectangles which can be formed having segments of these lines as sides is $\binom{m}{2}\binom{n}{2}$.

Year 1960

Problem 1

Find the distance between the points $(-1,-5)$ and $(-13,0)$.

Problem 2

Evaluate $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4$.

Problem 3

Find the area of an equilateral triangle of side 1.

Problem 4

One root of $2hx^2 + (3h - 6)x - 9 = 0$ is the negative of the other.
(a) Find the value of h.
(b) Solve the equation.

Problem 5

Find a cubic equation with integral coefficients which has as roots the numbers $2/3$, -2, -1.

Problem 6

Solve the equation $x^3 - 4x^2 + x + 6 = 0$.

Problem 7

(a) Prove that, for any positive number n that satisfies the equation $x^y = y^x$, $x = (1 + 1/n)^{n+1}$ and $y = (1 + 1/n)^n$.
(b) Do the formulae in (a) give all the positive solutions of $x^y = y^x$?

Problem 8

Given that $f(x) = 1/(x + 1/(x + 1/x))$ and $g(x) = x - 1/x$, find
(a) $f(g(x))$;
(b) $g(f(x))$.

Problem 9

Prove that $\log_a xy = \log_a x + \log_a y$.

Problem 10

Prove that $\log_a x^n = n \log_a x$.

Problem 11

From A, a pilot flies $12\sqrt{2}$ km in the direction N30°W to position B, and then $12\sqrt{2}$ km in the direction S60°E to position C. How far and in what direction must he now fly to again reach A?

Problem 12

Prove that $(1 - \cos x + \sin x)/(1 + \cos x + \sin x) = \tan(x/2)$.

Problem 13

There are ten points A, B, ... in a plane, no three in the same straight line.
(a) How many lines are determined by the points?
(b) How many of the lines pass through A?
(c) How many triangles are determined by the points?
(d) How many of the triangles have A as a vertex?
(e) How many of the triangles have AB as a side?

Problem 14

In how many ways can the word PYRAMID be spelt out, using adjacent letters of the arrangement below?

```
D I M A R Y P Y R A M I D
  D I M A R Y R A M I D
    D I M A R A M I D
      D I M A M I D
        D I M I D
          D I D
            D
```

Problem 15

Prove that if one side of a triangle is greater than another, the angle opposite the greater side exceeds the angle opposite the shorter side.

Problem 16

Prove that if $AC = 2BC$ in triangle ABC, then angle B is more than twice angle A.

Problem 17

A ball is dropped from a height of 6 metres. Each time it strikes the ground after falling from a height of h metres, it rebounds to a height of $2h/3$ metres.
(a) How far has the ball travelled when it hits the floor for the fifth time?
(b) What is the total distance travelled by the ball before it comes to rest?

Problem 18

Three numbers are in geometric progression. If A, G and H are their arithmetic, geometric and harmonic means, respectively, prove that $G^2 = AH$.

Problem 19

Numerical calculation seems to show that the relation $8 - \sqrt{62} = {}^3\sqrt{2}/10$ is at least approximately true. Find whether the relation is exact.

Year 1961

Problem 1

Express $1 + 2/(x + 3/(x + 4/x)) - 3/(x - 3/(x + 4/x))$ as a simple fraction.

Problem 2

Express $\sqrt{(a - b)/(a + b)} + \sqrt{(a + b)/(a - b)} - \sqrt{c^2a^2 - c^2b^2}$ as a simple fraction.

Problem 3

Solve the equation $x - 7\sqrt{x - 4} - 12 = 0$.

Problem 4

Solve the equation $\sqrt{3x + 9} - \sqrt{x + 5} = \sqrt{2x + 8}$.

Problem 5

Solve the equation $2^x = 10$.

Problem 6

Solve the inequality $(3x - 2)/x > 1$.

Problem 7

In triangle ABC, the longest side BC is of length 20 and the altitude from A to BC is of length 12. A rectangle $DEFG$ is inscribed in ABC, with D on AB, E on AC and both F and G on BC. Find the maximum area of $DEFG$.

Problem 8

Prove that the largest triangle which can be inscribed in a circle is equilateral.

Problem 9

Find the locus of a point P such that $AP^2 - BP^2 = d^2$, where A and B are two fixed points and d is a real number.

Problem 10

Prove that the point of intersection of the lines $2y + x + 1 = 0$ and $y - 3x + 4 = 0$ lies on the line $A(2y + x + 1) + B(y - 3x + 4) = 0$, where A and B are real numbers.

Problem 11

Sketch the graph of $(x^2 + y^2)(x^2 + y^2 - 1) = 0$.

Problem 12

Let x take on a set of values which form a geometric progression. Prove that the corresponding values of $y = \log_d x$ form an arithmetic progression, where d is a positive real number not equal to 1.

Problem 13

Let $f(x) = \sqrt{4 - x^2}$. Consider $(f(x) - f(1))/(x - 1)$.
(a) Interpret this expression geometrically as x takes on values successively nearer to 1.
(b) Determine its limiting value.

Problem 14

Let $S_n = 1 + 1/2 + 1/4 + 1/8 + \cdots + 1/20^n$.
What is the least value of n such that $S_n > 127/64$?

Problem 15

Convert the recurring decimal 0.147147147... into a fraction.

Problem 16

The equation $ax^2 + bx + c = 0$ has two real roots. Prove that, if a is very small while b and c are of "moderate" size, then one of the roots is close to $-c/b$ while the other is very large numerically.

Problem 17

Prove that

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

for all non-negative integers n.

Problem 18

Prove that

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^{n-1}\binom{n}{n-1}$
$+(-1)^n\binom{n}{n} = 0$ for all positive integers n.

Problem 19

Nine students are to be assigned to three rooms, three students to a room.
(a) In how many ways can this be done?
(b) What if two particular students refuse to be assigned to the same room?

Problem 20

Fifteen passengers rode on a railway line which leads to 25 towns. If no two persons get off at the same town, what is the total number of ways in which they can get off?

Problem 21

Solve the equation $\sin x = \sin 2x$.

Problem 22

Solve the equation $\tan x = \cot 2x$.

Problem 23

In triangle ABC, $BC = a > b = AC$ and the difference between the angles A and B is x. Given a, b and x, construct ABC with ruler and compass.

Year 1962

Problem 1

Determine d such that when $-1/2 < x < 1/2$,
$((1 - 4x^2)^{1/2} - (x/2)(1 - 4x^2)^{-1/2}(-8x))/(1 - 4x^2)$
$= 1/(1 - 4x^2)^d$.

Problem 2

(a) Find numbers a and b such that
$a(3x + 5) + b(2x + 3) = 12x + 19$ for every x.
(b) Determine numbers A and B so that for every x except $-3/2$ and $-5/3$, $(12x + 19)/(3x + 5)(2x + 3)$
$= A/(2x + 3) + B/(3x + 5)$.

Problem 3

Solve the equation $x^2 - x - 20 = 0$.

Problem 4

Solve the equation
$1/(x - 2) + 1/(x + 2) = 4/(x^2 - 4)$.

Problem 5

Solve the equation $\sqrt{4 - 3x} - x = 12$.

Problem 6

For any two real numbers x and y, each greater than 1, prove that $\log_y x = 1/\log_x y$.

Problem 7

Solve the equation $2^x - 4(2^{-x}) + 3 = 0$.

Problem 8

Using only a T-square, construct the centre of a given circle.

Problem 9

Solve the equation $\cos 2x = \cos x$.

Problem 10

Solve the equation $\sin 5x - \sin x = \cos 3x$.

Problem 11

Solve the equation $\sec x - 2 \cos x - \tan x = 0$.

Problem 12

(a) Let $S_n = 1/1(2) + 1/2(3) + 1/3(4)$
$+ \cdots + 1/n(n + 1)$. Express S_n as a simple fraction in terms of n.
(b) Find the number n such that S_n in (a) is greater than 100/101.

Problem 13

Let m and n be the roots of the equation $ax^2 + bx + c = 0$. Find a quadratic equation with coefficients expressed in terms of a, b and c which has $m + 2$ and $n + 2$ as roots.

Problem 14

Let $T_n = 1 + 2r + 3r^2 + \cdots + nr^{n-1}$ where r is a real number not equal to 1. Prove that
$T_n = (1 - r^n)/(1 - r)^2 - nr^n/(1 - r)$.

Problem 15

An after-dinner speaker anticipates delivering 35 speeches during the next five years. So as not to become bored with his jokes, he decides to tell exactly three jokes in every speech, and in no two speeches to tell exactly the same three jokes.
(a) What is the minimum number of jokes that will accomplish this?
(b) What is the minimum number if he decides never to tell the same joke twice?

Problem 16

In a triangle ABC, side BC and the angles B and C are known. Prove that the length of the altitude from A to BC is $BC/(\cot B + \cot C)$.

Problem 17

Prove that $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$, where A, B and C are the angles of any triangle.

Problem 18

Prove that, for any two positive numbers whose sum is 8, their product is a maximum when they are equal.

Problem 19

Prove that, among all rectangles with a given area, the square has the least perimeter.

Problem 20

The inscribed circle of triangle ABC has centre O and touches BC at P. One of the escribed circles of triangle ABC has centre O' and touches BC at Q.
(a) Prove that $BP = CQ$.
(b) Prove that the points B, C, O and O' lie on a circle.

Problem 21

(a) For the quartic equation $Ax^4 + Bx^3 + Cx^2 + Bx + A = 0$, prove that, if r is a root, so also is $1/r$.
(b) If $t = x + 1/x$, express $x^2 + 1/x^2$ in terms of t.
(c) Solve the equation $x^4 - x^3 - 10x^2 - x + 1 = 0$ and verify that its roots occur in pairs as indicated in (a).

Problem 22

Let ABC be any triangle.
(a) Prove that $\sin A + \sin B + \sin C = 2\cos(C/2)(\cos((A - B)/2) + \sin(C/2))$.
(b) For a fixed value of C, what is the relation between the angles A and B in order that $\sin A + \sin B + \sin C$ should have its largest value?
(c) Prove that the largest value of $\sin A + \sin B + \sin C$ is $3\sqrt{3}/2$.

Problem 23

Let S be a finite set of points in the plane. There will be a smallest distance d between some pair of them (which may, of course, occur between several pairs). Prove that, for any point P of S, there cannot be more than six other points of S whose distance from P is d.

Year 1963

Problem 1

Express $1/(x - 1/x) - 1/(x + 1/x) - 2x/(x^2 - 1/x^2)$ as a simple fraction.

Problem 2

Express $a - 2ax + 4ax^2 - 8ax^3/(1 + 2x)$ as a simple fraction.

Problem 3

Express $(2x/(1 - x^2))/(2 + 2x^2/(1 - x^2))$ as a simple fraction.

Problem 4

Simplify $(2 + 5x)^2 + (5 - 2x)^2 - 13x^2$.

Problem 5

Express $a/(2 - \sqrt{3})^2 + b/(3 + 2\sqrt{2})^2$ as a simple fraction.

Problem 6

Solve the system of equations $x + y = 80$ and $x^2 + y^2 = 3250$.

Problem 7

If the sum of two numbers is 80, find the largest possible value of their product.

Problem 8

If the sum of two positive numbers is equal to N, what is the smallest value of the sum of their reciprocals?

Problem 9

Two cars, A and B, cover a distance of 200 km, each at constant speed, but with B travelling at a constant speed 25/6 kph faster and hence requiring 12 minutes less time. Find the speeds of the cars.

Problem 10

Solve the equation $\sqrt{8x} - \sqrt{x + 1} = 1$.

Problem 11

Two straight lines $y = 3x + 7$ and $y = 5x - 4$ meet in one point. Prove that all other lines passing through the same point have equations of the form $y = k(3x + 7) + (1 - k)(5x - 4)$.

Problem 12

A hall has dimensions 10 metres by 20 metres. At one end, in the middle, one metre from the floor, is a fly. At the other end, in the middle, one metre from the ceiling, is a spider. The spider, being hungry, wishes to take the shortest route possible to crawl from where it is to where the fly is. What is the length of the shortest route?

Problem 13

(a) Prove that the sum of the first n positive integers is $n(n + 1)/2$.
(b) Prove that the sum of the cubes of the first n positive integers is $n^2(n + 1)^2/4$.
(c) Simplify $n^3(n + 1)^3 - n^3 (n - 1)^3$.
(d) Prove that the sum of the fifth powers of the first n positive integers is $n^3(n + 1)^3/6 - n^2 (n + 1)^2/12$.

Problem 14

Prove that the perpendicular distance of any point (a,b) such that $3a + 4b > 10$ from $3x + 4y = 10$ is $(3a + 4b - 10)/5$.

Problem 15

Let S be the sum of the first n terms of the series $a + 4ax + 9ax^2 + \cdots + n^2ax^{n-1}$.
(a) Prove that $S - xS = a + 3ax + 5ax^2 + \cdots + (2n - 1)ax^{n-1} - n^2ax^n$.
(b) Express $(1 - x)^2S$ as a simple fraction.

Problem 16

From a point P on the circumference of a circle, a distance PT of 10 metres is laid out along the tangent. The shortest distance from T to the circle is 5 metres. A straight line is drawn through T cutting the circle at X and Y. The length of TX is 15/2 metres.
(a) Find the radius of the circle.
(b) Find the length of XY.

Problem 17

The equation $x^4 - 19x^2 + 20x - 4 = 0$ may be rewritten as $(x^2 + sx + p)(x^2 - sx + q) = 0$ for constants s, p and q.
(a) Prove that $(20/s)^2 = (s^2 - 19)^2 + 16$.
(b) Verify that $s = 4$ satisfies the equation in (a).
(c) Solve the original quartic equation.

Problem 18

The sides of a triangle a, b and c are related by $c^2(a + b) = a^3 + b^3$.
(a) Prove that one angle is exactly 60°.
(b) Express the area of the triangle in terms of a and b.

Problem 19

An isosceles triangle has an interior angle of 36° between two sides, each one metre long. One of the angles at the base is bisected by a line from that vertex to the opposite side. This line is x metres long.
(a) Prove that the base is also x metres long.
(b) Prove that one of the segments of the divided side is also x metres long.
(c) Prove that $x + x^2 = 1$.

Problem 20

(a) Prove that $\sin 2x = 2 \sin x \cos x$.
(b) Prove that $4 \sin 18° \sin 54° = 1$.
(c) Prove that $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$.
(d) Prove that $\sin 54° - \sin 18° = 1/2$.
(e) Prove that $\sin 18° = (\sqrt{5} - 1)/4$.

Problem 21

(a) Prove that $2/(\tan 2x) = 1/(\tan x) - \tan x$.
(b) Express $1/\tan x - \tan x - 2 \tan 2x - 4 \tan 4x$ as a simple fraction.

Problem 22

Prove that $2/(\sin 2x) = 1/(\tan x) + \tan x$.

Problem 23

(a) Prove that $(\sin(n + 1)x - \sin nx) /(\cos(n + 1)x + \cos nx) = \tan(x/2)$.
(b) Prove that $1 + 2 \cos x + 2 \cos 2x + 2 \cos 3x + \cos 4x = (\sin 4x)/(\tan(x/2))$.

Year 1964

Problem 1

If the distance s metres that a bomb falls vertically in t seconds is given by the formula $s = 16t^2/(1 + 3t/50)$, how many seconds are required for a bomb released at an altitude of 20000 metres to reach ground level?

Problem 2

(a) Find three consecutive positive even integers such that the square of the largest is equal to the sum of the squares of the other two.
(b) Prove that this is impossible for consecutive positive odd integers.

Problem 3

A man has 15878 equilateral triangular pieces of mosaic, all of side length one cm. He constructs the largest possible mosaic in the shape of an equilateral triangle.
(a) What is the side length of the mosaic?
(b) How many pieces will he have left over?

Problem 4

A, B, C and D are four points in a plane. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively.
(a) Prove that $PQRS$ is a parallelogram.
(b) How is this result modified if the four points A, B, C and D are not all in one plane?

Problem 5

Solve the equation $9(10)^{2x} - 6(10)^x + 1 = 0$.

Problem 6

Explain how logarithms may be used to compute the fifth root of a real number.

Problem 7

Let p, q and r be three positive numbers such that $p + q + r = 12$.
(a) If p is held fixed while q and r are allowed to vary, prove that product pqr is greatest when $q = r$.
(b) What is the greatest possible value of pqr?

Problem 8

Find the value of the constant k so that the equation $kx^2 + 6x - 4 = 0$ has two equal roots.

Problem 9

Find a quadratic equation whose roots are the reciprocals of the roots of $x^2 + x + 4 = 0$.

Problem 10

Prove that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$, where A, B and C are the angles of any triangle.

Problem 11

Divide 100 loaves among five men so that the shares received shall be in arithmetic progression, and so that one-seventh of the sum of the largest three shares shall be equal to the sum of the smallest two shares. Individual loaves may be subdivided if necessary. What are the shares of the five men?

Problem 12

Assume that $\sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \cdots)}))}}}$, where the number of 2's and radical signs are infinite, is a meaningful expression and has a definite real value. Prove that this value is 2.

Problem 13

(a) Find the square root of the complex number $4 - 6\sqrt{5}i$.
(b) Represent graphically the given number and also the square roots.

Problem 14

A and B are two points on a circle which is divided into parts by the chord AB. The circle is then folded along AB so that both parts of the circle are on the same side of AB, and a line is drawn from B to cut the two circular arcs at P and Q, respectively. Prove that triangle PAQ is isosceles.

Problem 15

Prove that $1 - 1/2 + 1/4 - \cdots + (-1/2)^{n-1} = 2(1 - (-1/2)^n)/3$.

Problem 16

Two cyclists are 20 km apart on a straight road and, at the same moment, begin cycling towards each other at a speed of 10 kph. At the instant they begin moving, a fly which can travel at 20 kph leaves the nose of one of them and flies towards the other. As soon as it arrives at the second nose, it turns around and flies back to the first, continuing to go backwards and forwards until the cyclists meet. How far has the fly flown?

Problem 17

Use Newton's Binomial Theorem to compute the cube root of 63.9 to four decimal places.

Problem 18

Find the fifth term in the expansion of $(a^2 - 2b^2)^{3/2}$.

Problem 19

Four cards are drawn at random from an ordinary deck of 52. What is the probability that exactly three of these will be clubs?

Year 1965

Problem 1

Let $d = \sqrt{2}$. Find the value of $(d^d)^d$.

Problem 2

Prove that, for any positive number n that satisfies the equation $x^y = y^x$, $x = (1 + 1/n)^{n+1}$ and $y = (1 + 1/n)^n$.

Problem 3

Prove that $1 + 2x/(x^2 - x + 1) = (1 - 2/(x^3 + 1))(1 + 2/(x - 1))$ if $x \neq 1$ and $x^3 + 1 \neq 0$.

Problem 4

Find constants a and b such that the equation $\sqrt{(x + 1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} = 1$ may be rewritten in the form $(x/a)^2 - (y/b)^2 = 1$.

Problem 5

Solve the equation $2x - 5 = \sqrt{2x + 1}$.

Problem 6

Find the sum of the squares of the roots of the equation $x^3 - px = q$ in terms of p and q.

Problem 7

Solve the equation $x^3 - 13x = 12$.

Problem 8

Two parallel walls at some distance apart are perpendicular to the ground level and two ladders are placed one against each wall so that the other ends touch the bases of the opposite walls. The ladders touch each other at some point between the wall, h metres above the ground. The top of the ladder of length m metres is at a height of a metres above the ground. The height of the top of the ladder of length n metres is b metres above the ground.

(a) Prove that $h = ab/(a + b)$.

(b) Find an equation involving a, m, n and h but not b.

Problem 9

(a) Prove that there do not exist positive integers m and n such that $10^m = 2^n$.

(b) Prove that $\log_{10}2$ is not a rational number.

Problem 10

Prove that $\binom{2n}{n}$ is an even number where n is any positive integer.

Problem 11

Prove that $\binom{2n}{n-1} + \binom{2n}{n+1} = 2n\binom{2n}{n}/(n + 1)$.

Problem 12

A motorized column is advancing over flat country at the rate of 15 kph. It is one km long. A dispatch rider is sent from the rear to the front on a motorcycle travelling at a constant speed. He returns immediately at the same speed and his total time is 3 minutes. How fast is he going?

Problem 13

Through a point R outside a circle with centre O and radius r, a line is drawn cutting the circle in two distinct points P and Q. Prove that $RP{\cdot}RQ = OR^2 - r^2$.

Problem 14

Prove that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

Problem 15

(a) Prove that $1/n^2 - 1/(n + 1)^2 = (2n + 1)/n^2(n + 1)^2$.

(b) Prove that $3/1^2 2^2 + 5/2^2/3^2 + \cdots + (2n - 1)/n^2(n - 1)^2 = 1 - 1/n^2$.

(c) Express $1/n(n^2 - 1)$ as a sum of fractions with simpler denominators.

(d) Express $1/2(2^2 - 1) + 1/3(3^2 - 1) + 1/4(4^2 - 1) + \cdots + 1/n(n^2 - 1)$ as a simple fraction in terms of n.

Problem 16

The radius of the base of a right circular cone is 2 metres and the slant height from the edge of the base to the vertex is 6 metres. Find the total surface area of the cone.

Problem 17

The radius of the base of a right circular cone is 2 metres and the slant height from the edge of the base to the vertex is 6 metres. From a point A on the edge of the base one may proceed to a point B halfway up the cone towards the vertex. Consider the point C directly opposite B, also halfway up the cone. Find the shortest distance from A to C on the surface of the cone.

Year 1966

Problem 1

Factor $x^5 + x^4 + x^3 + x^2 + x + 1$ as far as possible into polynomials with integral coefficients.

Problem 2

Prove that the roots of the equation $bx^3 + a^2x^2 + a^2x + b = 0$ are in geometric progression.

Problem 3

If the product of two positive numbers is 36, prove that their sum is at least 12.

Problem 4

ABC is a triangle with $AB = AC$. D is a point on AB extended, and E is a point on CA (or CA extended) such that the angles BEC and BDC are equal. Prove that $BE = CD$.

Problem 5

Solve the inequality $1/(x - 1) + 1/(x + 1) > 1/2$.

Problem 6

Solve the system of equations $x + y + z = 7$, $3x + 2y - z = 3$ and $x^2 + y^2 + z^2 = 21$.

Problem 7

There are ten guests at a party. Assume that all acquaintances are mutual and that no one is considered an acquaintance of himself or herself. Prove that two of the guests are acquainted with the same number of guests at the party.

Problem 8

Indicate the region in the xy-plane for which $x + y$ takes values between -2 and 2 inclusive.

Problem 9

The line $y = 3x + b$ meets the parabola $2y = x^2 + 2x$ in two distinct points P and Q.
(a) What restriction does this place on b?
(b) Prove that the x coordinate of the midpoint of PQ is independent of b.

Problem 10

Prove that the area of a triangle inscribed in a parallelogram is at most one-half the area of the parallelogram.

Problem 11

Let S_n denote the sum of the first n terms of the series $1 + 2/2 + 3/4 + \cdots + n/2^{n-1} + \cdots$.
(a) Calculate S_5.
(b) Prove that $4 - S_n = (n + 2)\,(1/2)^{n-1}$.
(c) Find the "sum to infinity" of this series.

Problem 12

P is a point on the side CD of a parallelogram $ABCD$. AP and BC, extended if necessary, meet at Q. AD and BP, extended if necessary, meet at R. Prove that $1/BQ + 1/AR = 1/AD$.

Problem 13

Let $F(x,y)$ be a function of x and y such that for any x and y, (1) $F(x,y) = F(y,x)$;
(2) $F(x,y) = F(x,x - y)$. Prove that $F(x,y) = F(-x,-y)$.

Problem 14

A triangle has sides 20 cm, 20 cm and 5 cm. Find the lengths of its interior angle bisectors.

Problem 15

(a) Prove that two consecutive integers have no common divisors other than ± 1.
(b) Suppose $n + 1$ positive integers are taken, all different and none greater than $2n$. Prove that at least two of them have no common divisors other than ± 1.

Problem 16

(a) Express $x/(1 - x) - x/(1 + x)$ as a simple fraction.
(b) Prove that $x/(1 - x) = x/(1 + x) + 2x^2/(1 + x^2) + 4x^4/(1 + x^4) + \cdots$ for $-1 < x < 1$.

Problem 17

Prove that, however large the positive number N may be, one can always find a number whose logarithm to base 10 is greater than N.

Problem 18

Prove that $\sin A + \sin B + \sin C = 4\cos(A/2)\cos(B/2)\cos(C/2)$, where A, B and C are the angles of any triangle.