Counting It Twice

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He's making a list and checking it twice
Gonna find out who's naughty and nice
....

He's counting a set and doing it twice
Gonna find out if something is nice
....

Addition is commutative and associative—so what? It does not matter, when you sum several numbers, in which order you add them, or which subtotals you form and then add to get the total. It always comes out to the same result. This principle is used more often in everyday occurrences than we realize. Here are a few examples.

Taking attendance at a meeting. Typically, to assure there is a quorum, two or more counters tally the people and, almost without fail, the order in which they count is completely different. One may begin at the front of the auditorium, the other at the back. The counts are compared and, if they agree, that number is the official recorded attendance. (If the counts do not agree, then a new count is taken.)

Keeping track of a chequing account balance. As you write cheques, you subtract the amount of each cheque from your chequebook balance. As the cheques are cashed and clear the bank, the bank subtracts the amount of each cheque from its record of your balance. The order in which the cheques are written and the order in which they are cashed are almost never the same, yet the amount your chequebook shows and the amount the bank statement shows must agree. (When they do not, you or the bank should re-examine the records.)

One of the oldest ways of bookkeeping is to record figures in rows and columns and then to tally each row, tally each column and, finally, check that the total of the row sums equals the total of the column sums. The sales record for a company for a year can be laid out this way: each of 12 columns corresponds to a month and each row corresponds to a particular item sold. The tally of each row is the yearly sales for a particular item and the tally of each column is the monthly sales for all the items sold by the company. The grand total of all the rows is the yearly total sales of the company; this is also the grand total of all the columns.

This commonsense technique of checking bookkeeping records can also be used as an effective technique to achieve surprising mathematical results. This principle has been dubbed the *Fubini Principle* by S.K. Stein (1979), in honor of the theorem in analysis which bears G. Fubini's name. This theorem states that, for well-behaved functions of two variables, x and y, the double integral can be evaluated as an iterated integral, integrating first with respect to x, then with respect to y, or in the other order. In other words, the value of the integral does not depend on the order of integration of the variables. The "discrete sum" version of this principle says that, when summing a rectangular array of numbers, the value of the sum does not depend on whether one first sums over rows or first sums over columns. In sigma notation,

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij} .$$

In our bookkeeping example above, the number a_{ij} would represent the sales of item i in month j.

Acknowledgments—I wish to thank my friends who suggested examples for inclusion in this paper: Steve Maurer, Andy Liu and Martin Gardner.

The equation just says that finding the total yearly sales of each item and then adding these will give the same result as finding total sales of all items for each month and then adding these.

Many mathematical proofs involving entries in a matrix, or some other set of numbers with double subscripts, employ the Fubini principle to change the order of summation at a critical moment to achieve success. Here is a partial solution to a problem on the 1977 International Mathematical Olympiad which cleverly uses a matrix arrangement and counts twice to show an impossible situation. The problem is—*Find the longest sequence of real numbers such that the sum of every 7 consecutive terms is positive and the sum of every 11 consecutive terms is negative.* Here is how to show that the sequence cannot have length 17 (or more).

If there is such a sequence $a_1, a_2, \ldots a_{17}, \ldots$, then form an 11 x 7 matrix with entries as shown below—

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_7 \\ a_2 & a_3 & a_4 & \cdots & a_8 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{17}. \end{array}$$

If we calculate the sum of all the entries in the matrix row by row, then clearly the sum is positive. But if we calculate the same sum column by column, the sum must be negative. Thus the sequence with the desired properties cannot have length 17 or more. (A sequence of length 16 satisfying the stated condition can be constructed.)

Most often, sets of numbers (or other sets of objects) are not neatly displayed in rows and columns, yet any sum that accounts for every item will give the total. So we will adopt, with Stein, the following maxim as the Fubini principle—*When you count a set in two different ways, you get the same result.*

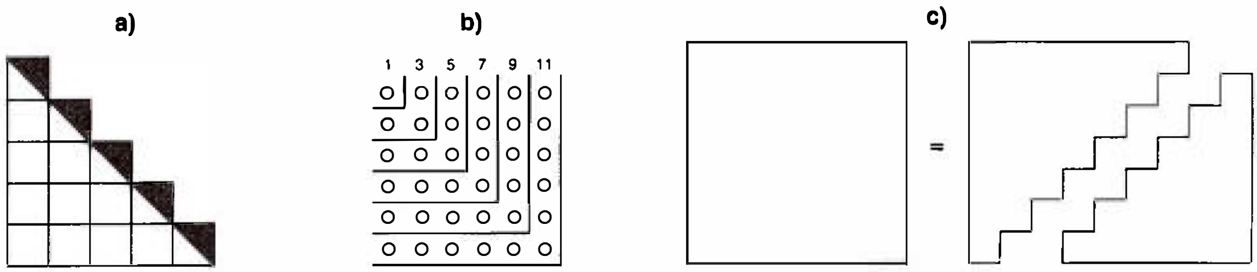
Of course. What could be more obvious? The surprising thing is that this simple statement is actually a powerful device in discovering and in proving formulae. Sometimes these formulae give succinct expressions for the number of objects in a given set; sometimes they show that two expressions which appear very different are actually equal.

Picture proofs: counting using partitions

Many of the "Proof without Words" picture proofs of counting formulae exemplify the use of the Fubini principle. A set of objects is shown and then partitioned (sometimes in two different ways); the partition yields an expression for the number of objects in the set (the whole is equal to the sum of its parts). Formulae for polygonal numbers are easily seen this way. The nth triangular number, t_n, is the number of dots in an array of n rows in which the first row has one dot and each succeeding row contains one dot more than the previous one; thus $t_n = 1 + 2 + \cdots + n$.

The sequence of triangular numbers begins 1,3,6,10,15,21,28, ... The nth square number, s_n, is the number of dots in a square array of dots with n rows and n columns; this is just the number n^2, so $s_n = n^2$.

Figure 1 shows some picture proofs which employ the Fubini principle to obtain some formulae for triangular numbers and square numbers which are not obvious from their definitions. In these pictures, the number 1 is sometimes a dot and sometimes the area of a small unit square.



a) $t_n = 1 + 2 + 3 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$

b) $s_n = 1 + 3 + 5 + \cdots + (2n - 1)$

c) $s_n = t_n + t_{n-1}$

Figure 1

Sources: a) Richards (1984); b) and c) Gardner (1986) and others.

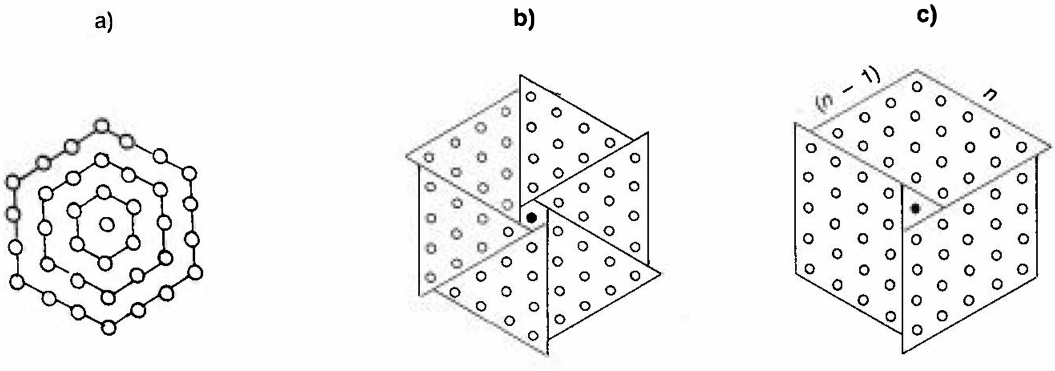


Figure 2

Another nice picture example, involving hex numbers, is given by Martin Gardner (1988). These are "centred" hexagonal numbers, obtained by arranging successive layers of dots in a concentric hexagonal array, as shown in Figure 2(a). The sequence of hex numbers begins: 1,7,19,37,61,91,127, . . .; the nth hex number is the number of dots in the first n layers of the hexagonal array. Two distinct ways of partitioning the array are shown in Figure 2(b) and 2(c); each gives an obvious formula for the nth hex number h_n. Applying the Fubini principle gives the identity,
$h_n = 6t_{n-1} + 1 = 3n(n - 1) + 1.$
Although defined by an entirely different configuration, this equation shows that the hex numbers lead to the identity for triangular numbers in Figure 1.

If you begin to sum consecutive hex numbers, you find the sequence 1,8,27,64, . . ., so a natural conjecture is that the sum of the first n hex numbers is n^3. This is true, and Gardner gives a nice visual illustration of this fact, again using the Fubini principle. He also discusses some surprising properties of "star" numbers, the number of dots in a hexagonal star configuration, like the holes on a Chinese Checkers board. Many more of these picture proofs which use the Fubini principle can be found in issues of *Mathematics Magazine* and in "Look-See Proofs" (Gardner 1986).

Counting and recounting—combinatorial argument

The art and science of counting is a major aspect of combinatorics. In this field, formulae and identities for counting are proved and often the key device used is the Fubini principle. In fact, counting a set in two distinct ways and then setting the expressions equal to each other is what is usually meant by the term "combinatorial argument." Here are a few examples. We use the usual notation $\binom{n}{k}$ to mean the number of ways of choosing k items from n items; the compact formula for this number is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If a set has n elements, then how many subsets does it have? One way to find this is to count the number of subsets with 0 elements, the number of subsets with 1 element, 2 elements, etc. and then to take the sum of all of these numbers. Another way is to think of the n elements as numbered from 1 to n, and each subset as an n-tuple whose ith coordinate is 1 if the ith element is in the subset and 0 if it isn't. Since there are 2^n such n-tuples (n slots, each of which can be filled in two ways), there are exactly 2^n subsets.

We have answered the original question and also have obtained a combinatorial identity by counting the collection of all subsets in two ways; the left side is the first way we counted, the right side is the second way—

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n .$$

If this identity looks like the binomial theorem, your perception is correct—just replace 2^n on the right side with $(1 + 1)^n$ and it is a special case of the binomial theorem—

$$\sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k} = (a + x)^n .$$

This more general theorem can also be proved by "counting it twice" (see Tucker 1984).

Here are two other quick proofs of combinatorial identities obtained by the Fubini principle. The totality of all possible committees of k persons chosen from a club of n persons can be partitioned into two classes—those committees which will contain the club president and those which will not. If the president is on a committee, then the remaining $k - 1$ committee members are chosen from $n - 1$ people; if the president is not on a committee, then its k members are chosen from $n - 1$ people. In symbolic form, this says

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} .$$

This argument is essentially the same one given by Blaise Pascal (1623 - 1662) in his book *Traité du Triangle Arithmétique,* published posthumously in 1665 (Edwards 1987).

Now let us count the number of all possible committees (of all sizes) chosen from n persons, where each committee has a chairman designated. If we think of picking the committee members, and then designating the chairman, and count the committees of size 0, size 1, size 2, etc, we obtain the sum on the left in the equation below. But if we first choose a chairman and then count all possible committees that can be formed with that chairman, we get the count on the right side of the equation (recall our count of all subsets of a set above)—

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}.$$

Tucker (1984) and other texts on combinatorics present many more such examples of combinatorial arguments. Marta Sved (1983) gives some striking examples of counting and recounting in her article by that title. She also plays a variation on the game of "here is the answer, what is the question?". She offers a combinatorial identity and challenges readers to provide an interpretation which proves it by "counting it twice." The sequel is in "Counting and Recounting: The Aftermath" (1984).

Indirect Counting

Often counting things directly is much more complicated than counting a "complementary" set. This is the simplest form of the "inclusion-exclusion" principle of counting—*the number of elements in a subset S of a set is the number of elements in the whole set minus the elements that are not in S.* So to get a count of a subset, you can count the number of elements in its complement and subtract that number from the count of the whole set.

Here is a typical example. How many different tosses of two standard dice (one red, one blue) will have the sum of their top faces less than 10? Since the sum cannot exceed 12, it is easier to count the ways in which the sum can be 10, 11, or 12. There are six ways in which this can happen (count them) and there are 36 different combinations of the two dice, so there are 30 different tosses in which the sum is less than 10.

One of the most basic combinatorial identities also recognizes complementation in counting: the number of ways of choosing k items from n items is the same as the number of ways of choosing $n - k$ items from n items. After all, once k items are chosen, that selection determines the complementary selection of $n - k$ items. In symbolic notation,

$$\binom{n}{k} = \binom{n}{n-k} .$$

Instead of using complementation, often a set can be counted indirectly by counting a related set. Here ingenuity in recognizing a related set plays a prime role! For example, if 75 people are to play in a tennis tournament (in which you are "out" as soon as you lose a game), how many matches must be scheduled? Forget about the matches and think about winners and losers. There are exactly 74 losers in the series of matches, so there are exactly 74 matches.

Double (or more) counting

A distinctly different way of "counting it twice" is to count the objects in a set in such a way that each object is counted twice (or even more than twice). This "double counting" (or multiple counting) is often coupled with the Fubini principle to yield surprising results. A classic example which illustrates this technique is the method of obtaining the formula for the sum of the first n consecutive positive integers (the number t_n) which is attributed to the young

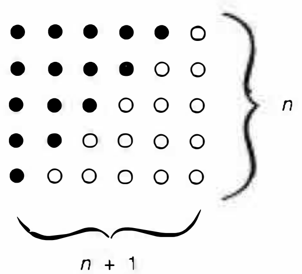
Gauss. Write down the sum twice (horizontally) as in Figure 3 and then note that all vertical sums equal $n + 1$. There are n of these sums, so that $2t_n = n(n + 1)$.

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \dots & + & n \\ n & + & n-1 & + & n-2 & + & \dots & + & 1 \end{array}$$

An easy way to see that $2t_n = n(n + 1)$

Figure 3

Using this same trick, you can easily obtain a formula for a portion of any arithmetic progression, like the sum of all the multiples of 3 from 60 to 333. Incidentally, you can also obtain the formula in Figure 3 by a picture proof of double counting: two copies of the geometric configuration for t_n fit together to form an $n \times n + 1$ rectangle. Figure 4 shows this configuration.



$2t_n = n(n + 1)$

Figure 4

Source: Gardner (1986)

To conclude this article, we give a more elaborate and sophisticated example of the power of "counting it twice."

Counting edges or counting vertices on polyhedra can be done by employing the multiple-counting principle and this leads quickly to inequalities which describe some of the geometric constraints which polyhedra must obey. Here is a quick review of the assumptions about faces, edges and vertices of a polyhedron—you may wish to test them on a familiar one, such as a cube or pyramid. Each face of a polyhedron is a polygon and each edge of a polyhedron is a common side of exactly two adjacent faces. A vertex of a polyhedron is a point where three or more edges meet; it is also a "corner" where the vertices of three or more faces coincide. Each edge of a polyhedron has as its endpoints vertices of the polyhedron; each edge touches exactly two vertices of the polyhedron.

Suppose we are given a polyhedron P which has v vertices, e edges and f faces. Without knowing the value of e, the number of edges, we can at least compare e to the values v and f, by counting. First, we use vertices to count edges. Each vertex has at least three edges which meet there, but also each edge meets exactly two vertices, so each edge is counted twice by vertices. This gives the inequality $3v \leq 2e$.

Next, we use faces to count edges. Each face is a polygon, so it must have at least three sides. Each edge is the side of exactly two polygonal faces, so counting edges which surround faces gives the inequality $3f \leq 2e$.

These constraints must be satisfied by any polyhedron—so they can be used to test if certain configurations of vertices and edges can be realized as polyhedra at all. If a polyhedron has some uniform features, such as having the same number of edges meeting at each vertex, or the same number of edges surrounding each face, then we have equations. For example, a cube satisfies $3v = 2e$ and $4f = 2e$, and an octahedron satisfies $4v = 2e$ and $3f = 2e$.

One of the most useful relationships between the numbers of vertices, edges and faces of a convex polyhedron was discovered by Leonard Euler over two hundred years ago and is known as *Euler's Formula:* $v + f = e + 2$. This formula, together with counting arguments on vertices, edges and faces, leads to many surprising and non-intuitive results. Before looking at a reference you may wish to try to prove: (1) *A convex polyhedron cannot have 7 edges;* (2) *If a convex polyhedron has only pentagons and hexagons as faces, then it has exactly 12 pentagonal faces.* (The second result tells us about the structure of geodesic domes and soccer balls.) For these and other implications of Euler's formula see Beck et al (1969).

Another very surprising result which can be proved from Euler's formula is due to René Descartes and is the three-dimensional analogue of the theorem which states that the sum of the exterior angles of a polygon equals 2π. Each vertex of a convex polyhedron is surrounded by the angles of all the polygonal faces that meet there. The sum of all of these face angles which meet at a vertex must be less than 2π in order for the polyhedron to be convex. The angular defect (or deficiency) of a vertex of the polyhedron is obtained by subtracting from 2π the sum of the face angles which meet at that vertex.

Thus, just as an exterior angle of a polygon measures how close the corresponding interior angle is to having measure π, the angular defect of a vertex of a polyhedron measures how close that corner of the polyhedron is to being flat or having measure 2π. Descartes' Theorem states that, *for any convex polyhedron, the sum of the angular defects of all of the vertices of the polyhedron is exactly* 4π.

George Pólya proved this theorem using an argument which can be used to establish an even more general result (Hilton and Pedersen 1987). The key technique is to count the sum A of all of the face angles of the polyhedron in two different ways (first using vertices, then using faces), then use the Fubini principle to equate the results. If a_n is the sum of all of the face angles about the vertex v_n, then since there are v vertices, the total defect D of the polyhedron is given by the sum,

$$D = \sum_{n=1}^{v}(2\pi - a_n) = 2\pi v - \sum_{n=1}^{v} a_n.$$

The sum of all the a_n on the right side of the equation is the sum of all face angles over the whole polyhedron and thus equals A. So we have

$$D = 2\pi v - A.$$

The sum A is also obtained if we first find the sum of the angles in each face and then sum over all of the faces of the polyhedron. If a face has m sides, then $(m - 2)\pi$ is the sum of the angles of that face; so it is efficient to lump together faces with the same number of sides as we calculate. Let f_3 denote the number of three-sided faces, f_4 the number of four-sided faces, and so on. Then f, the total number of faces of the polyhedron, is just the sum,

$$f = f_3 + f_4 + f_5 + \ldots = \Sigma f_m.$$

The sum of all of the face angles of all of the m-sided faces is $(m - 2)\pi f_m$, so the sum of all of the face angles of the whole polyhedron is

$$A = \pi f_3 + 2\pi f_4 + 3\pi f_5 + \ldots = \Sigma(m - 2)\pi f_m.$$

Even though we do not know what the largest m is, this sum is finite (since the polyhedron has only a finite number of faces). So the sum can be distributed; it equals

$$A = \pi\Sigma m f_m - 2\pi\Sigma f_m = (\pi\Sigma m f_m) - 2\pi f.$$

Since there are f_m faces with m sides, the sum of all of the terms mf_m is the total number of sides of all of the faces of the polyhedron. But since each edge of the polyhedron is the side of exactly two faces, this sum is just $2e$. So now we have

$$A = 2e\pi - 2\pi f = 2\pi(e - f).$$

If we substitute for A in our equation for D and then use Euler's formula, we have

$$D = 2\pi v - 2\pi(e - f) = 2\pi(v - e + f) = 4\pi,$$

which is Descartes' theorem.

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