Appendix V: Answers and Solutions

A. Supplementary Problems in Appendix I

Problem 1

Pretend that the pebbles have hands and that every two shake hands when they are separated. When a heap of pebbles is divided into two, the product generated simply counts the number of handshakes that are caused by that division. At the end, every two pebbles shake hands exactly once. Since there are 1,000 pebbles, the total number of handshakes is $1000 \cdot 1001/2 = 500500$. This is the desired sum, regardless of what sequence of divisions is used.

Problem 2

Consider an arbitrary problem and let it be solved by exactly r contestants. These contestants solve $6r$ other problems (counting multiplicities). Each of the remaining 27 problems is counted twice in $6r$, so that $r = 2 \cdot 27/6 = 9$. It follows that each problem is solved by exactly 9 contestants, and the number of contestants is $9 \cdot 28/7 = 36$.

Suppose every contestant solves one, two or three problems in Part I. Let n be the number of problems in Part I, and x, y and z be the respective numbers of contestants who solve one, two and three of these problems. Then

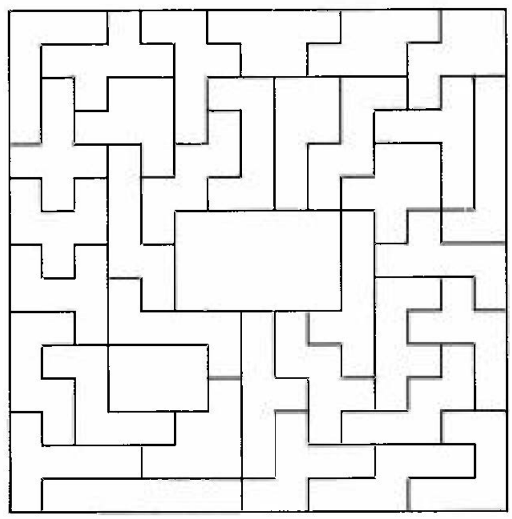
(1) $x + y + z = 36$,

(2) $x + 2y + 3z = 9n$,

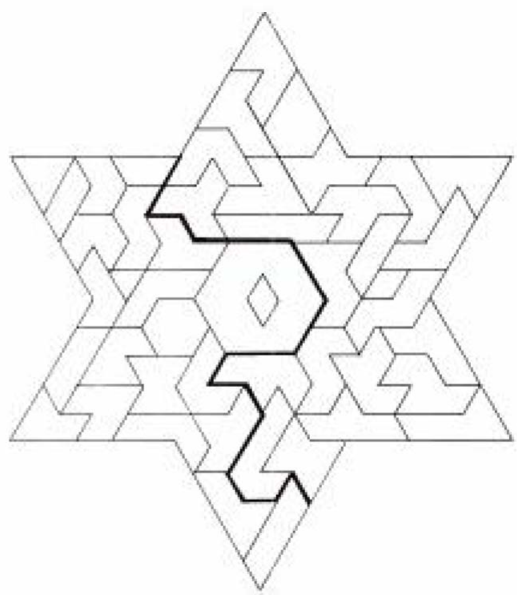
(3) $y + 3z = 2\binom{n}{2}$.

The last equation arises from the observation that, for every pair of problems in Part I, exactly two contestants solve both of them. Multiplying (1), (2) and (3), respectively, by -3, 3 and -2 and adding the resulting equations, we have $y = -2n^2 + 29n - 108 = -2(n - 29/4)^2 - 23/8 < 0$, an impossible situation. Hence at least one contestant solves four or more problems in Part I and consequently three or fewer problems in Part II.

Problem 3



Problem 4



Problem 5

<u>0</u>	5	10	15	20	25	30	35	...
1	6	11	<u>16</u>	21	26	31	36	...
2	7	12	17	22	27	<u>32</u>	37	...
3	<u>8</u>	13	18	23	28	33	38	...
4	9	14	19	<u>24</u>	29	34	39	...

We write the non-negative integers in five rows as shown in the above table. The first five multiples of 8 are underlined.

Note that since $\gcd(5,8) = 1$, there is one in each row. Clearly, a multiple of 8 is a sum of 8's. Moreover, any number to the right of a multiple of 8 can be expressed as a sum of 8's and 5's, since the numbers go up by 5 at a time in each row. In particular, all numbers in the first row can be so expressed. In each of the remaining rows, we claim that no number to the left of the first multiple of 8 can be expressed as a sum of 8's and 5's. Otherwise, there will be a smallest such number. The sum for this number cannot consist only of 8's, as we have not yet hit a multiple of 8. Hence there is at least one 5 in the sum. Knocking off a 5 yields a valid expression for the previous number in the row and our number could not have been the smallest in the first place. This contradiction justifies our claim.

Hence the numbers that cannot be expressed as a sum of 8's and 5's are 1, 2, 3, 4, 6, 7, 9, 11, 12, 14, 17, 19, 22 and 27.

Problem 6

(a) A prime triple with common difference 2 is (3,5,7).

(b) There are no others. If (x,y,z) is a prime triple with common difference d and d is not divisible by 3, then one of x, y and z must be divisible by 3.

(c) There are none. If (x,y,z) is a prime triple with common difference d and d is not divisible by 2, then one of x, y and z must be divisible by 2.

(d) A prime triple with common difference 4 is (3,7,11).

(e) There are no others. See (b).

(f) There are none. See (c).

(g) A prime triple with common difference 6 is (5,11,17).

(h) Another prime triple with common difference 6 is (7,13,19). There are lots of others.

Problem 7

The figure has 22 edge squares, but the pentominoes can cover at most 21 of them. Hence the construction is impossible.

Problem 8

The construction is impossible since the figure has only 59 squares.

Problem 9

We call an ordered pair (n,m) admissible if $m,n\epsilon[1,2,\cdots,1981]$ and $(n^2 - nm - m^2)^2 = 1$. If $m = 1$, then (1,1) and (2,1) are the only admissible pairs. Now for any admissible pair (n_1,n_2) with $n_2 > 1$, we have $n_1(n_1 - n_2) = n_2^2 \pm 1 > 0$, so that $n_1 > n_2$. Define $n_3 = n_1 - n_2$; then $n_1 = n_2 + n_3$, and substituting this into the original equation, we have $1 = (n_1^2 - n_1n_2 - n_2^2)^2 = ((n_2 + n_3)^2 - (n_2 + n_3)n_2 - n_2^2)^2 = (-n_2^2 + n_2n_3 + n_3^2)^2 = (n_2^2 - n_2n_3 - n_3^2)^2$. Thus (n_2,n_3) is also an admissible pair. If $n_3 > 1$, then, in the same way, we conclude that $n_2 > n_3$; and, letting $n_2 - n_3 = n_4$, we find that (n_3,n_4) is an admissible pair. Thus we have a sequence $n_1 > n_2 > n_3 > \cdots$ (necessarily finite) such that $n_{i+1} = n_{i-1} + n_i$, and where (n_i,n_{i+1}) is admissible for all i. The sequence terminates if $n_i = 1$. Since $(n_{i-1},1)$ is admissible and $n_{i-1} > 1$, we must have $n_{i-1} = 2$. Therefore, (n_i,n_{i-1}) consists of consecutive terms of the truncated Fibonacci sequence 1597, 987,..., 13, 8, 5, 3, 2, 1. Conversely, any such pair is admissible.

Every step of this construction is reversible; so, running it backwards from (2,1), determines uniquely the Fibonacci sequence 1, 2, 3, 5, 8, 13, ..., 987, 1597, which contains, as adjacent members, all admissible pairs. The largest such pair not exceeding 1981 is (1597,987); so the maximum value of $m^2 + n^2$ is $1597^2 + 987^2$.

Problem 10

We make the following two claims—

(1) There are at least three ways of placing 72 superknights on a superboard such that no two attack each other.

(2) If there exists a re-entrant superknight tour on a superboard, then there are at most two ways of placing 72 superknights on a superboard such that no two attack each other.

It will then follow that there are no re-entrant Superknight tours on a Superboard.

Proof of (1)—
Color the superboard in the usual checkerboard fashion.

We can then place the 72 superknights either on all the white squares or on all the black squares. In addition, we can place them on all the squares in the 1st, 2nd, 6th, 7th, 11th and 12th rows.

Proof of (2)—
Let 72 superknights be placed on the superboard such that no two attack each other. Then along the re-entrant superknight tour, there must be at least one vacant square between two occupied squares. Since there are 72 of each, vacant and occupied squares must alternate along the tour. Hence there are at most two possible placements of the superknights.

B. Contest papers in Appendix II

Year 1957

1, $(3 + x^2)/x(1 - x)(1 + x)$
2, 5
3, 29/2 metres
4, -2
5, $(-1\pm\sqrt{3}i)/2$
6, 2, $-1\pm 2i$
8, 2.4 by 2.4
9, $x = 0$, $y = 1$, $z = 2$
10, no solutions
11, (a) 3 milligrams
(b) $3/\sqrt{2}$ milligrams
12, $(4\pi-3\sqrt{3})/6$
13, 48 kph
14, 9
18, 36%

Year 1958

1, 241/168
2, $ac(2c^2 + ac + b)/(ac + b)(ac - b)$
3, 255
4, $(5\pm\sqrt{5})/2$
5, $(7\pm\sqrt{5})/2$
6, $x = 9/25$, $y = -14/25$, $z = 26/25$
8, 17
9, (a) 91390
(b) 84645
11, $(3^{10} + 1)/2 + 2^{10}$
12, 2.962
14, -1, 3/2(repeated)
18, $x = 3(-1\pm 4\sqrt{17}i)/13$,
$y = (1\pm 4\sqrt{17}i)/13$, or
$x = (27\pm 4\sqrt{3}i)/37$,
$y = 3(27\pm 4\sqrt{3}i)/37$
19, $pr(r^5 - 1)/(r - 1)$ dollars

The omission of a problem number or subsection signifies that the question requires a proof.

Year 1959

1, 2
2, 2, -2, -3
3, $x = 4$, $y = 6$, $z = 8$
4, (a) $a = c$
(b) $b = 0$
6, intersection of AC and BD
7, 480
10, yes
11, no
12, $(\sin^2\theta)/2$
13, 635/2
15, 216
16, (b) $\sqrt{ab}$
18, (c) $(2 + a/2, 0)$, $x = 2 - a/2$

Year 1960

1, 13
2, 194
3, $\sqrt{3}/4$
4, (a) 2
(b) $\pm 3/2$
5, $3x^3 + 7x^2 - 4 = 0$
6, -1, $(5\pm\sqrt{21})/2$
7, (b) no
8, (a) $x(x^4 - x^2 + 1)/(x^4 + 1)(x + 1)(x - 1)$
(b) $(1 - 2x^2 - 3x^4 - x^6)/x(x^2 + 1)(x^2 + 2)$
11, $12(\sqrt{3} - 1)$ km, S45°W
13, (a) 45
(b) 9
(c) 120
(d) 36
(e) 8
14, 127
17, (a) 682/27 metres
(b) 30 metres
19, no

Year 1961

1, $(x^5 - x^4 + 8x^3 - 24x^2 + 7x - 76)$
$/x(x^2 + 1)(x^2 + 7)$
2, $(2a - a^2c + b^2c)/\sqrt{(a + b)(a - b)}$
3, 68
4, no solutions
5, $-\log_{10}2$
6, $x > 1$ or $x < 0$
7, 36
9, a line perpendicular to AB
13, (b) $-1/\sqrt{3}$
14, 9
15, 49/333
19, (a) 1680
(b) 1280
20, 25!/10!
21, 0°, 60°, 180°, 300°
22, 30°, 150°, 210°, 330°

Year 1962

1, 3/2
2, (a) $a = 2$, $b = 3$
(b) $A = 2$, $B = 3$
3, -4, 5
4, no solutions
5, -7
7, 0
9, 0°, 120°, 240°
10, 15°, 30°, 75°, 90°, 150°, 195°, 210°, 255°, 270°, 330°
11, 90°, 210°, 270°, 330°
12, (a) $1 - 1/n$
(b) 102
13, $ax^2 - (4a - b)x + (4a - 2b + c) = 0$
15, (a) 7
(b) 105
21, (b) t^2-2
(c) $2\pm\sqrt{3}$, $(-3\pm\sqrt{5})/2$
22, (b) $A = B$

Year 1963

1, $-2x/(x^2 + 1)$
2, $a/(1 + 2x)$
3, x
4, $16x^2 + 29$
5, $(a(17 + 12\sqrt{2}) + b(7 - 4\sqrt{3}))/(7 - 4\sqrt{3})$
$(17 + 12\sqrt{2})$
6, $x = 35$, $y = 45$ or $x = 45$, $y = 35$
7, 1600
8, $4/N$
9, 125/2 kph, 200/3 kph
10, 8
12, $4\sqrt{52}$ metres
13, (c) $6n^5 + 2n^3$
15, (b) $a + ax(1 - x^{n-1})/(1 - x)$
$-(n^2 + 2n - 1)\, ax^n + n^2ax^{n+1}$
16, (a) 15/2 metres
(b) 35/6 metres
17, (c) $2\pm\sqrt{3}$, $-2\pm2\sqrt{2}$
18, $\sqrt{3}ab/4$
21, (b) $8/\tan 8x$

Year 1964

1, $25(3+\sqrt{17})/2$
2, (a) 6, 8, 10
3, (a) 126 cm
(b) 2
4, (b) no modification
5, $-\log_{10}3$
7, (b) 64
8, $-9/4$
9, $4x^2 + x + 1 = 0$
11, 5/3, 65/6, 20, 175/6, 115/3
13, (a) $\pm(3 - \sqrt{5}i)$
16, 20 km
17, 3.9979
18, $3b^8/8a^5$
19, 858/20825

Year 1965

1, 2
4, $a = 1/2$, $b = \sqrt{3}/2$
5, 4
6, $2p$
7, -1, -3, 4
8, (b) $n^2 - (ah/(a - h))^2 = m^2 - a^2$
12, 45 kph
15, (c) $1/2(n - 1) - 1/n + 1/2(n + 1)$
(d) $(n - 1)(n + 2)/4n(n + 1)$
16, $24\pi^2 + 4\pi$ metres^2
17, $3\sqrt{3}$ metres

Year 1966

1, $(x + 1)(x^2 - x + 1)(x^2 + x + 1)$
5, $1 < x < 2 + \sqrt{3}$ or $-1 < x < 2-\sqrt{3}$
6, $x = 1$, $y = 2$, $z = 4$ or $x = 16/13$,
$y = 22/13$, $z = 53/13$
9, (a) $b > -2$
11, (a) 57/16
(c) 4
14, 6 cm, 6 cm, $15\sqrt{7}/2$ cm
16, (a) $2x^2/(1 - x)(1 + x)$

C. Sample Problems in Appendix III

Problem 1

Just above the South Pole, there is a circle C_1 with circumference one mile. If the explorer starts from any point X one mile north of C_1, walking one mile south will take him to a point Y on C_1. Walking one mile east will take him once around C_1 back to Y, and walking one mile north will return him to X. We can use the same argument with the circle C_2 with circumference 1/2, the circle C_3 with circumference 1/3, and so on.

Problem 2

The distance between the two missiles is diminishing at a rate of 30000 miles per hour or 500 miles per minute. Thus one minute before they collide, they must have been 500 miles apart. The information that they were 1317 miles apart initially is redundant.

Problem 3

Pretend that there is a second monk who ascends the mountain on the day the first monk comes down and that his ascent duplicates exactly that of the first monk. Somewhere along the path, the two monks must pass each other. This is the spot we seek.

Problem 4

(Answer) 29786 + 850 + 850 = 31486.

Problem 5

Let the six stars be A, B, C, D, E and F. Suppose A loves at least three of the others, say B, C and D. If any pair of B, C and D loves each other, these two with A will form a love triangle. If not, then B, C and D form a hate triangle. If A loves at most two of the others, then A hates at least three of the others, and the same argument yields the desired result.

Problem 6

Let the distances be as indicated in the diagram (not drawn to scale). By Pythagoras' Theorem, $18000^2 + 21000^2 = (x^2 + y^2) + (w^2 + z^2)$ $= (w^2 + y^2) + (x^2 + z^2) = 6000^2 + d^2$. From this, we obtain $d = 27000$.

Problem 7

Since the number of points selected is finite, we can find a direction such that no two of the selected points lie on a line in that direction. Draw a line in that direction. If more than 500000 of the selected points lie on one side of it, move the line toward that side by parallel displacement. Since the moving line passes over the selected points one at a time, it eventually arrives at a position with exactly 500000 of the selected points on each side.

Problem 8

Deal from the bottom of the deck, first card to himself and continue counterclockwise.

Problem 9

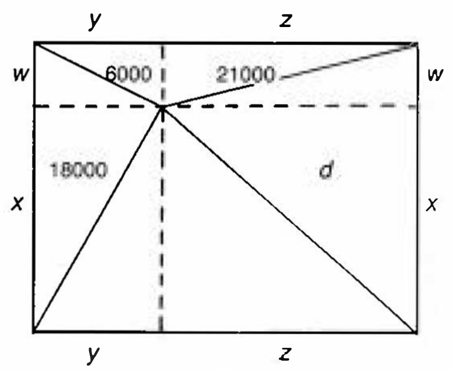
Let the weights be R_1, R_2, W_1, W_2, B_1 and B_2. Weigh R_1 and W_1 against R_2 and B_1. If they balance, weigh R_1 against R_2. If R_1 is heavier, then the heavy ones are R_1, W_2 and B_1. Otherwise, they are R_2, W_1 and B_2. If R_1 and W_1 are heavier than R_2 and B_1 in the first weighing, weigh W_1 against B_2. If they balance, the heavy ones are R_1, W_1 and B_2. If W_1 is heavier than B_2, the heavy ones are R_1, W_1 and B_1. If W_1 is lighter than B_2, the heavy ones are R_1, W_2 and B_2. The case where R_1 and W_1 are lighter than R_2 and B_1 in the first weighing can be dealt with similarly.

Problem 10

(Answer) $n < 1000000$.

Problem 11

Since there are 10 people at the party and nobody shakes hands with himself or herself or with his or her spouse, the number of handshakes is at most 8. Since each of nine people gives a different answer,



Problem 6

the answers are, collectively, 0, 1, 2, 3, 4, 5, 6, 7 and 8. Clearly, the person who shakes 8 hands is married to the person who shakes 0 hands. Eliminating this couple, we can see that the one who shakes 7 hands is married to the one who shakes 1 hand, and so on. Thus the narrator's wife shakes 4 hands.

Problem 12

(Answer) The seven points divide the perimeter into seven equal parts.

Problem 13

Suppose A does say that there is exactly one knight among them. Then B's statement is true and C's is false, making the former a knight and the latter a knave. However, we now see that this leads to a contradiction. If A is a knight, then there will be two knights and A's statement will be false. If A is a knave, then there will be one knight and A's statement will be true. It follows that B is a knave and C is a knight. We cannot determine whether A is a knight or a knave.

Problem 14

Consider all the pieces (on both sides) that move on black squares. It is possible for a pawn to be captured *en passant* by a pawn moving on white squares, but the captured pawn itself could not have made any captures. Hence the last piece moving on black squares that has captured another piece moving on black squares is still on the board. It cannot be either of the white pawns as they obviously have not moved. If the white king has moved, it must be by castling to *g*1, but then it can never get back to *e*1. Thus the white king also has not moved. Therefore, the white bishop must be on a black square since the black king is not. Hence the white bishop is on square *e*3.

Problem 15

(Answer) The white king is on square *c*3. Two moves ago, the white king was on *b*3, a white pawn was on *c*2 and a black pawn was on *b*4. As the black bishop gave check, the white pawn blocked and was captured *en passant*.

Problem 16

If the first sign is true, then the second must also be true. Hence the first sign is false and the lady is in the second room.

Problem 17

If the Cook is mad, then one of the two is indeed mad and the mad Cook will be believing something true. Hence the Cook is sane and at least one of the two is mad. The mad one must be the Cheshire Cat.

Problem 18

For B and C, there is a bird E such that $B(C(x)) = E(x)$ for every x. For A and E, there is a bird D such that $A(E(x)) = D(x)$ for every x. Hence $A(B(C(x))) = A(E(x)) = D(x)$.

Problem 19

(Answer) "I will not get exactly one prize."

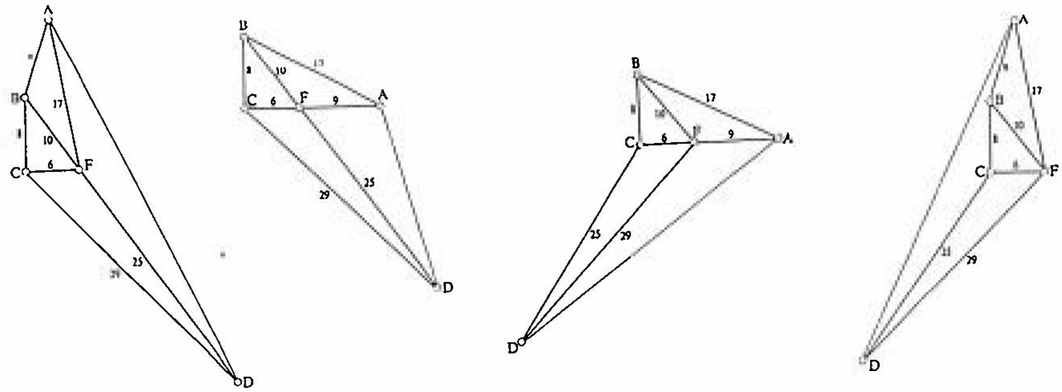
Problem 20

Let a, b and c be the sides of a triangle whose area is numerically equal to its perimeter. If s denotes the semi-perimeter, then the area is given by $\sqrt{s(s-a)(s-b)(s-c)}$ and the perimeter by $2s$. Setting $x = s-a$, $y = s-b$ and $z = s-c$, we have $xyz = 4(x + y + z)$. We may assume that $x \leq y \leq z$. If $x \geq 4$, then $xyz \geq 16z$ while $4(x + y + z) \leq 12z$. Hence $x \leq 3$. If $x = 3$, we have $3xy - 4y - 4z = 12$ or $(3y - 4)(3z - 4) = 52$. We cannot have $3y - 4 = 1$ and $3z = 52$, nor can we have $3y - 4 = 4$ and $3z - 4 = 13$. If $3y - 4 = 2$ and $3z - 4 = 26$, then $y = 2$ and this contradicts $x \leq y$. Thus there are no integral solutions in this case.

If $x = 2$, we have $(y - 2)(z - 2) = 8$ and (y,z) is one of (3,10) and (4,6). If $x = 1$, then $(y - 4)(z - 4) = 20$ and (y,z) is one of (5,24), (6,14) and (8,9). Now the corresponding values for (a,b,c) are (13,12,5), (10,8,6), (29,25,6), (20,15,7) and (17,10,9).

We must use three of these five triangles for forming the Beeling market place. Since we need two pairs of common sides, the triangles must be (17,10,9), (10,8,6) and (29,25,6).

Let $ABCD$ be the marketplace and F be the flagpole. There are four possible configurations as shown (next page), but the last is inadmissible. In each of the remaining cases, either C, F and A are collinear or B, F and D are collinear. The area of the remaining triangle is easily found to be 90 square metres. Hence the overall area is $36 + 24 + 60 + 90 = 210$ square metres.



Problem 20

Problem 21

We label the holes A, B or C in diagonal fashion as shown in the first illustration. Let a, b and c denote the numbers of men in holes with labels A, B and C, respectively. Initially, $a = b = 11$ and $c = 10$. In each jump, two of these numbers go down by 1 while the third goes up by 1, so that each goes from odd to even or vice versa. After 31 jumps, a and b will be even and c will be odd. Since by then only the marked man is left, $a = b = 0$ and $c = 1$, and it stands on a C-hole. If we label the holes D, E or F as shown in the second illustration, the same argument indicates that the marked man ends on a D-hole. Now the only CD-holes are $d2$, $a5$, $d5$ and $g5$. Since the marked man starts on $d7$, the only one of these holes it can get to is $d5$, and that is where it ends.

		A	B	C		
		B	C	A		
A	B	C	A	B	C	A
B	C	A	B	C	A	B
C	A	B	C	A	B	C
		C	A	B		
		A	B	C		

(1)

		D	E	F		
		F	D	E		
F	D	E	F	D	E	F
E	F	D	E	F	D	E
D	E	F	D	E	F	D
		E	F	D		
		D	E	F		

(2)

Problem 21

Problem 22

(Answer)

Top layer	*Middle layer*	*Bottom layer*
F E C	I H B	A A A
E F C	E B G	I H D
F C G	B G D	H I D

Problem 23

(Answer) Let the rows be labelled a, b and c and the columns 1, 2 and 3. Let the white knights be on $a1$ and $c1$ and the black knights on $a3$ and $c3$ initially. A seven-move solution is $a1-b3$, $a3-c2-a1$, $c3-b1-a3-c2$, $c1-a2-c3-b1-a3$, $b3-c1-a2-c3$, $a1-b3-c1$ and $c2-a1$.

Problem 24

Two of the numbers will be consecutive and therefore relatively prime.

Problem 25

Divide the 10 by 10 by 10 box into 125 2 by 2 by 2 portions; color each portion black or white so that the whole box resembles a three-dimensional chessboard. Now 63 portions are of one color while the remaining 62 are of the other color. Since each 1 by 1 by 4 block occupies two unit cubes of each color, there will be eight unit cubes of the same color left over. Thus the task is impossible.

Problem 26

The tournament should be held at New York. Pair each non-New York master with a New York master. For this pair, the optimal place is anywhere on the straight line between the first master's home town and New York. There are more masters in New York than elsewhere and, for those who have not been paired, their optimal place is obviously New York.

Problem 27

It does not matter what color the first card is. Once drawn, only one of the remaining three cards will match it in color; thus the desired probability is 1/3.

Problem 28

(Answer) Draw $AB = p$ and on AB draw a semicircle. Draw a line at a distance q from AB, cutting the semicircle at C. Draw a line through C perpendicular to AB, cutting AB at D. Then AD and BD are the desired line segments.

Problem 29

(Answer) It is one-sixth of a circle with one of the vertices of the equilateral triangle as centre.

Problem 30

Draw a line through P perpendicular to l, intersecting l at a point D. Draw the bisector of $\measuredangle DPF$. Now every point on this bisector is equidistant from D and F. Hence apart from P, every point is closer to l than to F. Hence this bisector has only one point P in common with the parabola and is therefore the tangent to the parabola at P.

Problem 31

Suppose in a combinatorially regular polyhedron, each face has x sides and each vertex is the endpoint of y sides. Let there be V vertices, E sides and F faces.

By counting the sides of each face, we obtain a total count of xF. Since each side is counted exactly twice, we have $xF = 2E$. Similarly, $yV = 2E$. Substituting into Euler's Formula $V - E + F = 2$, we have $E(2/y - 1 + 2/x) = 2$.

Since each of x and y is at least 3, this equation shows that each is at most 5. If (x,y) is (4,4), the equation becomes $0 = 2$. If (x,y) is (4,5) or (5,4), the equation becomes $E = -20$. If (x,y) is (5,5), the equation becomes $E = -10$. Ruling out these four cases, we have five types of combinatorially regular polyhedra. If the faces are regular polygons, these polyhedra correspond to the tetrahedron (3,3), the cube (4,3), the dodecahedron (5,3), the octahedron (3,4) and the icosahedron (3,5).

Problem 32

(Answer) Let the rows be labelled a, b, c, d, e, f and g and the columns 1, 2, 3, 4, 5, 6 and 7. We can take the 21 points $a1$, $a2$, $a3$, $b1$, $b4$, $b5$, $c1$, $c6$, $c7$, $d2$, $d4$, $d6$, $e2$, $e5$, $e7$, $f3$, $f4$, $f7$, $g3$, $g5$, and $g6$.

Problem 33

Let $x = \sin 10°$. Then $1/2 = \sin 30°$ $= \sin 10° \cos 20° + \cos 10° \sin 20° = x(1 - 2x^2)$ $+ 2x(1 - x^2) = 3x - 4x^3$ or $8x^3 - 6x + 1 = 0$. The only rational numbers that can possibly be roots of this equation are ± 1, $\pm 1/2$, $\pm 1/4$ and $\pm 1/8$, but none checks out. Hence $\sin 10°$ is irrational.

Problem 34

We divide a hemisphere with horizontal base into n horizontal slices with uniform thickness. The i-th slice from the base is roughly a circular cylinder of height r/n and radius $\sqrt{r^2-(ir/n)^2}$. Hence its volume is $\pi r^3(1/n - i^2/n^3)$. Summing from $i = 1$ to n, we have $\pi r^3(1 - n(n + 1)(2n + 1)/6n^3)$ which tends to $2\pi r^3/3$ as n tends to infinity. Thus the volume of the sphere is $4\pi r^3/3$.

Problem 35

Let a, b and c be the sides of a triangle and let s be the semi-perimeter. Now the area A of the triangle is maximum if and only if $A^2/s = (s - a)(s - b)(s - c)$ is maximum. Since $(s - a) + (s - b) + (s - c) = s$ is constant, the product is maximum if and only if $s - a = s - b = s - c$. Thus the equilateral triangle has the greatest area among all triangles with fixed perimeter.

Problem 36

Let Q be the reflection of P across the bisector of $\measuredangle BAC$. Then Q is at a distance p_c from AC and a distance p_b from AB, so that the total area of the triangles QAB and QAC is $(p_bAB + p_cAC)/2$. Now these two triangles have a common base $AQ = AP$ and a combined height of at most BC. Hence $p_bAB + p_cAC \leq PA \cdot BC$. Similarly, $p_aAB + p_cBC \leq PB \cdot AC$ and $p_aAC + p_bBC \leq PC \cdot AB$. Hence
$PA + PB + PC \geq p_a\,(AB/AC + AC/AB)$
$+ p_b(AB/BC + BC/AB)$
$+ p_c(AC/BC + BC/AC) \geq 2(p_a + p_b + p_c)$
since $x + 1/x \geq 2$ for all positive numbers x.

Problem 37

(Answer) (d).

Problem 38

Since the last digit is 4, the number is divisible by 2. Since the last digits are 3 and 4, the number is not divisible by 4. Hence it cannot be a square.

Problem 39

For $k > 1$, $k > k - 1$ so that $1/k^2 < 1/k(k-1) = 1/(k-1) - 1/k$. It follows that $1/1^2 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 < 1 + (1 - 1/2) + (1/2 - 1/3) + \cdots + (1/(n-1) - 1/n) = 2 - 1/n$, which is strictly less than 2.

Problem 40

(Answer) Extend AJ to K so that $AJ = AK$. On the same side as A of BK, draw a circular arc subtending an angle supplementary to ACB. Let this arc cut CD at F and let the extension of BF cut the original circle at X. This is the desired point.

Problem 41

(Answer) $68/77 = 3/7 + 5/11$.

Problem 42

Let A, B, C and D be the first four houses. We may assume that they are connected in such a way that ABC is a triangle and D is inside. If the fifth house E is outside ABC, it cannot be connected to D. If it is inside ABD, it cannot be connected to C. If it is inside ACD, it cannot be connected to B. If it is inside BCD, it cannot be connected to A.

Problem 43

Suppose $2x + 3y$ is divisible by 17. Then so is $9(2x + 3y) - 17y = 2(9x + 5y)$. Since 17 and 2 are relatively prime, $9x + 5y$ is divisible by 17. Conversely, suppose $9x + 5y$ is divisible by 17. Then so is $2(9x + 5y) + 17y = 9(2x + 3y)$. Since 17 and 8 are relatively prime, $2x + 3y$ is divisible by 17.

Problem 44

Let n, $n+1$, $n+2$ and $n+3$ be the four consecutive positive integers. Since $n(n+1)(n+2)(n+3) = (n^2 + 3n + 1)^2 - 1$, and the only two consecutive squares are 0 and 1, $n(n+1)(n+2)(n+3)$ is not a square.

Problem 45

We have $DE = OD = OB$. Hence, $\measuredangle AOB = \measuredangle OBD + \measuredangle BEA = \measuredangle ODB + \measuredangle BEA = 2\measuredangle BEA + \measuredangle DOE = 3\measuredangle BEA$.

Problem 46

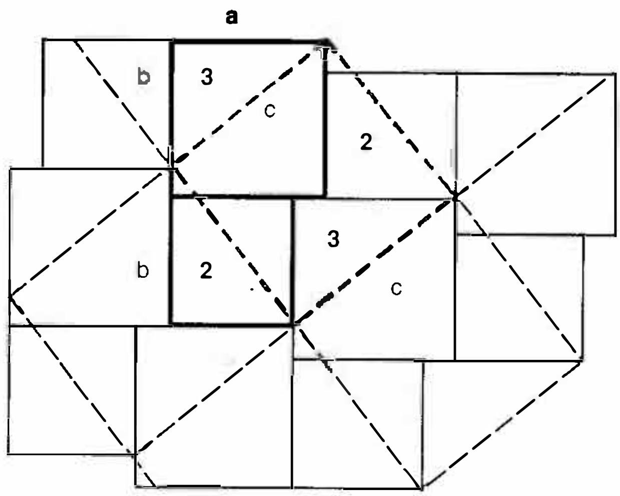
Consider the elements $x, x^2, \ldots, x^p$. Since there are p elements and only $p-1$ modulo classes, we must have $x^i \equiv x^j \pmod p$ for some $1 \le i < j \le p$. Hence $x^{j-i} \equiv 1 \pmod p$ and we can take y to be the element in the set congruent to x^{j-i-1}.

Problem 47

(Answer) 45 handshakes.

Problem 48

(Answer) See illustration.



Problem 48

Problem 49

(Answer) (a).

Problem 50

Any line passing through the centre O of the circular pancake bisects its area. Draw a directed line through O. If it also bisects the area of the other pancake, all is well. Suppose there is more of the other pancake on the right side of this directed line. Rotate it about O. After a 180° rotation, it will return to its original position but in the opposite direction. Now there is more of the other pancake on its left. Hence somewhere during the rotation, the amount of the other pancake on each side of the line must equalize.

Problem 51

Let $ABCD$ be the quadrilateral, E be the point of intersection of AC and BD, and F be a point on AC such that $ABF = EBC$. Then the triangles ABF and DBC are similar, as are BFC and BAD. Hence $AB/AF = BD/DC$ and $BC/FC = BD/AD$, and $AB \cdot CD + BC \cdot AD = BD \cdot AF + BD \cdot FC = BD \cdot AC$.

Problem 52

(Answer) 45360.

Problem 53

(Answer) We use a transformation called homothety. Take any point P on AC and draw the square $PQRS$ with Q and R on AB. Draw the line AS intersecting BC at Z. Draw a line through Z parallel to AB, cutting AC at W. The rectangle $WXYZ$ with X and Y on AB is easily seen to be the desired square.

Problem 54

We consider all possible shifting of the first word:

FGHIJKLMNOPQRSTUVWXYZABCDE
RSTUVWXYZABCDEFGHIJKLMNOPQ
ZABCDEFGHIJKLMNOPQRSTUVWXY
DEFGHIJKLMNOPQRSTUVWXYZABC
UVWXYZABCDEFGHIJKLMNOPQRST
GHIJKLMNOPQRSTUVWXYZABCDEF
VWXYZABCDEFGHIJKLMNOPQRSTU

The only meaningful word is COWARDS. Thus each letter has been shifted three places forward and the original message is COWARDS DIE MANY TIMES BEFORE THEIR DEATHS.

Problem 55

Let $d_1, d_2, \ldots, d_m$ be the positive divisors of an abundant number n. Then $kd_1, kd_2, \ldots, kd_m$ are among the positive divisors of kn. Since $d_1 + d_2 + \cdots + d_m > 2n$, $kd_1 + kd_2 + \cdots + kd_m > 2kn$ and kn is abundant.

Problem 56

Take a point C on the opposite side of AB of the line parallel to AB. Join AC and BC, cutting the line at E and D, respectively. Join AD and BE, intersecting at G. Join CG and extend it to intersect AB at F. We claim that F is the midpoint of AB. By Ceva's Theorem, we have $BD \cdot CE \cdot AF/DC \cdot EA \cdot FB = 1$. Since ED and AB are parallel, we have $CE/EA = DC/BD$. Hence $AF/FB = 1$ or $AF = FB$ as claimed.

Problem 57

(Answer) (a).

Problem 58

Let $x_0 = 2$. Define $y_0 = 2/x_0 = 1$.
Take $x_1 = (x_0 + y_0)/2 = 1.5$.
Define $y_1 = 2/x_1 \approx 1.333$ to three decimal places.
Take $x_2 = (x_1 + y_1)/2 \approx 1.416$.
Define $y_2 = 2/x_2 \approx 1.412$.
Take $x_3 = (x_2 + y_2)/2 \approx 1.414$.
Define $y_3 = 2/x \approx 1.414$.
Since x_3 and y_3 agree to three decimal places, we have $\sqrt{2} \approx 1.41$ rounded off to the required two decimal places.

Problem 59

We have $2(21n + 4) + 1 = 3(14n + 3)$. Hence any number dividing both $21n + 4$ and $14n + 3$ must also divide 1. It follows that $21n + 4$ and $14n + 3$ are relatively prime so that there is no reduction for $(21n + 4)/(14n + 3)$.

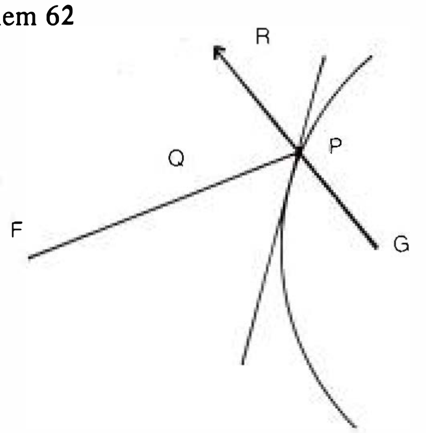
Problem 60

The chance of not getting any 6's in three rolls of an honest die is $(5/6)^3 = 125/216 > 1/2$. Thus the chance of getting at least one 6 is less than 1/2.

Problem 61

(Answer) (d).

Problem 62



Problem 62

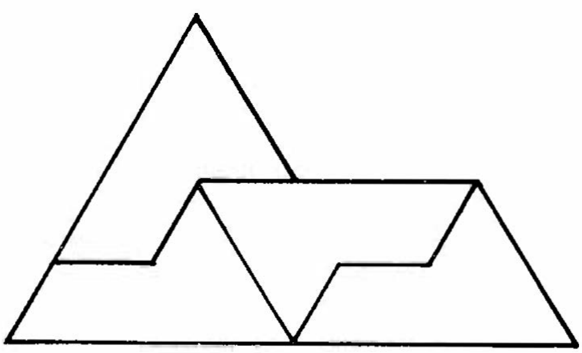
Let F and G be the foci of the hyperbola. Let the branch closer to G be the mirror, so that the light source is placed at F. For any point P on the mirror, $PF - PG$ is a constant k. We claim that the bisector l of $\measuredangle FPG$ is the tangent to the hyperbola at P. Let Q be the reflection of G across l. Then Q lies on PF. For any point T other than P on l, $FT - GT = FT - QT < FQ = k$, so that it is not on the hyperbola. It follows that l is indeed a tangent as claimed. Now the light ray from F falling on P will reflect along the ray PR which is equally inclined to l as FP. Since l bisects $\measuredangle FPG$, G, P and R are collinear so that the reflected ray will appear to originate from G.

Problem 63

Take a circle and divide it into $n + 1$ arcs, numbered 0, 1, ..., n. Take 1 red and r green markers. Consider the ways of placing the markers on the arcs so that no arc has no more than one marker. For each possible position of the red marker, each way of placing the green marker corresponds to a subset of r elements of 1, 2, ..., n, namely, the distances of the green markers from the red one, say, measured in the clockwise direction. We want to find the average distance from the red marker to the next marker in the clockwise direction. We consider the configurations in which a given set of arcs is occupied. There are $r + 1$ possible places for the red marker when the occupied arcs are given. The sum of the distances from the red marker to the next marker, taken over all these possibilities, is just the sum of the distances from one occupied arc to the next taken around the whole circle, which is $n + 1$. Hence the average distance over each of these groups is $(n + 1)/(r + 1)$, which must also be the overall average.

Problem 64

(Answer) See illustration.



Problem 64

Problem 65

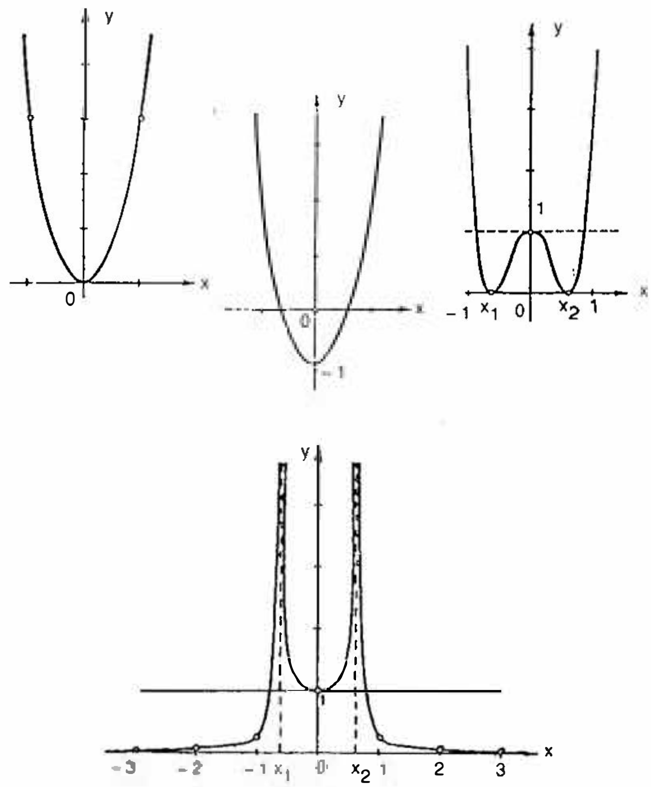
Let $f(x) = x - 5x + 2x + 1$. Then $f(-1) = -7$, $f(0) = 1, f(1) = -1, f(2) = -7, f(4) = -7$ and $f(5) = 11$. Hence the equation $f(x) = 0$ has three real roots, one between -1 and 0, one between 0 and 1 and one between 4 and 5.

Problem 66

Divide each area into n vertical strips of uniform width. The i-th strip from the left in the first area is roughly a rectangle of width $2/n$ and height $1/(1+2i/n)$, while that in the second area is roughly a rectangle of width $6/n$ and height $1/(3 + 6i/n)$. Thus the corresponding rectangles have the same area and so will their sums and the limiting values.

Problem 67

(Answer) See illustrations.



Problem 67

Problem 68

(Answer) The transformation is a 45° counterclockwise rotation. The image is a triangle with vertices at 0, $\sqrt{2}$ and $\sqrt{2}i$.

Problem 69

Let r and h, respectively, be the radius and height of the cylinder. The $V = \pi r^2 h$ and the surface area is given by $2\pi r^2 + 2\pi rh = 2\pi r^2 + V/r + V/r$. The product of the three terms is $2\pi V^2$, a constant. Hence the minimum sum occurs when $2\pi r^2 = V/r$ or $h = 2r$.

Problem 70

(Answer) Draw any ray from A and mark off on it points P, Q, R, S and T so that $AP = PQ = QR = RS = ST$. Draw a line through S parallel to BT, intersecting AB at C. Draw a line through S parallel to BR, intersecting the extension of AB at D.

Problem 71

(Answer) We should always use weapon A_3 and the enemy should always use aircraft B_2.

Problem 72

Let $x = 1/(1 + 1/(1 + 1/(1 + \cdots)))$. Then $x = 1/(1 + x)$ or $x^2 + x - 1 = 0$. By the Quadratic Formula, $x = (-1 \pm \sqrt{1+4})/2$. Since $x > 0$, $x = (\sqrt{5}-1)/2$.

Problem 73

If there are numbers which have no prime factorizations, then there must be a smallest such number x. Now x cannot be a prime, as otherwise x itself constitutes a prime factorization. Hence $x = ab$ for smaller positive integers a and b. Being smaller than x means that each of a and b has a prime factorization, but then the concatenation of the prime factorizations of a and b will be a prime factorization, contradicting the assumption that x has no prime factorizations. Hence every integer greater than 1 has a prime factorization.

Problem 74

(Answer) With A as centre, draw an arc with AC as radius. With B as centre, draw an arc with BC as radius, intersecting the first arc in a second point D. With C and D as respective centres, draw two circles with radius r. Their points of intersection are the desired points.

Problem 75

Let the elements of the first set be $f(1)$, $f(2)$, $f(3)$, . . . and the elements of the second set be $g(1)$, $g(2)$, $g(3)$, . . ., where f and g are the respective counting functions. A counting function h for the union of these two sets may be defined by $h(2n-1) = f(n)$ and $h(2n) = g(n)$ for $n = 1, 2, 3, \ldots$.

Problem 76

Let A and B be two points on a line projected from a point V onto a plane Π. If V, A and B are collinear, the image of the line AB is clearly the point of intersection of AB with Π. Otherwise, V, A and B determine a plane Π' which intersects Π in a line l. It is easy to see that l is the image of AB on Π.

Problem 77

We use induction on n. For $n = 1$, the result is trivial. Suppose the task is accomplished for a particular value of n. Consider now an $(n+1)$st square. This can be combined with the composite square obtained from the first n squares using the construction given in Problem 48. Since only a finite number of steps are involved, the number of pieces generated is clearly finite.

Problem 78

By the Arithmetic-Mean-Geometric-Mean Inequality, we have $n! = (1 \cdot n)(2(n-1))(3(n-2)) \cdots < ((n+1)/2)^n$.

Problem 79

(Answer) The treasure can be found. The hunter can take any spot as the location of the missing gallows.

Problem 80

(Answer) The image is a straight line perpendicular to the line joining O and the centre of the circle passing through O.

Problem 81

(Answer) $x/|x| + y/|y| = 2$.

Problem 82

Postage amounting to 8¢ can be made up of one 3¢ and one 5¢ stamps. Postage amounting to 9¢ can be made up of three 3¢ stamps. Postage amounting to 10¢ can be made up of two 5¢ stamps. For any amount over 10¢, an inductive argument shows that it can be made up using only 3¢ and 5¢ stamps, because the amount 3¢ less can be so made up.

Problem 83

Let $f(x) = x^3 + 3x - 1$. Then $f(0) = -1$ while $f(1) = 3$. Thus the equation $f(x) = 0$ has a solution between 0 and 1. Now the graph of $y = f(x)$ is not a straight line. Approximating the crucial portion of it between $x = 0$ and $x = 1$ by a straight line joining $(0,-1)$ to $(1,3)$, we see that this line $y = 4x - 1$ intersects the x-axis at the point $(1/4,0)$. Thus 1/4 is an approximate solution to the equation $x^3 + x - 1 = 0$.

Problem 84

Since 139 of 400 points fall within the figure, 139/400 would be a good approximation of the area of the figure.

Problem 85

Let l be the tangent to a circle with centre O, at the point P. Draw the diagram on a vertical plane so that l is horizontal and below O. Then P is the lowest point on the circle. Attach one end of a string of length OP to O and the other end to a weight. When released, the weight will settle in its lowest possible position. Since the weight is on the circle, it must be at P. Since the string is vertical and l is horizontal, the desired conclusion follows.

Problem 86

(Answer) The tapehead is back at cell 0 and there are no changes other than cell 0 being marked and then unmarked.

Problem 87

Let BE be the bisector of $\measuredangle ABC$ and CF be the bisector of $\measuredangle ACB$. Suppose $AB > AC$. Then $\measuredangle ABC < \measuredangle ACB$. Complete the parallelogram $BEGF$. Since $FG = BE = FC$, $\measuredangle FGC = \measuredangle FCG$. Since $\measuredangle FGE = \measuredangle FBE < \measuredangle FCE$, $\measuredangle CGE > \measuredangle GCE$ so that $CE > GE = BF$. In triangles BCF and CBE, we have $BC = CB$ and $BF = CE$. Since $\measuredangle EBC < \measuredangle FCB$, $EC < FB$, and we have a contradiction.

Problem 88

(Answer) $s_n = (3^{n+1} - 1)/2$.

Problem 89

Let C be the point on AB and D be the point on AB extended so that $AC/BC = AD/BD = 3$. We claim that the circle with CD as diameter is the desired locus. Draw BE parallel to CP and BF parallel to DP for an arbitrarily chosen point P on the circle, with E and F on AP. Since $\measuredangle CPD = 90°$, $\measuredangle EBF = 90°$. Now $AP/EP = AC/BC = AD/BD = AP/FP$. Hence $EP = FP = BP$ and $AP/BP = AC/BC = 3$. Conversely, for any point P such that $AP/BP = 3 = AC/BC$, PC bisects $\measuredangle APB$ and PD bisects the exterior $\measuredangle APB$. Hence $\measuredangle CPD = 90°$ and P lies on the circle.

Problem 90

If we unroll the cylindrical surface into a plane, the shortest distance between A and C is obviously the straight line. Since $BC = 8$ and $AB = 6$, $AC = \sqrt{8^2+6^2} = 10$. When rolled back onto the cylindrical surface, this shortest path is part of a spiral.

Problem 91

If $d = c^2$ for some integer c, then $(x + cy)(x - cy) = 1$. Hence both factors are 1 or -1. From $x + cy = x - cy$, we have $y = 0$. It follows easily that $x = \pm 1$.

Problem 92

Let T be the point diametrically opposite S. Then $\measuredangle STM' = \measuredangle SMT = 90°$. Hence $\measuredangle SM'N' = \measuredangle STM = \measuredangle SNM$. Similarly, $\measuredangle SN'T = \measuredangle STN$. It follows easily that $\measuredangle SN'M' = \measuredangle SMN$.

Problem 93

(Answer) The minimum value of 3 is attained at the point $(x,y) = (0,3)$.

Problem 94

We have
$A \cap (A \cup B) = (A \cup \phi) \cap (A \cup B) = A \cup (\phi \cap B)$
$= A \cup \phi = A$.

Problem 95

(Answer) Make the first two cuts as shown in the third frame. Make the third cut horizontally halfway between the top and the bottom of the cake.

Problem 96

(Answer) There are no missing dollars. The price structure has changed, even though it does not appear to be the case, as the clerk believes.

Problem 97

Let ABC be the equilateral triangle of side s. Let h_a, h_b and h_c be the distances of a point H to BC, CA and AB, respectively. Now the area of the triangle ABC is equal to the sum of the areas of the triangles HAB, HBC and HCA, which is $s(h_a + h_b + h_c)/2$. Since this area is constant and so is s, $h_a + h_b + h_c$ is also a constant. Hence any point on the island is as good as any other.

Problem 98

Adding the two equations yields $100x + 100y = 1000$ or $45x + 45y = 450$. Hence $10x = 70$ and $10y = 30$, so that $x = 7$ and $y = 3$.

Problem 99

(Answer) Draw a marble from the box labelled red/white.

Problem 100

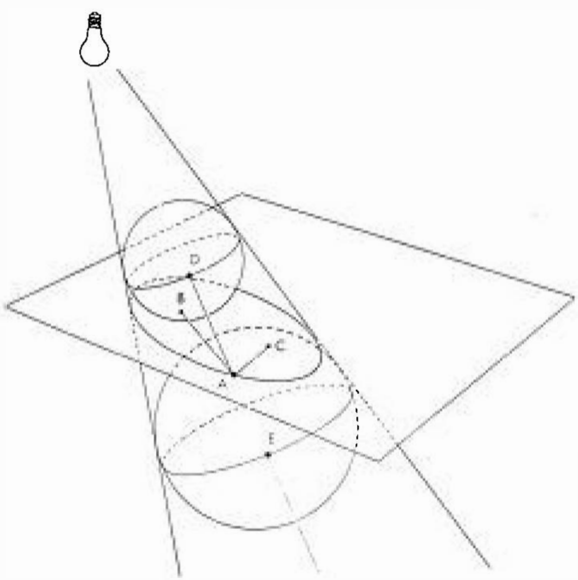
The computation on the left-hand column expresses a number as a sum of powers of 2. In the example featured in the statement of the problem, $35 = 17 \times 2 + 1 = 8 \times 2^2 + 2 + 1$ $= 4 \times 2^3 + 2 + 1 = 2 \times 2^4 + 2 + 2$ $= 2^5 + 2 + 1$. The computation on the right-hand column multiplies a number by powers of 2. In the featured example, $2 \times 56 = 112$, $2^2 \times 56 = 224$, $2^3 \times 56 = 448$, $2^4 \times 56 = 896$ and $2^5 \times 56 = 1792$. The overall computation works because multiplication is distributive over addition. In the featured example, we have 35×56 $= (2^5 + 2 + 1) \times 56 = 2^5 \times 56 + 2 \times 56 + 56$ $= 1792 + 112 + 56 = 1960$.

Problem 101

(Answer) $9567 + 1085 = 10652$.

Problem 102

Pretend that there is a larger sphere beneath the level surface inside the cone, tangent to both the level surface and the cone, as shown in the illustration. Let B and C be the points of tangency with the level surface of the original sphere and the larger sphere, respectively. Let A be any point on the boundary of the shadow. Draw a line passing through A and the apex of the cone. This line will be tangent to the spheres at D and E, respectively. By symmetry, DE has constant length regardless of the position of A. Now $AB = AD$ since both are tangents from A to the original sphere. Similarly, $AC = AE$. Hence $AB + AC = AD + AE = DE$ is constant. It follows that the boundary of the shadow is an ellipse with B and C as foci.



Problem 102

Problem 103

Let ω be a root of the equation $x^3 - 1 = 0$ not equal to 1. Since $(\omega - 1)(\omega^2 + \omega + 1) = 0$ and $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$. In the binomial expansion for $(1 + x)^n$, substitute $x = 1$, $x = \omega$ and $x = \omega^2$ in turn to obtain $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, $\binom{n}{0} + \binom{n}{1}\omega + \binom{n}{2}\omega^2 + \cdots + \binom{n}{n}\omega^n = (1 + \omega)^n$ and $\binom{n}{0} + \binom{n}{1}\omega^2 + \binom{n}{2}\omega^4 + \cdots + \binom{n}{n}\omega^{2n}$ $= (1 + \omega^2)^n$. Note that $1 = \omega^3 = \omega^6 = \cdots$, $\omega = \omega^4 = \omega^7 = \cdots$ and $\omega^2 = \omega^5 = \omega^8 = \cdots$. Adding the three identities and using $\omega^2 + \omega + 1 = 0$, we have $3[\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots] = 2^n + (-1)^n\omega^{2n}$ $+ (-1)^n\omega^n$. Hence $S = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots$ $= 2^n + (-1)^n(\omega^n + \omega^{2n}))/3$. It follows that
if $n \equiv 0 \pmod 6$, $S = (2^n + 2)/3$;
if $n \equiv 1$ or 5, $S = (2^n + 1)/3$;
if $n \equiv 2$ or $4 \pmod 6$, $S = (2^n - 1)/3$;
and if $n \equiv 3 \pmod 6$, $S = (2^n - 2)/3$.

Problem 104

Let the bus routes be represented by straight lines, no two parallel and no three concurrent. Let the bus stops be at all the points of intersection of two routes and nowhere else. Suppose a route connecting two stops P and Q is closed. Now P lies on another route, as does Q, and these two routes have a common stop R. Hence it is possible to get from P to Q via R. On the other hand, a stop P is on exactly two routes. If both are closed, it is no longer possible to get from P to any other stop Q.

Problem 105

Let the sides of the smaller triangle be a, b and c with $a \le b \le c$. Then the sides of the larger triangle must be b, c and d with $a/b = b/c = c/d$. Let the common value be denoted by m/n in the lowest term. Then $a/d = m^3/n^3$. Hence $a = km^3$ and $d = kn^3$ for some integer k. Now

$k(n - m)(n^2 + nm + m^2) = 387 = 3^2 43$. Since $(n - m)^2 < n^2 + nm + m^2$, $n - m$ is either 1 or 3. If $n - m = 1$, $k(3m^2 + 3m + 1) = 387$. The only divisor of 387 greater than 1 that is congruent to 1(mod 3) is 43. However, we cannot have $3m^2 + 3m + 1 = 43$ as m would not be integral. Hence $n - m = 3$ and $k(m^2 + 3m + 3) = 43$. We must have $k = 1$ and $m^2 + 3m + 3 = 43$. Hence $m = 5$ or -8, and the negative root is rejected. It follows that $a = 125$, $b = 200$, $c = 320$ and $d = 512$.

Problem 106

Let such a polygon have n sides. Then the sum of its interior angles is given by $(n-2)180°$ and also by $120° + 125° + \cdots + (120° + (n - 1)5°) = n \cdot 120° + (n(n - 1)/2)5°$. It follows that $72(n - 2) = 48n + n(n - 1)$ or $n^2 - 25n + 144 = (n - 9)(n - 16) = 0$. Hence $n = 9$ or 16, and the polygon we seek has 16 sides. Note, however, that it is degenerate, because one of its angles is 180°.

Problem 107

Think of the three circles as the equators of three spheres. Then the three points in question are the apexes of the pairwise common tangent cones of these spheres. Hence, they all lie in a common external tangent plane of the three spheres. Since they also lie in the original plane and two planes intersect in a line, they are all on a line.

Problem 108

Let the triangle be ABC with $AB = AC$ and $BC = a$. Let E be the point on AC such that BE bisects $\measuredangle ABC$, with $BE = t$. Let $CE = x$. The construction is easy once x is determined. Extend BC to D so that $CD = x$. Then triangles BED and ECD are similar and we have $t^2 = x(a + x)$. On a circle with centre O and diameter a, take a point P and draw a tangent $PQ = t$. Join OQ, cutting the circle at R and S with R closer to Q. Then $PQ^2 = QR \cdot QS = QR(QR + a)$ so that $QR = x$.

Problem 109

We claim that no two consecutive terms are both odd. Otherwise, there must be a first such pair (x,y). It is not at the beginning of the sequence because the first two terms are 2 and 3. Hence this pair is generated. Now at least one of x and y is the last digit of a previous product but, since the product is odd, we must have an earlier odd pair, contradicting our assumption that (x,y) is the first such pair. This justifies our claim. Now if a 9 appears in the sequence, it must appear as the tens digit of a product, but this is impossible. If a 7 appears, it must be generated by $9 \times 8 = 72$, but this is impossible since there are no 9s. Finally, if a 5 appears, it must be generated by either $9 \times 6 = 54$ or $7 \times 8 = 56$, but neither is possible.

Problem 110

The two conscientious jurors in the three-person jury agree on the correct decision with a probability of p^2. They will disagree with a probability of $2p(1-p)$ and, after consulting the flippant juror, the probability of a correct decision is halved to $p(1-p)$. Hence the overall probability of a correct decision is $p^2 + p(1-p) = p$, the same as that of a one-person jury.

Problem 111

We use the substitution $x = t - 1$. Then the equation becomes $t^4 - 9t^3 + 17t^2 - 18t + 13 = 0$. If $t \leq 0$, the polynomial is clearly positive. For $0 < t < 1$, $t^4 - 9t^3 + 17t^2 - 18t + 4 = t^2(t - 1)(t - 9) + 4t^2 + (5t - 13)(t - 1) > 0$. Hence $t^4 - 9t^3 + 17t^2 - 18t + 13 = 0$ has no roots less than 1, and $x^4 - 5x^3 - 4x^2 - 7x + 4 = 0$ has no negative roots.

Problem 112

(Answer) A cylindrical spiral.

Problem 113

(Answer) The farmer can catch the hen and his wife can catch the rooster.

Problem 114

Since oil sells for twice as much as vinegar, twice as much vinegar as oil is sold. Thus the total number of gallons of oil and vinegar sold is a multiple of 3. When divided by 3, the numbers 15, 8, 17, 13, 19 and 31 leave remainders of 0, 2, 2, 1, 1 and 1, respectively. It follows that the barrel that is left contains 13, 19 or 31 gallons. If it is the 13-gallon barrel, then 30 gallons of oil is sold, but the total of 30 cannot be made up from the barrels. If it is the 31-gallon barrel, we have to make up a total of 24, which again is impossible. Hence the 19-gallon barrel is left. The customer buys $15 + 13 = 28$ gallons of oil and $17 + 8 + 31 = 56$ gallons of vinegar.

Problem 115

Each outside triangle has a pair of sides equal to a pair of sides of the central triangle and the two angles included by these pairs are supplementary. Hence all four triangles have equal area. To compute the area of the central triangle, consider a rectangle $ABCD$ with $AB = 4$ and $BC = 5$. Let E be a point on AB with $AE = 1$ and F be a point on BC with $BF = 3$. Then $DE = \sqrt{26}$, $EF = \sqrt{18}$ and $FD = \sqrt{20}$, so that DEF is congruent to the central triangle. The area of DEF is easily computed to be 9. Hence the total area of Farmer Wurzel's estate is $26 + 18 + 20 + 9 + 9 + 9 + 9 = 100$.

Problem 116

(Answer) $74369053 \times 87956 = 6541204425668$.

Problem 117

(Answer) Let the girls be A, B, C, D, E, F, G, H, I, J, K, L, M, N and O. A possible seven-day schedule is:
first day—(ABI)(CEM)(DHJ)(FGK)(LNO),
second day—(ACJ)(DFN)(EBK)(GHL)(MOI),
third day—(ADK)(EGO)(FCL)(HBM)(NIJ),
fourth day—(AEL)(FHI)(GDM)(BCN)(OJK),
fifth day—(AFM)(GBJ)(HEN)(CDO)(IKL),
sixth day—(AGN)(HCK)(BFO)(DEI)(JLM) and
seventh day—(AHO)(BDL)(CGI)(EFJ)(KMN).

Problem 118

Caius provided seven dishes and ate five, so that two were given to Titus. Titus got three dishes from Sempronius and should give him 18 of the 30 denarii.

Problem 119

The first player wins by taking 4 matches from the larger pile. After this move, the numbers of matches in the two piles are congruent (modulo 6). Whatever move the second player makes, this congruence cannot be maintained. On the other hand, the first player can always restore it on his subsequent move. Since at the end of the game, we have $0 \equiv 0 \pmod 6$, the first player must win.

Problem 120

(Answer) Label the coins 001, 010, 011, 012, 112, 120, 121, 122, 200, 201, 202, 220. First weigh 001, 010, 011 and 012 against 200, 201, 202 and 220. Then weigh 001, 200, 201 and 202 against 120, 121, 122 and 220. Finally, weigh 010, 120, 200 and 220 against 012, 112, 122 and 202.

Problem 121

Draw DE equal to the given perimeter. Draw a circular arc on DE subtending an angle of $90° + \alpha/2$, where α is the given angle. Draw a line parallel to DE and at a distance from DE equal to the given altitude, intersecting the circular arc at a point A. Let B and C be points on DE such that $BA = BD$ and $CA = CE$. Then triangle ABC has the correct perimeter and altitude from A. Note that $90° + \alpha/2 = \measuredangle DAE = \measuredangle BAD + \measuredangle BAC + \measuredangle CAE = (\measuredangle ABC)/2 + \measuredangle BAC + (\measuredangle ACB)/2 = 90° + (\measuredangle BAC)/2$. Hence $\measuredangle BAC = \alpha$ as desired.

Problem 122

(Answer) $(n + 1)/2n$.

Problem 123

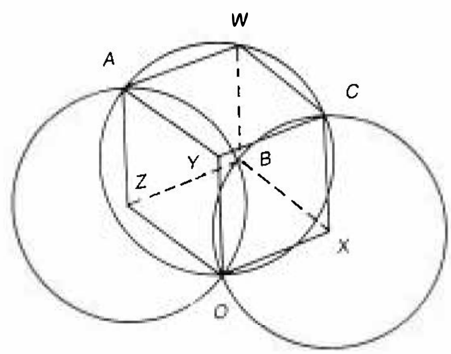
Of a total of 20 patients, 8 died. If there is really no difference between the two treatments, we may distribute the patients at random, provided that 9 received the old treatment and 11 the new one. The number of distributions in which none of the patients who died received the new treatment is $\binom{12}{11} = 12$. The number in which only 1 of those treated the new way died is $\binom{8}{1}\binom{12}{10} = 528$, and the number in which exactly two died is $\binom{8}{2}\binom{12}{9} = 6160$. Thus the desired probability is $(12 + 528 + 6160)/\binom{20}{11} = 335/8398$.

Problem 124

Ask each chicken to raise one foot and each rabbit to raise two feet. Then there are 70 feet touching the ground. Since there are only 50 heads, the 20 extra feet must be rabbit feet. Hence there are 20 rabbits and 30 chickens.

Problem 125

Let X, Y and Z be the circumcentres of triangles OBC, OCA and OAB, respectively. Join the lines as shown in the illustration.



Problem 125

Let W be the point other than Z such that $WA = WB = r$. Then $ZAWB$ is a rhombus. Hence WB is parallel to AZ, which is in turn parallel to YO and CX. Hence $WBXC$ is a parallelogram and $WC = BX = r$. Thus A, B and C lie on a circle of radius r and centre W.

Problem 126

(Answer) 10989.

Problem 127

(Answer) 25 x 3 = 75.

Problem 128

For such a number, n, $n^2 - n = n(n - 1)$ must be divisible by 1000. Since n and $n-1$ are relatively prime, one of them is divisible by 125 and the other by 8. If n is divisible by 125, it can only be 125, 375, 625 or 875. Since 124, 374 and 874 are not divisible by 8, we must have $n = 625$. If $n-1$ is divisible by 125, then n is 126, 376, 626 or 876, but only 376 is divisible by 8. Conversely, if $n(n-1)$ is divisible by 1000, so will n^k-n for $k > 2$ since $n(n-1)$ is a factor of n^k-n. Thus the only numbers we seek are 376 and 625.

Problem 129

(Answer) (e).

Problem 130

Let $n = 10x + y$ where $0 \leq y \leq 9$. Then $n^2 = 100x^2 + 20y + y^2$. The term $100x^2$ makes no contribution to the tens digit of n^2. The term $20xy$ contributes an even amount to the tens digit of n^2. Since 7 is odd, the odd amount must come from the term y^2.

It is routine to check that the only values of y for which the tens digit of y is odd are 4 and 6, with y^2 equal to 16 and 36, respectively. It follows that the unit digit of n^2, which is the same as the units digit of y^2, is 6.

Problem 131

Polynomials are continuous functions and compositions of polynomials are polynomials. Since $P(x) = Q(x)$ has no solutions, we may assume that $P(x) > Q(x)$ for all x. Suppose $P(P(x)) = Q(Q(x))$ for some x. Then for this x, $P(Q(x)) > Q(Q(x)) = P(P(x)) > Q(P(x)) = P(Q(x))$. This is a contradiction.

Problem 132

Represent the number n by n objects in a row. An expression of n as an ordered sum of k positive integers may be represented by the insertion of $k - 1$ partition-markers, each between two adjacent objects. Since there are $n - 1$ such spaces, there are 2^{n-1} ways of inserting from 0 to $n - 1$ partition-markers. It follows that n can be expressed as an ordered sum of positive integers in 2^{n-1} ways.

Problem 133

(Answer) The first adviser wins by recommending the recruiting of an officer who takes over all but one of the soldiers.

Problem 134

(Answer) Denote by A a unit cube which forms part of a 1 by 1 by 3 block; denote by B a unit cube which forms part of the 1 by 2 by 2 block; by C for the 2 by 2 by 2 block; by D for any of the 1 by 2 by 4 blocks.

First layer

D	D	A	A	A
D	D	B	B	D
D	D	C	C	D
D	D	C	C	D
D	D	D	D	D

Second layer

D	D	D	D	D
D	A	B	B	D
D	A	C	C	D
D	A	C	C	D
D	D	D	D	D

Third layer

D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
A	D	D	D	D

Fourth layer

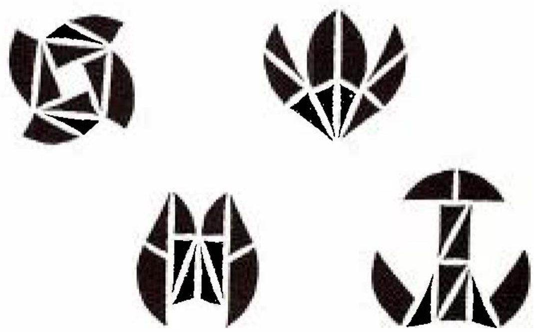
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
A	D	D	D	D

Fifth layer

D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
A	D	D	D	D

Problem 135

(Answer) See illustration.



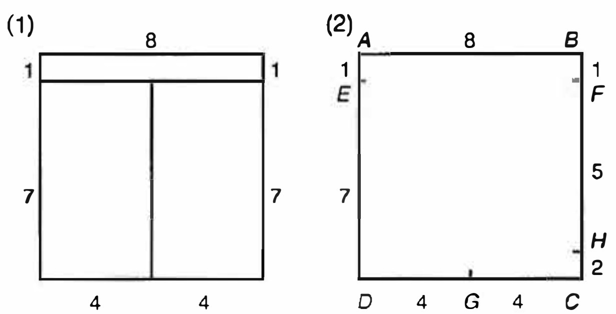
Problem 135

Problem 136

If n is divisible by 2 or 3, the tiling is clearly possible and we only need one kind of tiles. We now show that in all other cases the tiling is impossible. Color each cell of the n by n square black or white, black if the cell is on an odd-numbered row and white otherwise. Since n is not divisible by 2, the difference d between the numbers of visible black and white cells is equal to n at this stage. We now place the 2 by 2 and 3 by 3 squares to cover up the cells. A 2 by 2 square always covers up 2 cells of each color and has no effect on d. The placement of a 3 by 3 square either raises d by 3 or lowers it by 3. Since n is not divisible by 3, d can never be reduced to 0. This means that at least one cell is visible and a tiling is therefore impossible.

Problem 137

The first illustration shows that it is possible to divide the square into three rectangles each with diagonal $\sqrt{65}$. We now suppose that there is a division into three rectangles X, Y and Z each with diagonal less than $\sqrt{65}$. Of the four vertices of the square, two of them must belong to the same rectangle. Label the points as shown in the illustration, with A and B belonging to X. Since all of BE, AF and AG are at least $\sqrt{65}$, none of E, F and G can belong to X. G cannot belong to the same rectangle as E or F since both EG and FG are equal to $\sqrt{65}$. We may therefore put E and F in Y and G in Z. Since both BD and FD are greater than $\sqrt{65}$, D must belong to Z. Now all of AH, EH and DH are greater than $\sqrt{65}$. Hence H cannot belong to any of X, Y and Z, a contradiction.



Problem 137

Problem 138

(Answer) The claim is correct provided that the tripod's centre of gravity does not project outside the triangle determined by its three feet.

Problem 139

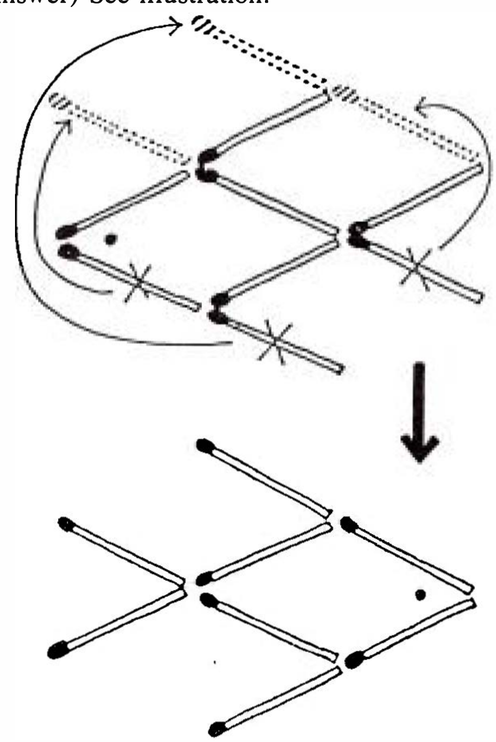
The division is impossible, because the sum of 25 odd numbers is odd, but 100 is even.

Problem 140

(Answer) The prisoner should get the keys in the following order: d, e (or e, d), c, a, b, f and g.

Problem 141

(Answer) See illustration.



Problem 141

Problem 142

Let x and y be the respective numbers of dollars and cents on the man when he entered the store. Then he had $100x + y$ cents, spent half of it and was left with $50y + x$. From $100x + y = 2(50y + x)$, we have $98x = 99y$. Since 98 and 99 are relatively prime, and x and y are positive integers under 100, we must have $x = 99$ and $y = 98$.

Problem 143

(Answer) 49 hours.

Problem 144

(Answer) There are no palindromic primes with an even number of digits apart from 11, because all palindromic numbers with an even number of digits are divisible by 11.