delta-k

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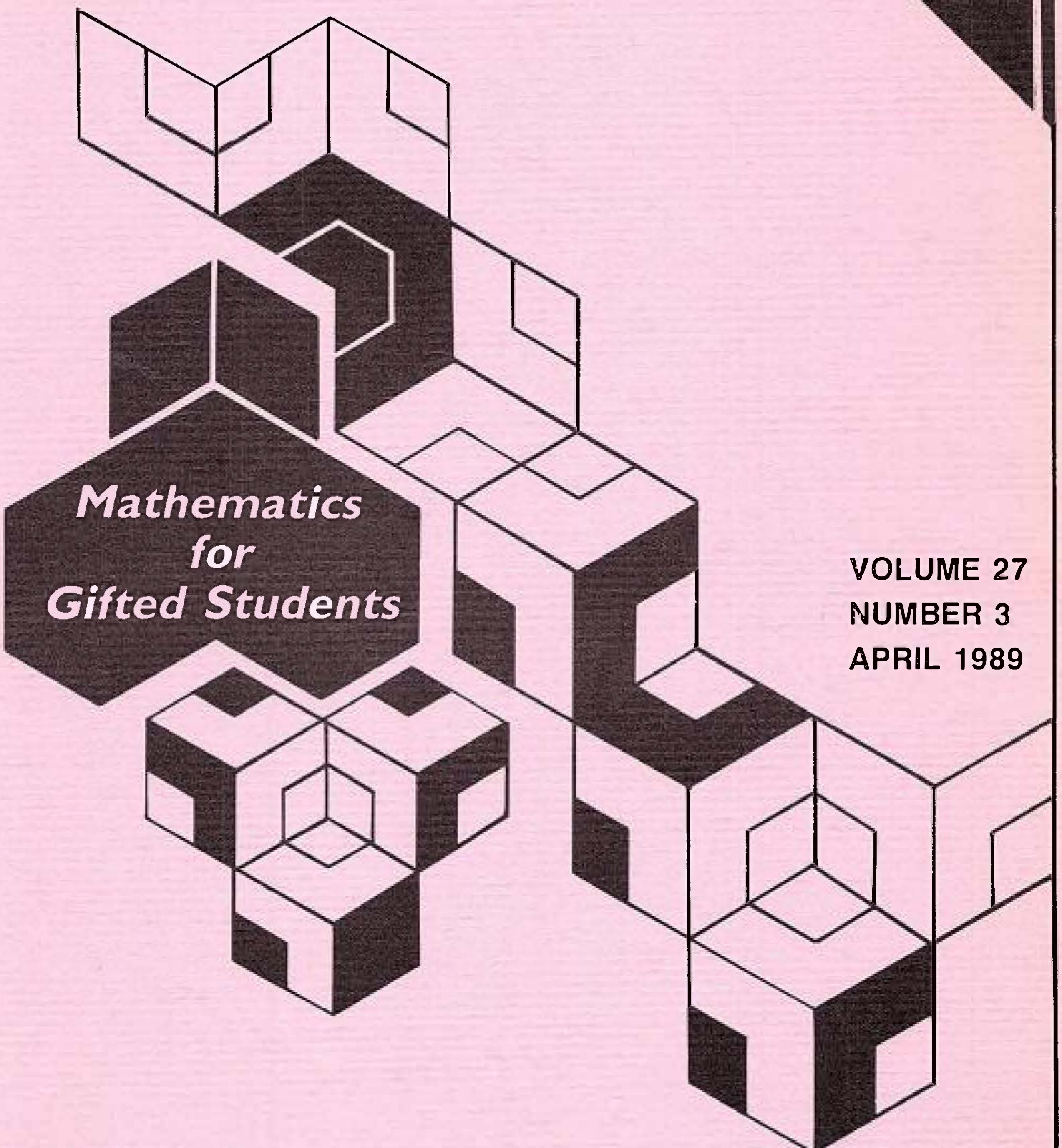

Mathematics for Gifted Students

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Mathematics Council and
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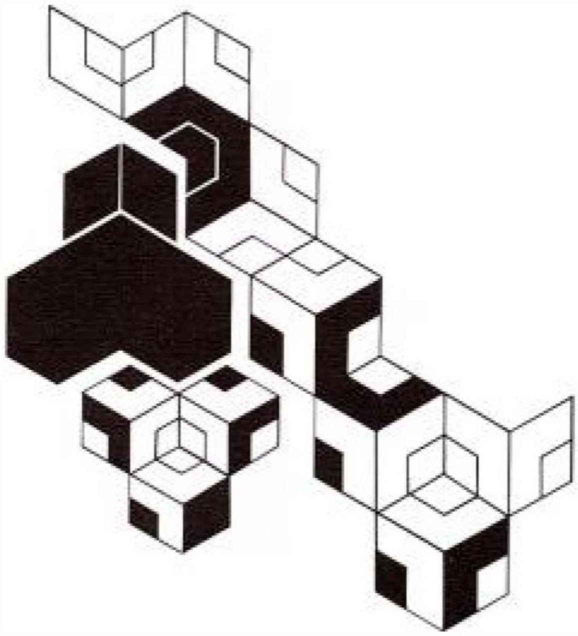
of The Alberta Teachers' Association



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Guest Editor, Andy Liu

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About the Contributors

Doris Schattschneider holds a Ph.D. in mathematics from Yale University and is professor of mathematics at Moravian College, Bethlehem, Pennsylvania, where she has taught for the last 20 years. An advocate for geometry, she has lectured widely and written several articles—on polyhedra, tiling problems, interaction of geometry and art and the work of the Dutch artist, M.C. Escher. Her popular book, *M.C. Escher Kaleidoscycles*, was recently republished by Pomegranate Artbooks, California.

Kathy Jones, a Hungarian-born inventor, is the president and chief executive officer of Kadon Enterprises, Inc., a company which produces top quality mathematical games and puzzles. Her background is in philosophy and commercial arts. Before venturing into her own business, she was a ballroom dance instructor and competition dancer. She is a member of the Imperial Society (London) of Teachers of Dancing. She regards artistic expression, whether through dance movement or mathematical symmetry, as a way of creating order and harmony amid the apparent chaos and discord of the world. Her game and puzzle designs are a celebration of the mind's ability to see beauty in the systems and structures of both nature and the mind's own creations. She is the inventor of "Lemma," a logical system-building game and the developer of many combinatorial puzzle sets.

Tony Gardiner combines research in combinatorics and group theory with a wider interest in mathematics, in its history and in how human beings—young and old—make the subject their own. His three books, *Infinite Processes* (Springer), *Discovering Mathematics* (Oxford) and *Mathematical Puzzling* (Oxford), reflect this breadth of interest. He has worked with many groups of interested youngsters and is currently establishing a national mathematics competition which is aimed at 30 percent of the fourteen-year-olds in the United Kingdom.

Mogens Esrom Larsen was born in 1942, graduated from the University of Copenhagen in 1965 and has been an associate professor of the department of mathematics there since, except for one year, 1969-70, when he visited the Massachusetts Institute of Technology. His interests cover most of mathematics, including functions of several complex variables, differential equations, finite groups and Rubik's cubes, pre-Euclidean history of mathematics, numerical analysis and the teaching and applications of modern mathematics. He writes a monthly problem column on popular science, where pentominoes frequently appear, for a Scandinavian journal. A game enthusiast, he was the founder of the Copenhagen Go Club in 1972.

Jan van de Craats was born in 1944 and received his Ph.D. in mathematics (numerical analysis) from Leiden University. Since graduation, his interests have shifted to geometry, problem solving and recreational mathematics. From 1973 to 1985, he was a member of the jury of the International Mathematical Olympiad and leader or deputy leader of the Dutch I.M.O. team. He trained the team by correspondence. He has been professor of mathematics at the Royal Military Academy in Breda since 1983.

Andy Liu (guest editor) is associate professor of mathematics at the University of Alberta. He obtained his doctor of philosophy degree and a professional diploma in elementary education from the University of Alberta in 1976 and has been a member of the faculty since 1980. His research interests are combinatorics, geometry and mathematical recreations. His ongoing professional activities include serving on the Alberta High School Mathematics Competition Board (since 1981) and editing its newsletter, *Postulate*, and involvement with the Edmonton Junior High Mathematics Contest (from 1983). Since 1986 he has been co-director of a year-round training program (by correspondence) for prospective members of the Canadian I.M.O. team.

Guest Editor's Comments

I am very pleased that both *AGATE* and *delta-k* are devoting a regular issue to mathematics education for the gifted and talented and honored to have been approached to be the guest editor. I was glad to accept the challenge, despite my lack of experience in the editorial process, because I feel strongly about the need for such an issue.

I want to emphasize that this issue is of interest not only to those involved in mathematics education for the gifted and talented. Alberta's mathematics curriculum is primarily content-oriented and mathematics teachers should find Doris Schattschneider's article of great value. It is a masterful exposition on one example of very useful techniques which apply across the whole spectrum of mathematics.

The mathematics curriculum is also heavily utilitarian. Teachers of the gifted and talented will enjoy Kathy Jones' article which highlights the aesthetic aspects of mathematics.

Tony Gardiner's article raises the important questions of the definition of the gifted and talented and of the role of the classroom teacher in their mathematics education. Such students should study Mogens Esrom Larsen's article, a beautiful example of the inspiration and perspiration needed to tackle a difficult problem. Jan van de Craats' article urges teachers not to overlook the social needs of exceptional students.

I am grateful to The Alberta Teachers' Association for getting me involved in this special project, which I hope the rank and file will find worthwhile, and to my friends and colleagues for sharing their valuable experience and insight on a very important education issue of today.

—*Andy Liu*

Counting It Twice

Doris Schattschneider

He's making a list and checking it twice
Gonna find out who's naughty and nice
....

He's counting a set and doing it twice
Gonna find out if something is nice
....

Addition is commutative and associative—so what? It does not matter, when you sum several numbers, in which order you add them, or which subtotals you form and then add to get the total. It always comes out to the same result. This principle is used more often in everyday occurrences than we realize. Here are a few examples.

Taking attendance at a meeting. Typically, to assure there is a quorum, two or more counters tally the people and, almost without fail, the order in which they count is completely different. One may begin at the front of the auditorium, the other at the back. The counts are compared and, if they agree, that number is the official recorded attendance. (If the counts do not agree, then a new count is taken.)

Keeping track of a chequing account balance. As you write cheques, you subtract the amount of each cheque from your chequebook balance. As the cheques are cashed and clear the bank, the bank subtracts the amount of each cheque from its record of your balance. The order in which the cheques are written and the order in which they are cashed are almost never the same, yet the amount your chequebook shows and the amount the bank statement shows must agree. (When they do not, you or the bank should re-examine the records.)

One of the oldest ways of bookkeeping is to record figures in rows and columns and then to tally each row, tally each column and, finally, check that the total of the row sums equals the total of the column sums. The sales record for a company for a year can be laid out this way: each of 12 columns corresponds to a month and each row corresponds to a particular item sold. The tally of each row is the yearly sales for a particular item and the tally of each column is the monthly sales for all the items sold by the company. The grand total of all the rows is the yearly total sales of the company; this is also the grand total of all the columns.

This commonsense technique of checking bookkeeping records can also be used as an effective technique to achieve surprising mathematical results. This principle has been dubbed the *Fubini Principle* by S.K. Stein (1979), in honor of the theorem in analysis which bears G. Fubini's name. This theorem states that, for well-behaved functions of two variables, x and y, the double integral can be evaluated as an iterated integral, integrating first with respect to x, then with respect to y, or in the other order. In other words, the value of the integral does not depend on the order of integration of the variables. The "discrete sum" version of this principle says that, when summing a rectangular array of numbers, the value of the sum does not depend on whether one first sums over rows or first sums over columns. In sigma notation,

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij} .$$

In our bookkeeping example above, the number a_{ij} would represent the sales of item i in month j.

Acknowledgments—I wish to thank my friends who suggested examples for inclusion in this paper: Steve Maurer, Andy Liu and Martin Gardner.

The equation just says that finding the total yearly sales of each item and then adding these will give the same result as finding total sales of all items for each month and then adding these.

Many mathematical proofs involving entries in a matrix, or some other set of numbers with double subscripts, employ the Fubini principle to change the order of summation at a critical moment to achieve success. Here is a partial solution to a problem on the 1977 International Mathematical Olympiad which cleverly uses a matrix arrangement and counts twice to show an impossible situation. The problem is—*Find the longest sequence of real numbers such that the sum of every 7 consecutive terms is positive and the sum of every 11 consecutive terms is negative.* Here is how to show that the sequence cannot have length 17 (or more).

If there is such a sequence $a_1, a_2, \ldots a_{17}, \ldots$, then form an 11 x 7 matrix with entries as shown below—

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_7 \\ a_2 & a_3 & a_4 & \cdots & a_8 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{17}. \end{array}$$

If we calculate the sum of all the entries in the matrix row by row, then clearly the sum is positive. But if we calculate the same sum column by column, the sum must be negative. Thus the sequence with the desired properties cannot have length 17 or more. (A sequence of length 16 satisfying the stated condition can be constructed.)

Most often, sets of numbers (or other sets of objects) are not neatly displayed in rows and columns, yet any sum that accounts for every item will give the total. So we will adopt, with Stein, the following maxim as the Fubini principle—*When you count a set in two different ways, you get the same result.*

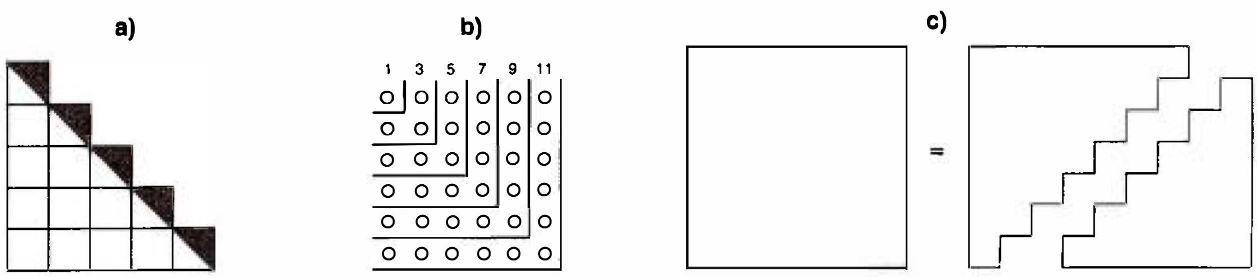
Of course. What could be more obvious? The surprising thing is that this simple statement is actually a powerful device in discovering and in proving formulae. Sometimes these formulae give succinct expressions for the number of objects in a given set; sometimes they show that two expressions which appear very different are actually equal.

Picture proofs: counting using partitions

Many of the "Proof without Words" picture proofs of counting formulae exemplify the use of the Fubini principle. A set of objects is shown and then partitioned (sometimes in two different ways); the partition yields an expression for the number of objects in the set (the whole is equal to the sum of its parts). Formulae for polygonal numbers are easily seen this way. The nth triangular number, t_n, is the number of dots in an array of n rows in which the first row has one dot and each succeeding row contains one dot more than the previous one; thus $t_n = 1 + 2 + \cdots + n$.

The sequence of triangular numbers begins 1,3,6,10,15,21,28, ... The nth square number, s_n, is the number of dots in a square array of dots with n rows and n columns; this is just the number n^2, so $s_n = n^2$.

Figure 1 shows some picture proofs which employ the Fubini principle to obtain some formulae for triangular numbers and square numbers which are not obvious from their definitions. In these pictures, the number 1 is sometimes a dot and sometimes the area of a small unit square.



$t_n = 1 + 2 + 3 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$

$s_n = 1 + 3 + 5 + \cdots + (2n - 1)$

$s_n = t_n + t_{n-1}$

Figure 1

Sources: a) Richards (1984); b) and c) Gardner (1986) and others.

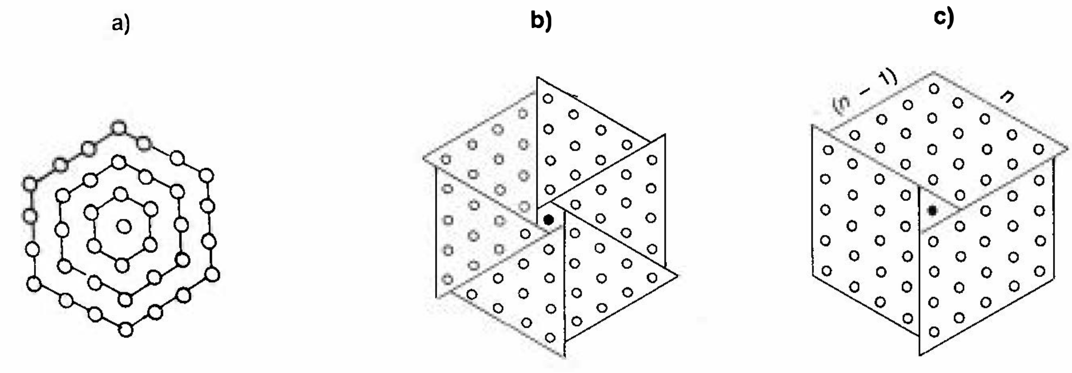


Figure 2

Another nice picture example, involving hex numbers, is given by Martin Gardner (1988). These are "centred" hexagonal numbers, obtained by arranging successive layers of dots in a concentric hexagonal array, as shown in Figure 2(a). The sequence of hex numbers begins: 1,7,19,37,61,91,127, . . .; the nth hex number is the number of dots in the first n layers of the hexagonal array. Two distinct ways of partitioning the array are shown in Figure 2(b) and 2(c); each gives an obvious formula for the nth hex number h_n. Applying the Fubini principle gives the identity,
$h_n = 6t_{n-1} + 1 = 3n(n - 1) + 1$.
Although defined by an entirely different configuration, this equation shows that the hex numbers lead to the identity for triangular numbers in Figure 1.

If you begin to sum consecutive hex numbers, you find the sequence 1,8,27,64, . . ., so a natural conjecture is that the sum of the first n hex numbers is n^3. This is true, and Gardner gives a nice visual illustration of this fact, again using the Fubini principle. He also discusses some surprising properties of "star" numbers, the number of dots in a hexagonal star configuration, like the holes on a Chinese Checkers board. Many more of these picture proofs which use the Fubini principle can be found in issues of *Mathematics Magazine* and in "Look-See Proofs" (Gardner 1986).

Counting and recounting—combinatorial argument

The art and science of counting is a major aspect of combinatorics. In this field, formulae and identities for counting are proved and often the key device used is the Fubini principle. In fact, counting a set in two distinct ways and then setting the expressions equal to each other is what is usually meant by the term "combinatorial argument." Here are a few examples. We use the usual notation $\binom{n}{k}$ to mean the number of ways of choosing k items from n items; the compact formula for this number is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If a set has n elements, then how many subsets does it have? One way to find this is to count the number of subsets with 0 elements, the number of subsets with 1 element, 2 elements, etc. and then to take the sum of all of these numbers. Another way is to think of the n elements as numbered from 1 to n, and each subset as an n-tuple whose ith coordinate is 1 if the ith element is in the subset and 0 if it isn't. Since there are 2^n such n-tuples (n slots, each of which can be filled in two ways), there are exactly 2^n subsets.

We have answered the original question and also have obtained a combinatorial identity by counting the collection of all subsets in two ways; the left side is the first way we counted, the right side is the second way—

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n .$$

If this identity looks like the binomial theorem, your perception is correct—just replace 2^n on the right side with $(1 + 1)^n$ and it is a special case of the binomial theorem—

$$\sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k} = (a + x)^n .$$

This more general theorem can also be proved by "counting it twice" (see Tucker 1984).

Here are two other quick proofs of combinatorial identities obtained by the Fubini principle. The totality of all possible committees of k persons chosen from a club of n persons can be partitioned into two classes—those committees which will contain the club president and those which will not. If the president is on a committee, then the remaining $k - 1$ committee members are chosen from $n - 1$ people; if the president is not on a committee, then its k members are chosen from $n - 1$ people. In symbolic form, this says

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} .$$

This argument is essentially the same one given by Blaise Pascal (1623 - 1662) in his book *Traité du Triangle Arithmétique*, published posthumously in 1665 (Edwards 1987).

Now let us count the number of all possible committees (of all sizes) chosen from n persons, where each committee has a chairman designated. If we think of picking the committee members, and then designating the chairman, and count the committees of size 0, size 1, size 2, etc, we obtain the sum on the left in the equation below. But if we first choose a chairman and then count all possible committees that can be formed with that chairman, we get the count on the right side of the equation (recall our count of all subsets of a set above)—

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}.$$

Tucker (1984) and other texts on combinatorics present many more such examples of combinatorial arguments. Marta Sved (1983) gives some striking examples of counting and recounting in her article by that title. She also plays a variation on the game of "here is the answer, what is the question?". She offers a combinatorial identity and challenges readers to provide an interpretation which proves it by "counting it twice." The sequel is in "Counting and Recounting: The Aftermath" (1984).

Indirect Counting

Often counting things directly is much more complicated than counting a "complementary" set. This is the simplest form of the "inclusion-exclusion" principle of counting—*the number of elements in a subset S of a set is the number of elements in the whole set minus the elements that are not in S.* So to get a count of a subset, you can count the number of elements in its complement and subtract that number from the count of the whole set.

Here is a typical example. How many different tosses of two standard dice (one red, one blue) will have the sum of their top faces less than 10? Since the sum cannot exceed 12, it is easier to count the ways in which the sum can be 10, 11, or 12. There are six ways in which this can happen (count them) and there are 36 different combinations of the two dice, so there are 30 different tosses in which the sum is less than 10.

One of the most basic combinatorial identities also recognizes complementation in counting: the number of ways of choosing k items from n items is the same as the number of ways of choosing $n - k$ items from n items. After all, once k items are chosen, that selection determines the complementary selection of $n - k$ items. In symbolic notation,

$$\binom{n}{k} = \binom{n}{n-k} .$$

Instead of using complementation, often a set can be counted indirectly by counting a related set. Here ingenuity in recognizing a related set plays a prime role! For example, if 75 people are to play in a tennis tournament (in which you are "out" as soon as you lose a game), how many matches must be scheduled? Forget about the matches and think about winners and losers. There are exactly 74 losers in the series of matches, so there are exactly 74 matches.

Double (or more) counting

A distinctly different way of "counting it twice" is to count the objects in a set in such a way that each object is counted twice (or even more than twice). This "double counting" (or multiple counting) is often coupled with the Fubini principle to yield surprising results. A classic example which illustrates this technique is the method of obtaining the formula for the sum of the first n consecutive positive integers (the number t_n) which is attributed to the young

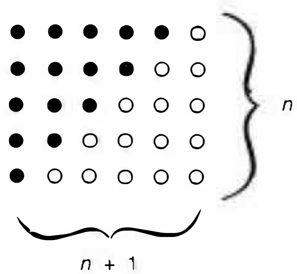
Gauss. Write down the sum twice (horizontally) as in Figure 3 and then note that all vertical sums equal $n + 1$. There are n of these sums, so that $2t_n = n(n + 1)$.

$$\begin{array}{l} 1 + 2 \quad + 3 \quad + \dots + n \\ n + n-1 + n-2 + \dots + 1 \end{array}$$

An easy way to see that $2t_n = n(n + 1)$

Figure 3

Using this same trick, you can easily obtain a formula for a portion of any arithmetic progression, like the sum of all the multiples of 3 from 60 to 333. Incidentally, you can also obtain the formula in Figure 3 by a picture proof of double counting: two copies of the geometric configuration for t_n fit together to form an $n \times n + 1$ rectangle. Figure 4 shows this configuration.



$2t_n = n(n + 1)$

Figure 4

Source: Gardner (1986)

To conclude this article, we give a more elaborate and sophisticated example of the power of "counting it twice."

Counting edges or counting vertices on polyhedra can be done by employing the multiple-counting principle and this leads quickly to inequalities which describe some of the geometric constraints which polyhedra must obey. Here is a quick review of the assumptions about faces, edges and vertices of a polyhedron—you may wish to test them on a familiar one, such as a cube or pyramid. Each face of a polyhedron is a polygon and each edge of a polyhedron is a common side of exactly two adjacent faces. A vertex of a polyhedron is a point where three or more edges meet; it is also a "corner" where the vertices of three or more faces coincide. Each edge of a polyhedron has as its endpoints vertices of the polyhedron; each edge touches exactly two vertices of the polyhedron.

Suppose we are given a polyhedron P which has v vertices, e edges and f faces. Without knowing the value of e, the number of edges, we can at least compare e to the values v and f, by counting. First, we use vertices to count edges. Each vertex has at least three edges which meet there, but also each edge meets exactly two vertices, so each edge is counted twice by vertices. This gives the inequality $3v \leq 2e$.

Next, we use faces to count edges. Each face is a polygon, so it must have at least three sides. Each edge is the side of exactly two polygonal faces, so counting edges which surround faces gives the inequality $3f \leq 2e$.

These constraints must be satisfied by any polyhedron—so they can be used to test if certain configurations of vertices and edges can be realized as polyhedra at all. If a polyhedron has some uniform features, such as having the same number of edges meeting at each vertex, or the same number of edges surrounding each face, then we have equations. For example, a cube satisfies $3v = 2e$ and $4f = 2e$, and an octahedron satisfies $4v = 2e$ and $3f = 2e$.

One of the most useful relationships between the numbers of vertices, edges and faces of a convex polyhedron was discovered by Leonard Euler over two hundred years ago and is known as *Euler's Formula:* $v + f = e + 2$. This formula, together with counting arguments on vertices, edges and faces, leads to many surprising and non-intuitive results. Before looking at a reference you may wish to try to prove: (1) *A convex polyhedron cannot have 7 edges;* (2) *If a convex polyhedron has only pentagons and hexagons as faces, then it has exactly 12 pentagonal faces.* (The second result tells us about the structure of geodesic domes and soccer balls.) For these and other implications of Euler's formula see Beck et al (1969).

Another very surprising result which can be proved from Euler's formula is due to René Descartes and is the three-dimensional analogue of the theorem which states that the sum of the exterior angles of a polygon equals 2π. Each vertex of a convex polyhedron is surrounded by the angles of all the polygonal faces that meet there. The sum of all of these face angles which meet at a vertex must be less than 2π in order for the polyhedron to be convex. The angular defect (or deficiency) of a vertex of the polyhedron is obtained by subtracting from 2π the sum of the face angles which meet at that vertex.

Thus, just as an exterior angle of a polygon measures how close the corresponding interior angle is to having measure π, the angular defect of a vertex of a polyhedron measures how close that corner of the polyhedron is to being flat or having measure 2π. Descartes' Theorem states that, *for any convex polyhedron, the sum of the angular defects of all of the vertices of the polyhedron is exactly* 4π.

George Pólya proved this theorem using an argument which can be used to establish an even more general result (Hilton and Pedersen 1987). The key technique is to count the sum A of all of the face angles of the polyhedron in two different ways (first using vertices, then using faces), then use the Fubini principle to equate the results. If a_n is the sum of all of the face angles about the vertex v_n, then since there are v vertices, the total defect D of the polyhedron is given by the sum,

$$D = \sum_{n=1}^{v}(2\pi - a_n) = 2\pi v - \sum_{n=1}^{v} a_n.$$

The sum of all the a_n on the right side of the equation is the sum of all face angles over the whole polyhedron and thus equals A. So we have

$$D = 2\pi v - A.$$

The sum A is also obtained if we first find the sum of the angles in each face and then sum over all of the faces of the polyhedron. If a face has m sides, then $(m - 2)\pi$ is the sum of the angles of that face; so it is efficient to lump together faces with the same number of sides as we calculate. Let f_3 denote the number of three-sided faces, f_4 the number of four-sided faces, and so on. Then f, the total number of faces of the polyhedron, is just the sum,

$$f = f_3 + f_4 + f_5 + \ldots = \Sigma f_m.$$

The sum of all of the face angles of all of the m-sided faces is $(m - 2)\pi f_m$, so the sum of all of the face angles of the whole polyhedron is

$$A = \pi f_3 + 2\pi f_4 + 3\pi f_5 + \ldots = \Sigma(m - 2)\pi f_m.$$

Even though we do not know what the largest m is, this sum is finite (since the polyhedron has only a finite number of faces). So the sum can be distributed; it equals

$$A = \pi\Sigma m f_m - 2\pi\Sigma f_m = (\pi\Sigma m f_m) - 2\pi f.$$

Since there are f_m faces with m sides, the sum of all of the terms mf_m is the total number of sides of all of the faces of the polyhedron. But since each edge of the polyhedron is the side of exactly two faces, this sum is just $2e$. So now we have

$$A = 2e\pi - 2\pi f = 2\pi(e - \mathrm{f}).$$

If we substitute for A in our equation for D and then use Euler's formula, we have

$$D = 2\pi v - 2\pi(e - f) = 2\pi(v - e + f) = 4\pi,$$

which is Descartes' theorem.

References

Beck, A., M. Bleicher and D. Crowe. *Excursions Into Mathematics*, Chapter 1, Worth, 1969.

Edwards, A. W. F. "Pascal's Triangle—and Bernoulli's and Vieta's", *Mathematical Spectrum* 20, no.2(1987):33-37.

Gardner, M. "Hexes and Stars," in *Time Travel and Other Mathematical Bewilderments,* W. H. Freeman, 1988, 15-25.

Gardner, M. "Look-See Proofs," in *Knotted Doughnuts and Other Mathematical Entertainments,* W. H. Freeman, 1986, 192-204.

Hilton, P., and J. Pedersen. *Build Your Own Polyhedra,* Addison-Wesley, 1987, 160-162.

Richards, I. "Proof Without Words: Sum of Integers," *Mathematics Magazine* 57(1984): 104.

Stein, S. K. "Existence out of Chaos," in *Mathematical Plums,* Ross Honsberger (editor), Mathematical Association of America, Dolciani Mathematical Expositions no. 4(1979): 62-93.

Sved, Marta. "Counting and Recounting," *The Mathematical Intelligencer* 5, no.4 (1983): 21-26.

Sved, Marta. "Counting and Recounting: The Aftermath," *The Mathematical Intelligencer* 6, no.2 (1984): 44-45.

Tucker, A. *Applied Combinatorics,* second edition, John Wiley & Son, 1984.

The Psycho-Aesthetics of Combinatorial Sets

Kathy Jones

Psycho-aesthetics is the concept of how the human mind observes, perceives and appreciates an achieved design, from the thrill of discovery through the pleasant contemplation of its beauty. The pursuit of knowledge, the wresting of order from chaos and the completion of a harmonious unity are the mind's most satisfying activities. Focusing these on the exploration of combinatorial sets can be a highly rewarding exercise that can carry over into more effective problem solving in the larger world.

Combinatorial sets are groups of geometric shapes formed by joining various multiples of the same basic building block along their unit edges, in all their possible relative positions. The simplest, best-known and most fascinating groups are those that consist of the three regular, convex polygons that can tile the plane. Here they are with their "family" names:

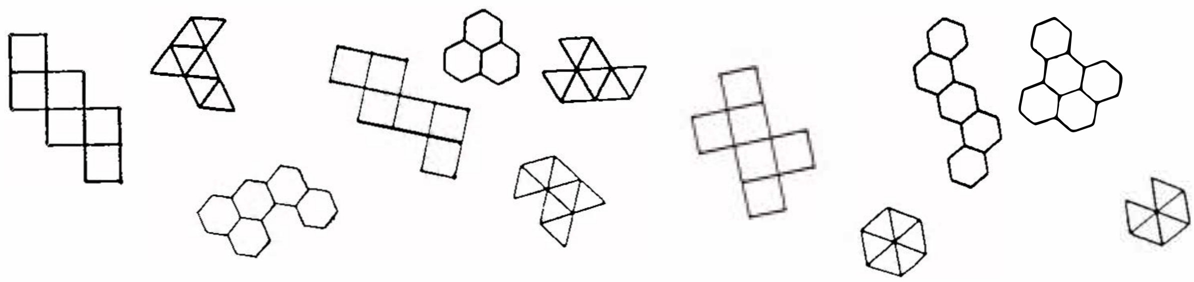
SQUARES—polyominoes
TRIANGLES—polyiamonds
HEXAGONS—polyhexes.

Combining ascending multiples of any one of these polygons yields progressively more intricate and more numerous distinct shapes. Figures 1a, 1b and 1c show the members of the three families, from unity up to the generally familiar levels or "orders" of size, together with their respective designations, as introduced in the landmark book, *Polyominoes,* by Solomon W. Golomb.

The amazing feature of all these shapes is that, for all their diversity and peculiarities, they can be fitted together with each other in coherent patterns, thus offering countless mathematical puzzles of surprising beauty and intellectual stimulation. Some unusual assemblies are shown in succeeding figures.

In the mind's eye, the rarer and more difficult to achieve are also the more precious and valued and the more attractive. Among the astronomical numbers of combinations possible with any polyform set, the most beautiful are those that have the elusive and special features of symmetry, regularity or self-replication. Symmetries are generally much more difficult to find and are more pleasing visually, sometimes to the point of approaching works of art.

Let it be borne in mind that the examples shown here are just as much the products of Nature's playfulness and ordered variety as are crystals, snowflakes, the DNA helix and the periodic table of elements, to name just a few. Readers are invited to judge whether their own minds don't resonate to these phenomena in a way best expressed by the colloquialism, "Oh wow!"



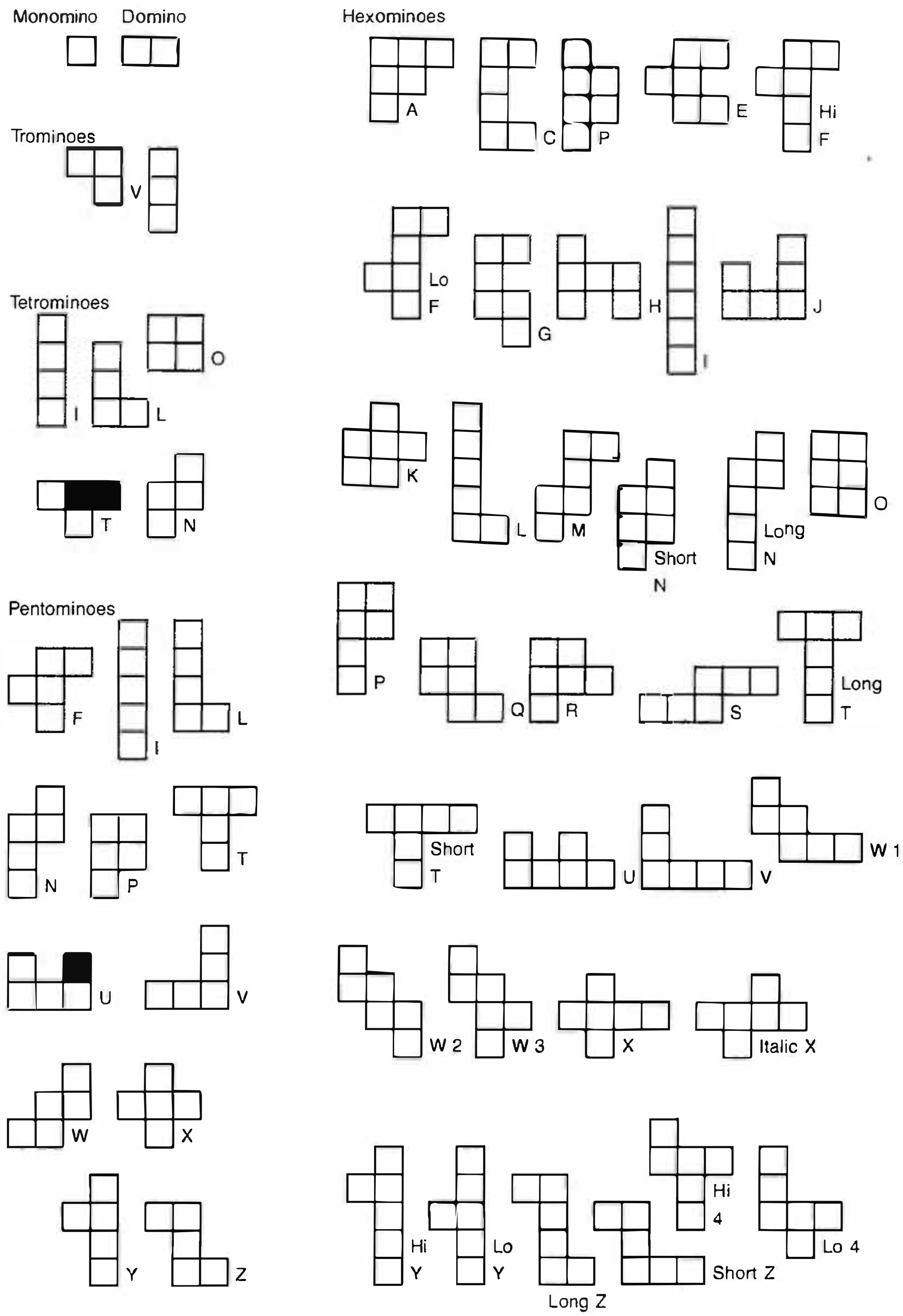


Figure 1a Polyominoes

Note—PENTOMINO is a registered trademark of Solomon W. Golomb.

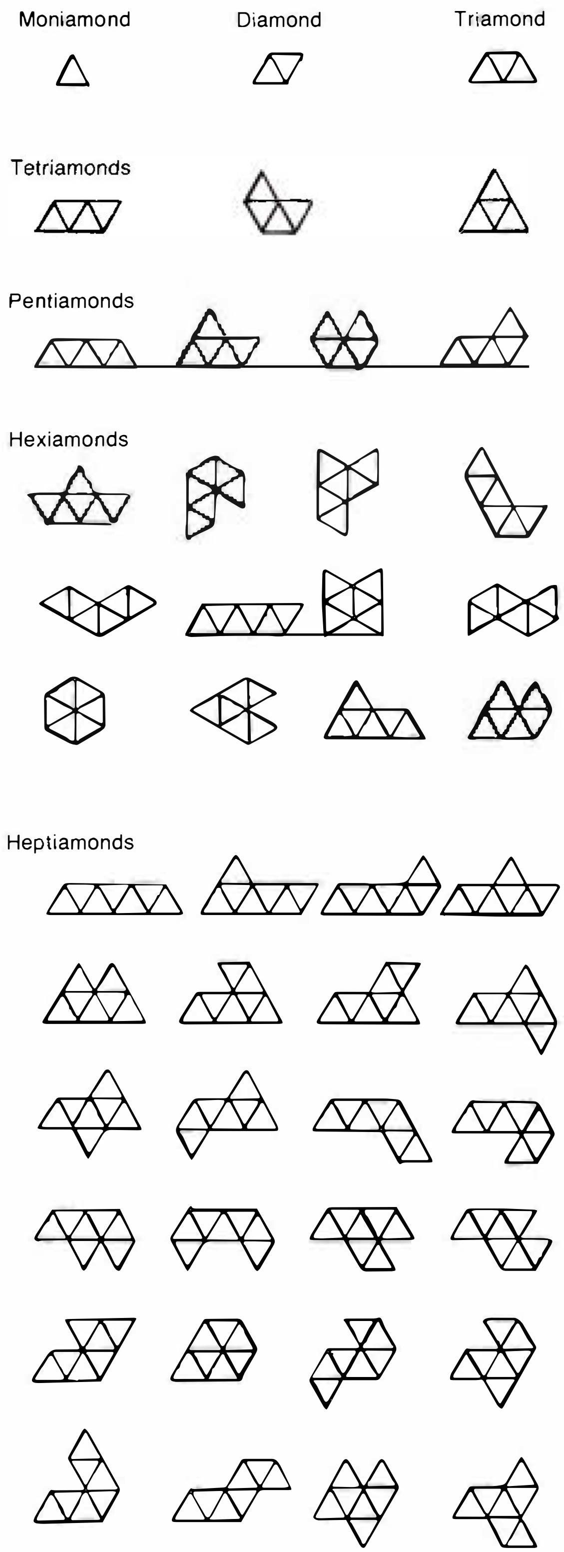


Figure 1b
Polyiamonds

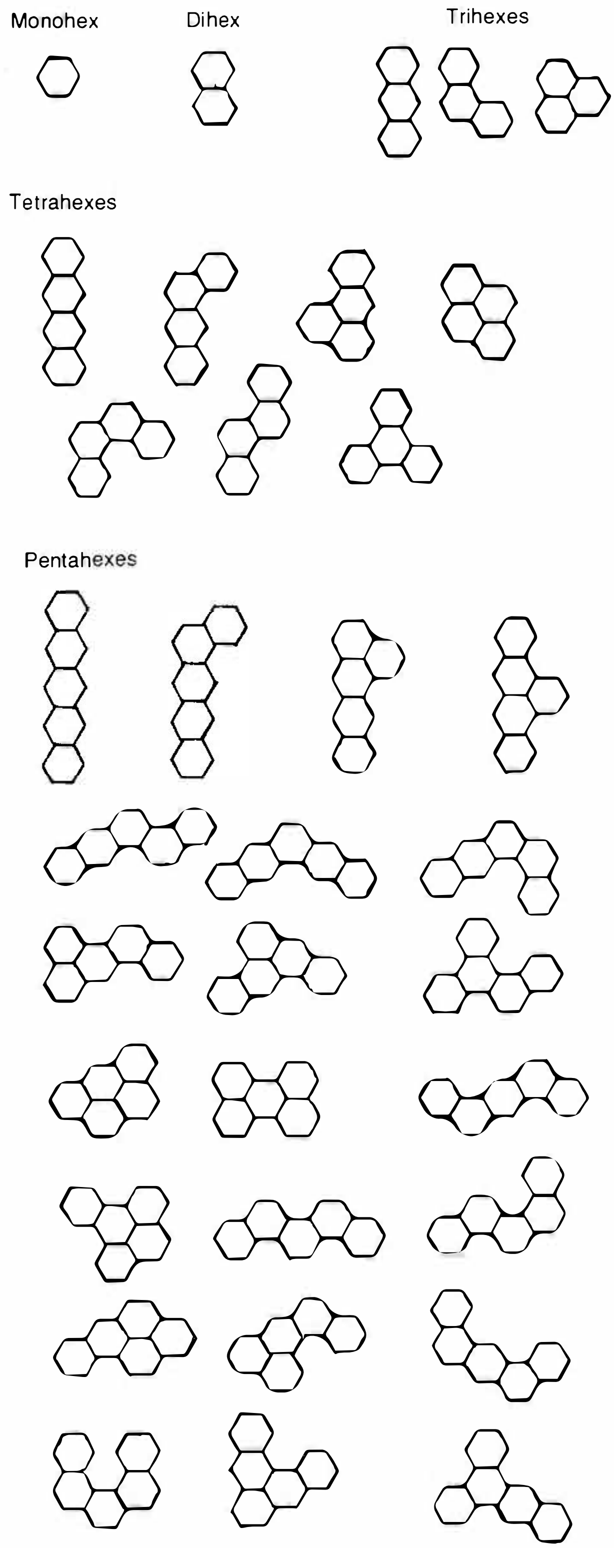


Figure 1c
Polyhexes

Figure 2 shows the most compact four-way symmetry with polyominoes of orders 1 through 4. We could ask that the monomino be in the centre or that a maximum number of pieces be totally enclosed.

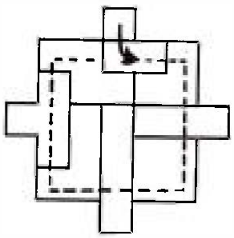


Figure 2

The line traces clockwise from the monomino around the outer row of the 5 x 5 square and passes through each polyomino in ascending order of sizes.

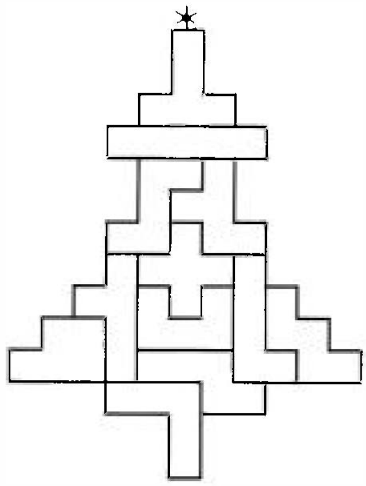


Figure 3

A symmetrical pentomino construction containing six symmetrical pairs. Can you spot them all?

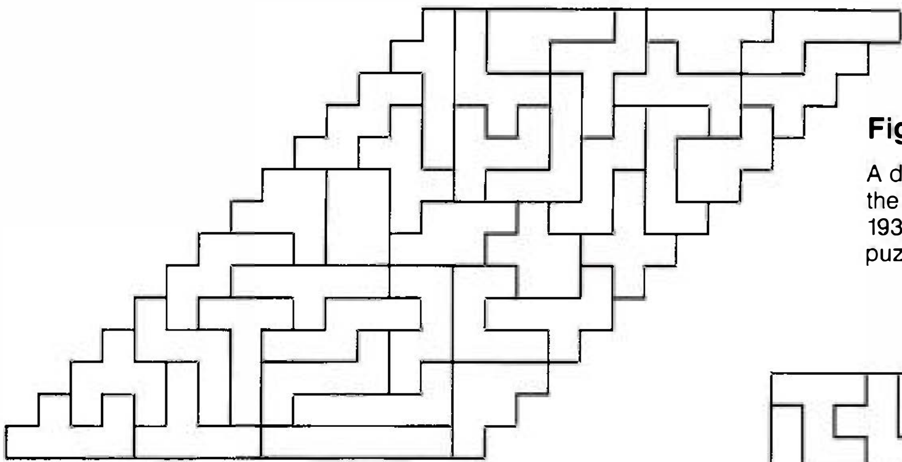


Figure 4

A design with rotational symmetry with the 35 hexominoes (first published in the 1930s in *Fairy Chess Review*, a British puzzle journal).

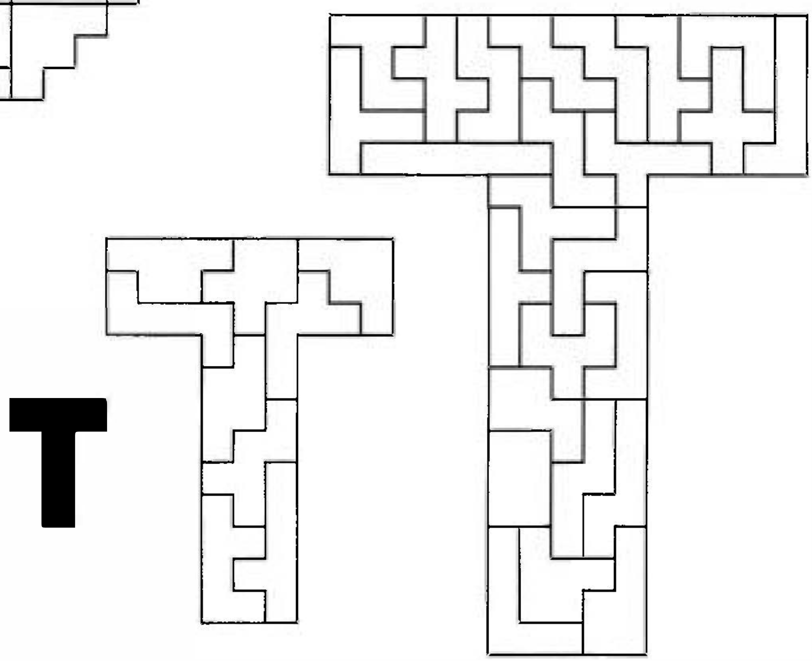


Figure 5

A replication solution of a hexomino, forming a triple and quintuple enlargement of the selected piece, thus involving all 35 pieces (*solution by the author*).

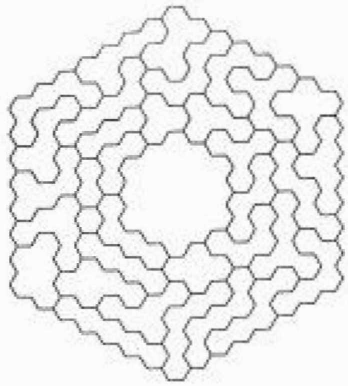


Figure 6a

A "Hex-Nut" arrangement of the polyhexes, sizes 1 through 5, in which the ascending sizes spiral from the inside to the outer ring, with the pentahexes forming a separate ring (*solution by Michael Keller*).

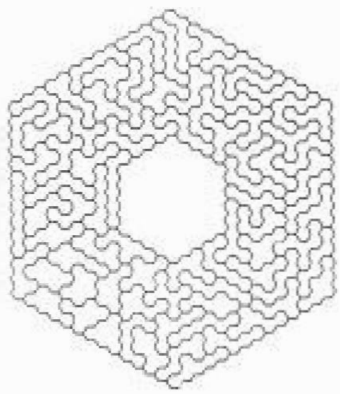


Figure 6b

A remarkable perfect ring formation of the 82 hexahexes (*solution by Michael Keller*).

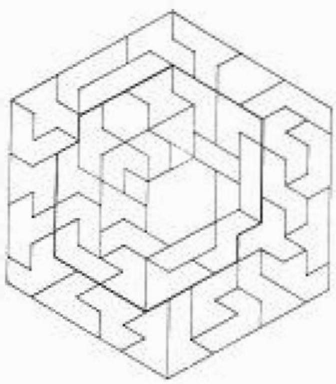


Figure 7

An "Iamond Ring" formation of the polyiamonds sizes 1 through 7, with the heptiamonds forming a symmetrical outer ring (*solution by the author*).

Combinatorial sets larger than those shown get somewhat unwieldy and overly complex for easy visualization and handling, although solutions have been found for rectangles with the 108 heptominoes (by David Klarner) and the 369 octominoes (by Michael Keller). Such enormous sets are no longer "user-friendly" for manipulation of the entire set except as *tours de force* by hardy investigators, such as the dazzling hexahex array in Figure 6b. Interesting explorations can, however, be done with selected subsets, perhaps in combination with smaller-order sets.

The restless search for greater challenges leads to problems with unique solutions. Computer programs and clever algorithms have made it possible to search for and verify unique solutions. Generally, the more pieces are used, the more solutions exist, so to tighten the problem the number of pieces is reduced or the shape of the region to be solved is constrained. The uniqueness of the letters in Figure 8 made with the 12 pentominoes was authenticated by Professor Yoshio Ohno of Tokyo. (The other 21 letters in this alphabet series have multiple solutions.)

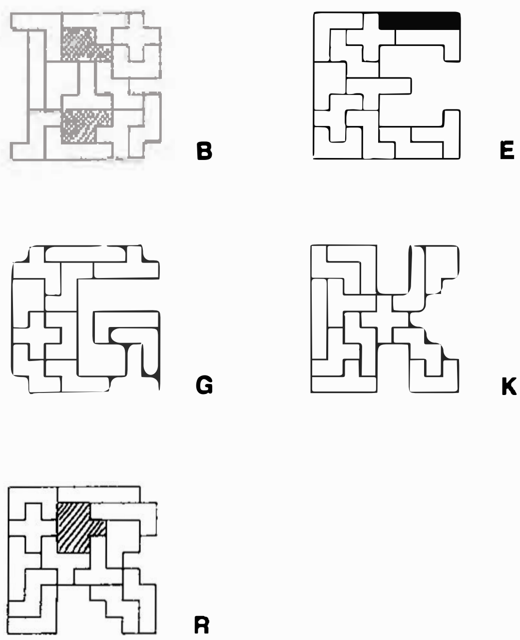


Figure 8

The B and E have unique solutions and the G, K and R have a minor variant (subsection flipped) in their otherwise unique arrangement.

Another approach to pursuing more difficult problems is to enter the third dimension. Solid pentominoes (made of cubes instead of squares) can form three different shapes of blocks: 3 x 4 x 5, 2 x 5 x 6 and 2 x 3 x 10. An astonishing discovery was that the latter two sizes can be accomplished by "folding" 6 x 10 rectangles, as shown in the following two figures.

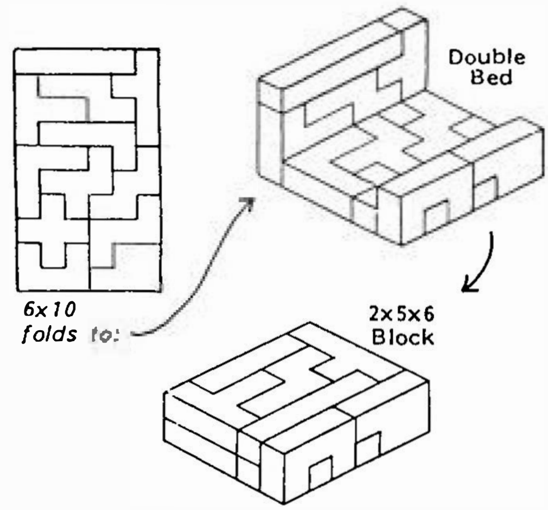


Figure 9

A foldable pentomino construction turns a 6 x 10 rectangle into a 2 x 5 x 6 block.

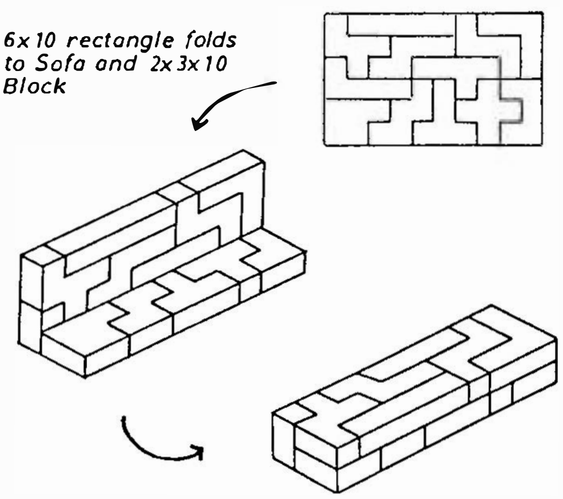


Figure 10

A unique solid pentomino conversion of a 6 x 10 rectangle into a 2 x 3 x 10 block.

In the three-dimensional realm, additional complexity can be introduced by joining five cubes in every 3-D combination, yielding 17 "pentacubes" (first constructed by David Klarner). These non-planar shapes, combined with the 12 solid pentominoes, provide inexhaustible possibilities for exploration and discovery.

Expanding the concept one more step, we find there are 166 hexacubes (the planar and non-planar joinings of six cubes). Supplemented by four single cubes, these 166 shapes will pack a 10 x 10 x 10 cube—one more exercise in heroic dimensions. Assaults on smaller subset problems, such as fitting 36 hexacubes into a 6 x 6 x 6 cube, are themselves thoroughly challenging.

It is actually in the smaller sets, which are easier to manage and for the mind to encompass, that we find the greatest popular appeal, accounting for the continuing charm of the now-classic Soma Cube, invented by Piet Hein many years ago. It consists of six tetracubes and one tricube. Its versatility and transformability have provided joy and delight to a generation of puzzlers.

The quest for novelty and new areas to conquer with combinatorial sets has inspired the creation of many variants on the theme of shape and color combinations. Ever since Major Percy MacMahon introduced his famous Three-Color Squares in the 1920s, the concept of color or contour adjacency has become an adjunct of groups defined by special characteristics. Figure 11 is one of the classic solutions with MacMahon Squares:

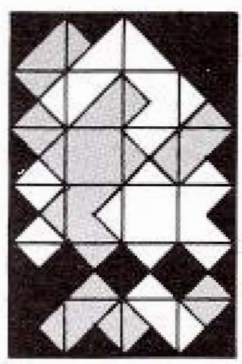


Figure 11

The 24 distinct three-color squares form a 4 x 6 rectangle where the border color is constant and every adjacent edge meets only a matching color.

MacMahon extended his research to four-color equilateral triangles (now known appropriately as MacMahon Triangles) and discovered that the 24 distinct pieces would indeed form a hexagon with constant border color *and* with matching color adjacency on touching edges. The solution he published is shown in Figure 12a and a variation by the author in Figure 12b.

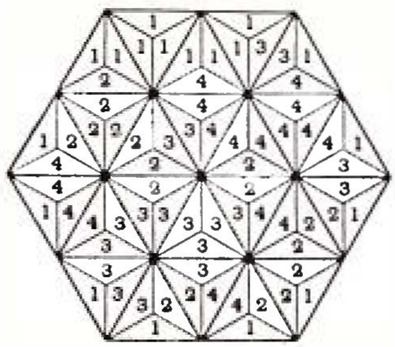


Figure 12a

The 24 distinct four-color triangles form an order-2 hexagon with constant border color and all adjacent edges touching only a matching color.

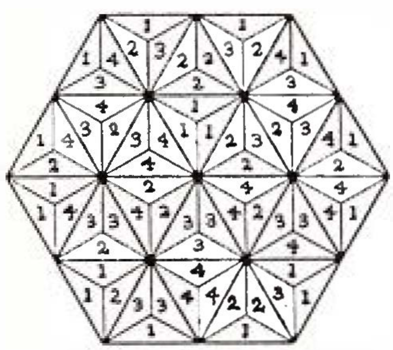


Figure 12b

This nearly symmetrical arrangement of the 24 MacMahon triangles has constant border color but *no* internal color adjacency.

Converting the edge coloring of MacMahon's Three-Color Squares to contours, where each square is a distinct combination of straight, convex and concave edges, we get 24 distinct shapes. Figure 13 is an array of them, supplemented by twelve duplicate pieces, forming a 6 x 6 square with a pronounced Escher flavor.

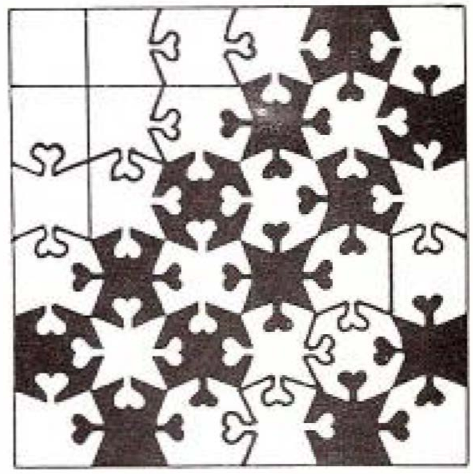


Figure 13

The topological equivalent of MacMahon squares (marketed under the product name, "Stockdale Super Square"); the dark pieces represent duplicates.

We can combine the concepts of combinatorial shapes and color permutations. One such set uses the shapes of right isosceles triangles joined (family name "polyaboloes" or, more commonly, "polytans"). Order-1 and order-2 polyaboloes are small triangle, double-size triangle, square, and left and right parallelogram. Permuted with two colors, the five shapes provide 15 distinct pieces that can tile an octagon and countless other symmetrical shapes with various color themes. Figure 14 is an array that has constant border color plus internal color symmetry.

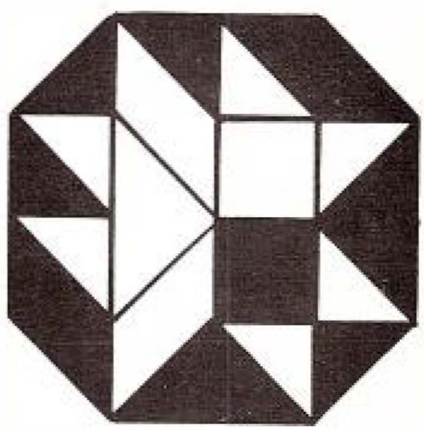


Figure 14

The fifteen order-1 and order-2 two-color polyaboloes with constant border color and overall color symmetry.

The next figure is a difficult arrangement of the same set as an octagon with all color-matched adjacent edges.

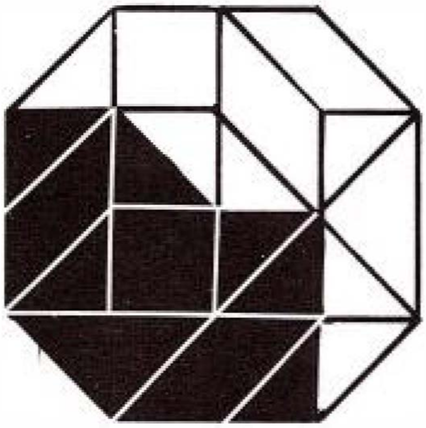


Figure 15
A color-matched solution of the 15 two-colored, order-1 and order-2 polyaboloes in their most compact form, an octagon.

Increasing the number of unit triangles per piece enlarges the polyabolo family to 4 triaboloes, 14 tetraboloes and 30 pentaboloes. These offer largely unexplored territory, but will allow themselves to be coaxed into two squares, supplemented by one unit triangle, as in Figures 16 and 17.

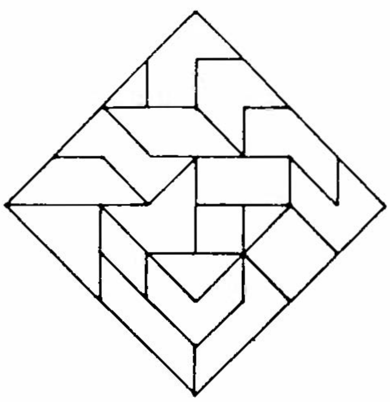


Figure 16
The polyaboloes, orders 1, 2 and 4, as a diagonal square (*solution by the author*).

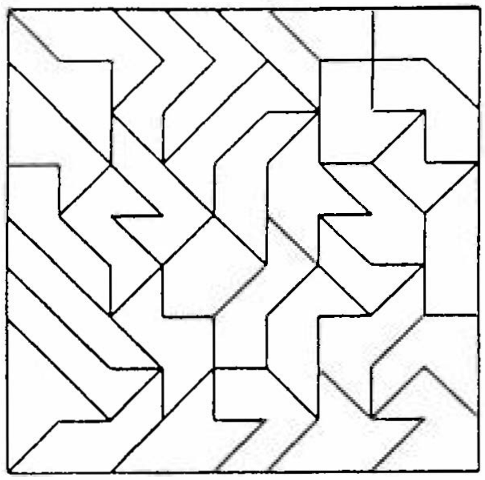


Figure 17
The polyaboloes, orders 3 and 5, as a 9 x 9 orthogonal square (*solution by Michael Keller*).

The 107 hexaboloes are still awaiting their conqueror.

It is easier to expand the multi-color order-2 set by adding more colors. There is a poetic progression of triangular numbers as the number of colors increases. Whereas the two-color set (shown in Figures 14 and 15) will form a 5 x 3 rectangle with color adjacency, a three-color set will tile a 5 x 6 rectangle; a four-color set will tile a 5 x 10 rectangle, and a five-color set will indeed fill a 5 x 15 rectangle—all with color adjacencies. Each new color added brings with it a single neutral triangle as a "filler" for the single triangle of each color. The neutral triangles act as wild cards that any color may touch. The 80 pieces of the five-color set skirt the limits of human endurance, yet the finished pattern is one of great abstract beauty, somewhat like a Vasarely painting. Figure 18 (overleaf) presents one solution found by the author. For best effects, it is recommended to color it in "by the numbers" with four colored pencils.

Moving further along the spectrum of shapes and colors, we enter the field of tilings and patterns, that is, completely covering a surface with multiple copies of the *same* shape or with combinations of two or more shapes. A wealth of research material is available, both historic and recent, and the subject of tessellations continues to fascinate with its kaleidoscopic variations and infinitely repeating patterns—inspirations for graphic artists and designers.

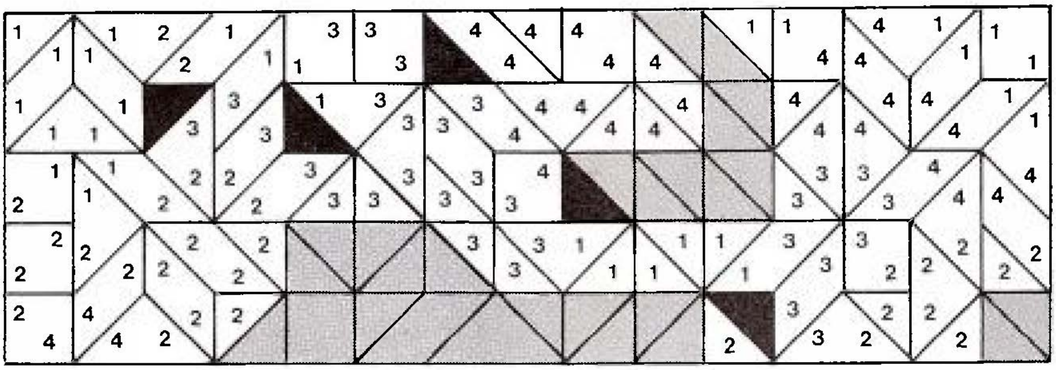


Figure 18

A five-color polyabolo set of orders 1 and 2, in a color-matched 5 x 15 rectangle.

A noteworthy tessellation problem concerns the pentominoes. Solomon Golomb has shown that any one of the 12 pentominoes can tile the plane with infinite copies of itself. A special case of this tiling ability is when multiples of a polyomino can form a square, and then multiples of this square can, in turn, form larger versions of the polyomino itself (Golomb calls them "rep-tiles"), and so on. It's a fine example of meta-thinking, like a box around a box around a box . . . . Combinatorics starts with a unit, a singularity; but there's no end in sight. A mini-version of rep-tiling can be seen in Figure 19.

A novel combination of tiling assembly and combinatorial shapes is the set shown in Figure 20. The shapes are formed by plotting circles on a square grid and using combinations of one, two and three circles and one, two and three "bridges", where the bridges are the connecting parts, somewhat like squares with curved-in sides, between the circles. The 7 x 7 grid is tiled with seven each of singles, doubles, triples and bridge pieces. The solution has the further interesting features that no two of the same shape touch each other and that there is a maximum amount of symmetry (solution found by Richard Grainger).

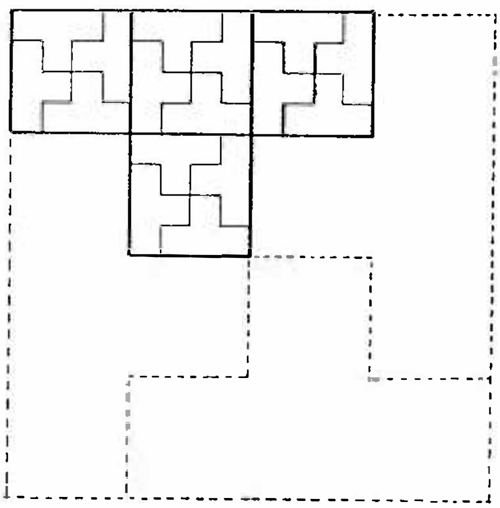


Figure 19

The T tetromino is replicated here in quadruple size. Four of *them* form a larger square, and four of those. . . .

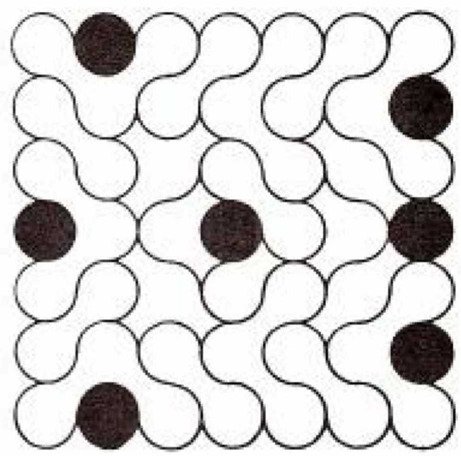


Figure 20

A lovely array of circle-based shapes on a square grid, with non-adjacency of similar shapes but with maximum symmetry.

The set when expanded to include combinations of four circles and four bridges will tile a 10 x 10 grid. The solution in Figure 21 has rotational symmetry as well as non-adjacency of similar shapes (solution by the author). The distribution of shapes makes for a dramatic visual impact.

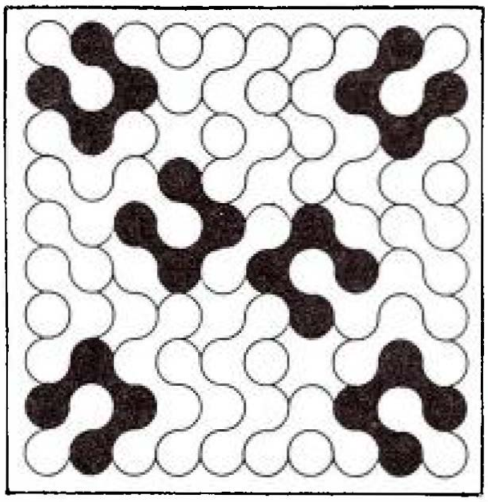


Figure 21
Multiples of 10 shapes of one, two, three and four rounds and connector bridges fill a 10 × 10 grid in a rare symmetry pattern.

Carrying the "round" idea into the third dimension leads us to the perennially popular ball pyramid puzzles. A close packing of spheres produces tetrahedra of various orders. The traditional simplest size is an order-4 pyramid with anywhere from four to seven pieces composed of various numbers of balls joined in a plane. If the balls are joined in a simple 60° or 120° relationship, the assembly is not usually difficult. When they are joined at 90° angles, the position of that piece within a triangular format becomes problematic and a real challenge for the puzzler. Len Gordon, today's foremost inventor of ball pyramid puzzles, has created three sizes of pyramids with computer-proven unique solutions. The component pieces range from two to four balls joined.

At the other end of the scale, Gordon has found thousands of solutions to an order-9 pyramid containing the 33 distinct planar shapes of five balls joined in every possible way, including those with 90° angles.

Figure 22 is a clever mini-pyramid using four copies of the same five-ball piece. It can be formed with two different arrangements, one of which allows the entire pyramid to be lifted by just its top. Can you visualize how the pieces must be joined?

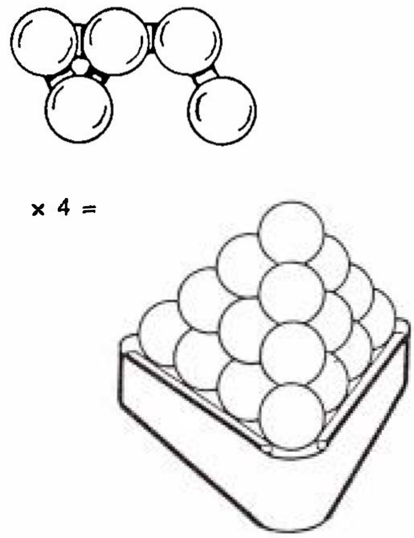


Figure 22
An order-4 tetrahedron formed with four of the pieces shown. They can interlock to hold together even when lifted by the top.

Some wonderful innovations in combinatorial sets involve hexagon-shaped tiles with special color patterns. One such set, sold under the tradename Kaliko and now sadly out of print, contains 85 distinct tiles with three colors permuted over the five distinct path patterns that join pairs of sides. Invented by Charles Titus and Craig Schenstedt, this magnificent set lends itself to the most wondrous and convoluted symmetric loop and color patterns. One is shown in Figure 23 (overleaf).

Another way to fit three colors on a hexagon was invented by Charles Butler. Here the tiles are divided into compartments of single and triple diamonds in each of three colors (the triples look like chevrons). The patterns give the impression of a perspective view of a 2 x 2 x 2 cube. Permuting the relative positions of the three colors produces 12 distinct patterns, which in turn combine into amazing symmetrical figures with color adjacency on all touching sides. Figure 24 is a triangle solution found by the author.

Figure 23
A Kaliko pattern with tri-color symmetry.

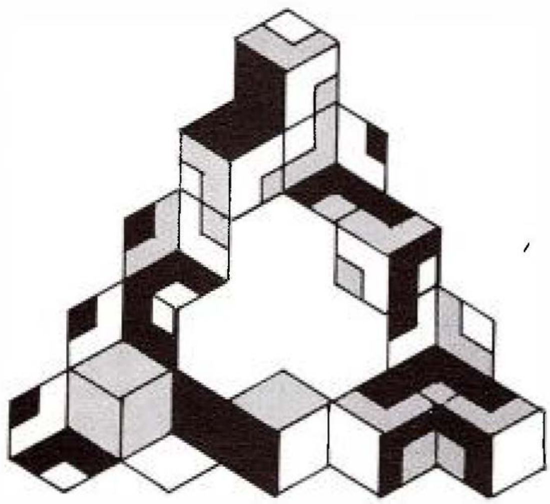


Figure 24
A color adjacency solution for the 12 tri-color Hexmozaix tiles.

John Horton Conway, England's great eccentric mathematician, discovered that regular pentagons, when edge-colored with five colors in every possible arrangement, produce 12 different tiles that will exactly cover a dodecahedron with color adjacency on every edge. We leave it to the nimble-fingered reader to construct and assemble this figure.

As human ingenuity knows no bounds, combinatorial sets can be derived from combinations of dissimilar unit shapes. A most exciting new multiform set was created by Dr. Andy Liu of the University of Alberta. Mapped onto the classic tiling pattern shown in Figure 25a, the members of the set are those that have not more than 4 adjoining cells and not more than 17 as the sum of their collective number of sides. The set so defined has 28 one-sided pieces and will exactly tile a triangular region. An elegant solution is shown in Figure 25b.

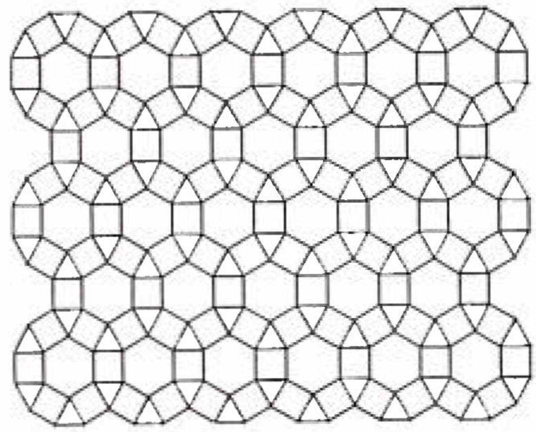


Figure 25a
A tessellation with the regular hexagons, squares and triangles.

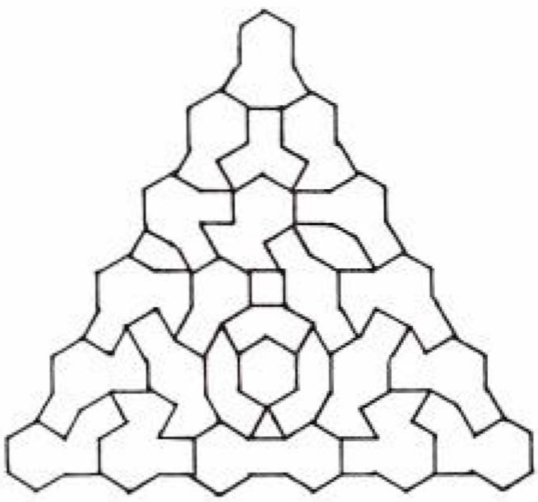


Figure 25b
The triangle formed with the 28 distinct tri-form shapes (*solution by Andy Liu*).

It is hoped that readers' scenic tour through some of the wonders of combinatorial diversity and harmony may have stimulated some appreciation for the aesthetic aspects of mathematical sets and kindled an interest in further study. Playing with combinatorial sets can develop an eye for spatial relationships, a facility for problem-solving and a deep pleasure in the process of research and discovery. The pride and satisfaction gained from successes in these pursuits, in turn, will serve as stimulus for future effort. The philosophical values gained from contemplating Nature's boundless variety of workable combinations are beyond measure.

An inherent logic and order in the unfolding of the genetic code, the processes at work in the interiors of stars, the passion of musical composition, the dynamics of economic striving—all are combinatorics. Only the level of complexity varies. Mathematics can model and make sense of a seemingly chaotic world—and keep things interesting with ever new challenges. For what is creativity but the combinatorics of the mind?

Bibliography

Gleick, James. *Chaos: Making a New Science.* New York: Viking Penguin Inc., 1987.

Golomb, Solomon W. *Polyominoes.* New York: Scribner's Sons, 1965.

Grunbaum, Branko, and G.C. Shephard. *Tilings and Patterns.* New York: W.H. Freeman, 1987.

MacMahon, Major P.A. *New Mathematical Pastimes.* London: Cambridge University Press, 1930.

World Game Review, No. 7, October 1987.

It All Depends What You Mean By . . .

Anthony Gardiner

Defer reading for a few minutes and have a go at this sequence of problems. We'll discuss these problems later.

Problem 1
Find a prime number which is one less than a perfect square.

Problem 2
Find another prime number which is one less than a perfect square. How many other such primes are there?

Problem 3
Find a prime number which is one more than a perfect square.

Problem 4
Find another prime number which is one more than a perfect square. How many others are there?

Problem 5
Find a prime number which is one less than a perfect cube.

Problem 6
Find another prime number which is one less than a perfect cube. How many others are there?

Problem 7
Find a prime number which is one more than a perfect cube.

Problem 8
Find another prime number which is one more than a perfect cube. How many others are there?

Introduction

Question 1
What exactly do we mean by "mathematical talent"?

Question 2
How can we recognize it?

Question 3
What can we do to encourage its development?

If mathematics teaching were a science, it might be reasonable to try to answer these three questions in the given order. As things are, mathematics teaching is not (yet)[1] a science: it remains a craft. So one should not be surprised at the suggestion that the questions may be best tackled in the reverse order: first look for rich, challenging material that encourages mathematical thinking; while using such material, observe the different approaches used by individual students and try to assess their requirements and talents in the light of their performance; finally, make use of this experience to refine one's ideas of what does, and what does not, constitute "mathematical talent." Those who work with young children often have their own tried and trusted ways of nurturing whatever mathematical talent is present in their classes, but are far less sure how one can reliably assess the degree of talent present in any given individual and are usually most reluctant to define exactly what they mean by "mathematical talent."

I have seen this kind of pragmatism work extremely well in individual classrooms and

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[1]This fact is reflected in the subtitle of Hans Frendenthal's thought-provoking book, *Weeding and sowing: preface to a science of mathematical education* (Reidel 1978).

schools. However, my thumbnail sketch has ignored the most basic question of all.

Question 3′
How do we distinguish between rich, challenging material that encourages mathematical thinking and material that is unsuitable?

Once this crucial question has been asked, it is clear that our response to the first three questions is bound to depend on our ideas of what mathematics itself is really about and of the kind of students we are inclined to call "talented."

Question 4 (a)
Which students are we mainly concerned about? That very select group, *la crême de la crême?* Or the much larger group of all those who belong to the "cream" and who crop up regularly in most high schools?

Question 4 (b)
What do we understand by "mathematics"? How do we decide whether an activity at a given level is or is not genuinely mathematical? How should the fact that one is working with youngsters affect the style and content of the mathematics?

One way forward?

The aspect of mathematics which appeals most strongly—perhaps at all ages—is the way in which elementary calculations and constructions can be used to resolve non-trivial problems.[2] All attempts to encourage students interested in mathematics must therefore exercise and extend students' ability to perform the relevant calculations or constructions. A good basic training in routine techniques is thus fundamental. Sadly, many of the talented students in our classes have only been expected to perform these routine techniques in the simplest imaginable contexts.

One cannot assume that "talented" students will somehow make up for our own limited expectations by making their own spontaneous generalizations. (For example, the resolution of the problem sequence above is entirely elementary, but seems to be totally unexpected—even for good college students majoring in mathematics. No one seems to have alerted them to even the most obvious connections, such as that between factorizing numbers and factorizing polynomials.) This immediately suggests one very simple way in which ordinary class teachers can make a significant contribution to the development of mathematical talent. We shall come back to this.

Our habit of teaching routine techniques in a very restricted way is one reason why problem competitions and enrichment materials which are officially aimed at mathematically talented students frequently turn out to be most unsuitable for our own talented students.[3] But the main reason for this mismatch seems to be that those

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[2]In his fascinating autobiography, *Disturbing the universe* (Harper and Row 1979) Freeman Dyson tells how, as a boy, he worked through the seven hundred or so problems in Piaggio's *Differential Equations* (G Bell 1920). Piaggio's book is quite different from modern texts. The author presents an absolute minimum of theory. Instead of waiting until a technique or method can be fully justified, he explains what he can and encourages the reader to "have a go"—the details of the complete picture becoming more clearly visible as one proceeds. Of course, one misses many important points first time through. But the book was important for Dyson not just as a way of mastering differential equations but also because it enabled him, through the joy of calculation, to fall in love with mathematics. Harold M. Edwards makes a related point in the Preface to his beautiful history of *Fermat's Last Theorem* (Springer 1977): "As even a superficial glance at history shows, Kummer and the other great innovators in number theory did vast amounts of computation and gained much of their insight in this way. I deplore the fact that contemporary mathematical education tends to give students the idea that computation is demeaning drudgery to be avoided at all costs."

[3]I am aware of two common strategies for stretching the most talented young mathematicians. Neither of these strategies seems to work very well with ordinary talented students. One approach involves presenting simplified treatments of selected topics from higher mathematics. But though the formal technical prerequisites may be kept well under control, such material often makes quite unreasonable assumptions about what is, and what is not, familiar, meaningful or interesting to bright high school students. Many such presentations have the additional weakness that the ratio of text to exercises is all wrong, as though talented students had less need of exercises! As a result these valiant efforts to make higher mathematics accessible are often best appreciated by adult mathophiles. The other approach involves competitions based on problems that are easy to state and whose content is elementary, but which are hard to solve (or hard to solve in the time allowed). I love these questions, but use them relatively rarely. They are a bit like those texts which claim to have no formal prerequisites other than a little "mathematical maturity." The trouble in both cases is that students with the necessary "maturity" are singularly hard to find.

who set the competition problems and write the booklets are chiefly interested in the "truly exceptional student." But this is, almost by definition, the kind of student most of us never see!

The answer of most teachers to Question 4(a) is likely to be determined as much by this simple fact of life as by any rigidly held educational principle: it is natural to be most concerned for the kind of talented students we come across regularly in our own institutions. Their talent may make little impression in competitions designed to identify *la crême de la crême*, but that does not lessen our responsibility for fostering the talent we know they have. If their talent is modest when compared with the very best, their numbers are so much greater that they present us with no less of a challenge. Moreover, given the right kind of material, such students are capable of some remarkable mathematics.

Most of the talented students I work with enjoy mathematics, but are not (perhaps never will be) ready to do battle with hard competition questions.[4] Instead they need lots of experience of tackling intuitively appealing problems which allow them to get started, which nevertheless remain strangely opaque, but slowly become first meaningful, then promising and finally transparent, as a result of intelligent groping.

The problem sequence at the beginning (taken from my book, *Mathematical Puzzling,* Oxford University Press 1987) illustrates one way of achieving this. Any student interested in mathematics immediately tries to answer the questions (especially if they are posed orally to a group of students). It is only when they fail to answer some of the harmless-looking questions (or when they discover that their own over-hasty answers are rejected by their peers), that they begin to realize there is something to explain—even if they are not at all sure what. By this point they are sufficiently committed not to back off in the face of a problem they would otherwise instinctively classify as being "too hard."

Mathematics is a much messier business than most textbooks are willing to admit. Basic techniques are important; so a part of mathematics instruction must certainly consist of applying standard procedures to solve familiar-looking problems or writing out solutions in a specified deductive form (as in geometry). But where did the solution come from in the first place? And how did one decide which standard algorithm to use?

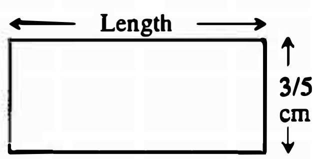
Much of the time we have very little idea how students find their solutions. As long as they continue to succeed one may argue that it does not matter. But once we find them beginning to struggle, it becomes all too tempting to cheat by making the route from the problem to its solution so short and direct that very little mathematical thinking is required. Students certainly need to master basic techniques, but mathematical thinking only really begins when the student has to select and coordinate a number of such basic steps to solve challenging multi-step problems.

A regular diet of even the simplest multi-step problems can have a dramatic effect on student perceptions and performance. As an indication of what happens when we fail to provide such a diet as part of our ordinary teaching, consider the following problems set to a large sample of 15-year-olds in the United Kingdom.

Question A

Area = 1/3 square centimetre

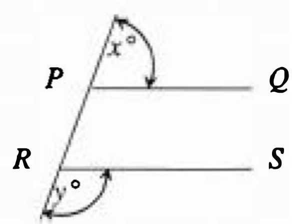
Length = ?



Question B

PQ is parallel to *RS*
$y = 2x$

What is the size of angle *PRS* in degrees?



Though each of these problems requires the student to identify an intermediate step, one

[4]Why not? The satisfaction which motivates those who do respond to hard competition problems stems from the prospect of occasional hard-won success. That, in its turn, presupposes extensive failure. Once the perceived prospect of success sinks below some personal threshold, the game loses its appeal. The talented student who has rarely been challenged by hard problems and who does not realize how important "failure" is in mathematics, only has to see a hard problem for the perceived prospect of success to sink way below his (inflated) threshold.

could scarcely call them hard! Yet success rates are abysmally low.[5] Such levels of incompetence are certainly not preordained: they are the result of years of systematic training in anti-mathematical thinking. Our persistent failure to set appropriately challenging problems ensures that many highly talented students simply lose interest in mathematics, while others perform so far below their potential level that their talent becomes almost invisible.

In my experience very few talented students—even those who have been well-taught—respond well to the kind of material that is often advocated for the most able.[6] These students need problems, or sequences of problems, which have a strong intuitive appeal, which make minimal technical demands while stretching students' own powers of calculation, and which above all force them to "think mathematically." (This kind of thinking is subtly different from the process of "seeing through" simple puzzles and generally requires extended periods of engagement.) My two books, *Mathematical Puzzling* and *Discovering Mathematics* (Oxford University Press 1987) represent two rather different responses to this challenge.

Discovering Mathematics is the more ambitious of the two in that it tries to convey to the talented high school student how one goes about exploring a substantial mathematical problem on one's own. It does this, not by talking about mathematics, but by involving the reader in extensive calculations, in making (often false!) conjectures and in checking and revising those initial guesses until the reader arrives at something requiring proof. Many teachers have enjoyed working through this material and have found the experience not only refreshing but also helpful in clarifying their own ideas in relation to Question 4(b).

Mathematical Puzzling has similar aims but a very different format. In spite of its emphasis on problems and on calculation, *Discovering Mathematics* presupposes a willingness to read "text." *Mathematical Puzzling* avoids "text" and consists largely of problems. The messages it seeks to convey (about the nature of mathematics, about the importance of looking for "connections," about being willing to experiment and explore, about the need to use one's judgment in making sense of a question, etc.) are therefore implicit in the choice and the wording of the problems and in the way they are grouped together. The material has been developed with various groups of 10- to 14-year-olds over the last ten years—though much of it has been used to good effect with much older students.

The opening example

At first sight, the opening example may look like a sequence of routine problems designed to test students' familiarity with prime numbers and powers. Though such an impression is superficial, it certainly reflects two very important general features of the problem sequence, namely,

—that each problem is accessible to and should evoke an immediate response from any student interested in mathematics, and

—that the content is entirely elementary and the initial demands on the student are restricted to calculating (preferably mentally) efficiently and accurately.

On a deeper level the problems emphasize the following points

- the importance of being willing to search systematically and intelligently for numbers with the required properties;
  (for example, Problem 2, is 8 prime? 15? 24? ...)
- the need for mental fluency;
  (what is 13^2? what is 6^3? how does one test quickly whether an unfamiliar number, like 143, is prime?)
- the inadequacy of merely guessing;
  (is 143 really prime? how about 217? or 511?)
- the virtue of reflecting on the results of one's own calculations.
  (why do powers of odd numbers obviously never work?)

However, there is far more to the problems than this, as the sting in the tail of the even-

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[5]Only one student in 20 managed to answer the first question correctly and one in 10 managed the second. Yet the studies from which these examples are taken steadfastly refuse to draw the obvious conclusion that these statistics say far more about the way they have been taught than about children's inherent ability.

[6]See note 3.

numbered parts soon shows. Problems 2, 6 and 8 are meant to make students suspicious. How they respond will depend very much on their previous experience. Many behave uncritically and pick the first vaguely prime-looking number as "the answer" (for example, with Problem 2, many able students choose 143—or even 63); they then simply ignore the crucial question "How many are there?" Others are much more careful, but (perhaps because they have never been challenged to look for something which may not exist) remain totally unsuspecting—even after two or three similar searches (such as Problems 2, 6 and 8) draw a complete blank.[7]

Many teachers object to the wording of Problems 2, 6 and 8. If only one such question were asked, I would probably agree that it would be thoroughly unfair (unless, of course, one knew that the students were used to keeping their wits about them). I see little educational value in trick questions designed to catch people out.

But in this setting, the doomed searches generated by Problems 2, 6 and 8 and the group discussion to which they should give rise represent one of the many ways in which one can help able students

- to see that there is more to mathematics than simply getting the right answer.

In this case it is precisely the three "rogue" Problems 2, 6 and 8 and the unexplained contrast between these and Problem 4 that force students

- to begin to think about the mathematics behind the problem sequence as a whole and to look for a genuine explanation which distinguishes between the two kinds of observed behavior.

The habit of "wanting to explain," rather than being content just to "get the right answer," is far from natural. It has to be educated. Without it students never develop that independence and autonomy which allows them to take control of their own activity: they remain dependent on others to provide them with problems to solve and to validate or correct, the answers they come up with. It is for this reason, rather than because of some belief in the deductive character of "real" mathematics, that one of our prime objectives should be to cultivate the habit of wanting to explain in our talented students. I would therefore restrict the use of Problems 2, 6 and 8 to students who are familiar with the basic factorization[8]: $(*)\ x^2 - 1 = (x - 1)(x + 1)$.

This is not to say that one expects such students to spontaneously translate the verbiage of the first two problems into symbolic form. They won't! At least, not until their failure to answer Problems 2, 6 and 8 has left them with a puzzle which commonsense methods have failed to resolve. Once the relevance of the familiar identity $(*)$ is noticed, it is but a short step to suspect, and then to discover, the less familiar algebraic factorizations for $x^3 \pm 1$.

Students who get this far can then be challenged to formulate, and to try to resolve, the two general questions of which Problems 1, 2, 5, and 6 and 3, 4, 7 and 8 are special cases.

- For which values of m, n is $m^n - 1$ prime?
- For which values of m, n is $m^n + 1$ prime?

These questions are on a much higher level than the original problems; but the earlier, more simple-minded problems do seem to help students to respond appropriately. The trivial case "$n = 1$" has to be noticed and excluded (usually later rather than sooner). The questions could then be tackled by generalizing the elementary algebraic factorizations alluded to above. However, even very able students are likely to spend quite a long time experimenting with special cases before they realize this.

Like most mathematical problems, the questions as stated are ambiguous. The colloquial formulation leads one to think in terms of "sufficient" conditions on m and n which will guarantee the primeness of $m^n \pm 1$, whereas mathematically one can only hope to obtain "necessary" conditions (which may or may not turn out to be "sufficient"). It may take some time for this distinction to emerge, but it leads naturally to a discussion of the fundamental method of analysis, in which one supposes that

[7]Hundreds of keen 17-year-old students specializing in mathematics who were given ten days to tackle Problems 1, 2, 5, 6 enlisted the help of their home computers. Faced with a negative output, they merely reported, "The prime numbers must be very large." Barely a dozen of these able students smelt a rat.

[8]Some very able students are perfectly capable of seeing that $3^2 - 1 = (3 - 1)(3 + 1)$, $4^2 - 1 = (4 - 1)(4 + 1)$, etc. is part of a general pattern, whether or not they have any formal acquaintance with algebra. But most talented students need some fluency in algebra if they are to discover the factorizations for $x^3 \pm 1$, and for $m^n \pm 1$, for themselves.

one has an entity of the required kind—for example, a prime number of the form "$m^n - 1$"—and proceeds to analyze the possibilities for the numbers m and n.

For example, $m^n - 1 = (m - 1)(m^{n-1} + m^{n-2} + \ldots + m + 1)$, so if $m^n - 1$ is prime then we must obviously have $m - 1 = 1$. Moreover, if $n = a \times b$ is composite, then $2^{a \times b} - 1 = (2^a)^b - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a + 1)$. Hence if $2^n - 1$ is prime, n must itself be prime.

The miracle of the method of analysis is that, if pushed far enough, the resulting "necessary" conditions often turn out to be "sufficient" as well (as in the classical analysis of primitive pythagorean triples: positive integers x, y, z with no common factors satisfying $x^2 + y^2 = z^2$). In our case, elementary algebra has led to the necessary condition that "if $m^n - 1$ is prime ($n \geq 2$), then $m = 2$ and n is prime." One naturally hopes that the condition "n is prime" may turn out to be sufficient to guarantee the primeness of $2^n - 1$. Well, does it?

A similar analysis of the second general question (When is $m^n + 1$ prime?) leads first to the observation that either $m = 1$ or m is even (Why?), and then to the observation that n must be a power of 2. (Suppose $n = a \times b$ with $b \geq 3$ odd, then $m^n + 1 = (m^a)^b + 1 = \ldots$.) Restricting to the simplest case where $m = 2$, we therefore know that, if $2^n + 1$ is to be prime, then n must be a power of 2. But is the condition $n = 2^k$ sufficient to guarantee the primeness of $2^n + 1$?

There is no need to stop there. A discussion of Mersenne and Fermat primes can lead on to more efficient ways of testing for primeness, based on *Fermat's Little Theorem* ($a^p \equiv a \pmod{p}$) or Lucas' test. All of this involves masses of calculation, but it is calculation that achieves results the students would have previously assumed to be beyond them. The limited vision of most talented students means that the initial phase of any activity needs to appear relatively straightforward; but if one is to broaden that vision, then one must somehow lead them on from these simple beginnings to higher things.

There Are No Holes Inside a Diamond

Mogens Esrom Larsen

A diamond is a girl's best friend! —Marilyn Monroe quote

There are twelve different ways of putting together five unit squares edge-to-edge. The results are the Pentominoes. (Pentomino is a registered trademark of Solomon W. Golomb who introduced them (Golomb 1954).) They are shown in Figure 1-1 with their single-letter names.

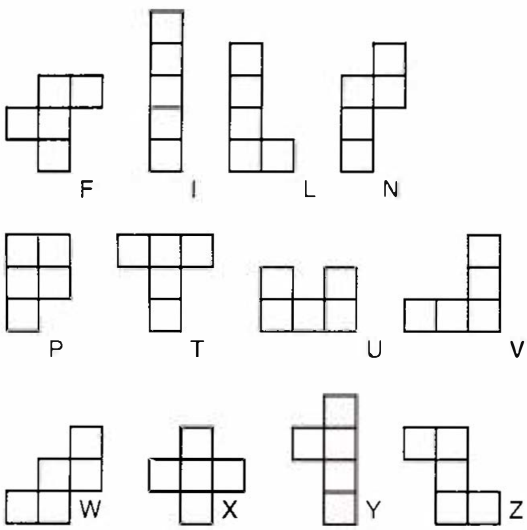


Figure 1-1

Note: Pentomino is a registered trademark of Solomon W. Golomb

The pentominoes can be used to construct many interesting shapes. On the other hand, there are shapes which are impossible to construct. The reason is sometimes obvious but often non-trivial. While some impossibility proofs are elegant, others are unavoidably complex.

One such shape is the diamond with a hole at its centre, as shown in Figure 1-2. Proposed by R. M. Robinson, it was proved by S. Earnshaw that its construction is impossible. The full proof was unpublished, but a four-page summary of the eight-step solution was presented by Golomb (1965, 69-73), who challenged his readers to find a simpler proof.

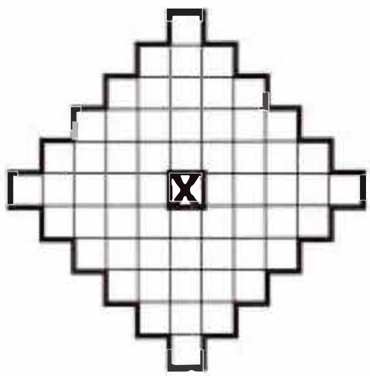


Figure 1-2

This paper presents a simpler argument which proves more—that the construction is impossible unless the hole is on the edge. In other words, there are no holes inside a diamond!

The approach is indirect. It will assume that a construction is possible with the hole inside and derive a contradiction.

Let us introduce some terminology. A pentomino is said to be *interior* if it is placed so that it does not cover any edge squares. A square is said to be *interior* if it is either the hole or covered by an interior pentomino. Note that the centre is always interior.

Acknowledgment—The author thanks the referee for valuable improvement to the presentation of this paper.

A pentomino is said to be *spectral* if it is placed so that it connects an edge square to one of the four squares adjacent to the centre. Only I, L, N, V, W and Z can be spectral. Finally, two spectral pentominoes are said to be *neighboring* if they cover at least one common point.

All 20 edge squares are to be covered. Each of F, W and X can cover three of them, each of N, P and Y two, and each of I, L, T, U, V and Z one. The total count is 21, so that there is no immediate contradiction, but there is some useful information.

☐ Observation 1.1. *There is at most one interior pentomino. There are at most six interior squares.*

☐ Observation 1.2. *If there is an interior pentomino, it is one of I, L, T, U, V, Z.*

We shall divide the proof of our main result into three parts, the first two devoted to proving the following auxiliary results.

☐ Theorem A. *There is exactly one interior pentomino.*

☐ Theorem B. *The interior pentomino is one of I, V, Z.*

Proof of Theorem A

Suppose there are no interior pentominoes. Then the hole must be at the centre and there will be four spectral pentominoes. Figure 2-1 illustrates a placing of L, N, V and W as the four spectral pentominoes.

Note that the remaining part of the diamond is partitioned into four regions. In order for them to be filled with the remaining pentominoes, the number of squares in each region must be divisible by five. While this is true for the regions between L and W, and between N and V, it is not the case for the regions between L and N, and between V and W.

Routine search reveals that only three pairs, L and W, N and V, and I and Z, can define regions with a number of squares divisible by five. It follows that no matter which four pentominoes are spectral, at most two of the regions can be filled with pentominoes.

Therefore, we must have at least one interior pentomino. By observation 1.1, there is at most one interior pentomino. Hence, there is exactly one interior pentomino and the proof of Theorem A is completed.

A most important result follows immediately from Theorem A.

☐ Corollary 2.1. *Each of F, W and X must cover three edge squares while each of N, P and Y must cover two edge squares.*

The technique used in proving Theorem A also yields an additional result.

☐ Lemma 2.2. *V and I cannot be neighboring spectral pentominoes, nor can V and Z.*

Proof: Routine search reveals that, no matter how V and I are placed as neighboring spectral pentominoes, the number of squares in the region between them is always two or three more than a multiple of five. Even if the hole is in this region, the rest of it still cannot be filled with pentominoes. The same holds for V and Z. Figure 2-2 shows two placings of V as spectral neighbors of I and Z, respectively.

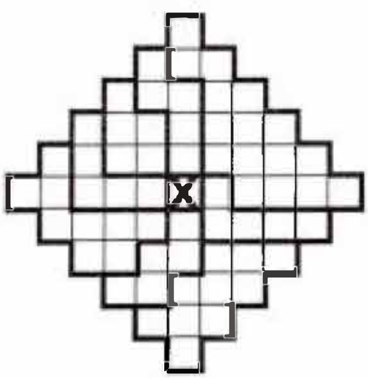


Figure 2-1

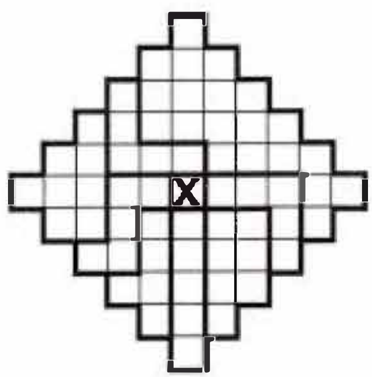


Figure 2-2

Proof of Theorem B

We shall prove that each of I, V and Z is either interior or spectral. It then follows that one of them must be interior. Otherwise, all three will be spectral and V must be a neighbor of I or Z. This is impossible by Lemma 2.2.

Let us now consider each of I, V and Z in turn. Z is the easiest to handle because it does not have any non-interior, non-spectral placings.

☐ Observation 3.1. *Z is either interior or spectral.*

☐ Lemma 3.2. *I is either interior or spectral.*

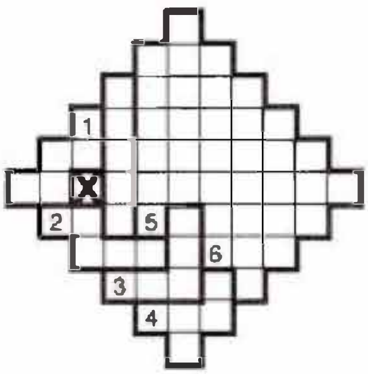


Figure 3-1

Proof: Figure 3-1 shows the only non-interior, non-spectral placing of I. Note that square 1 must be covered by W. We cannot cover square 2 with F by Corollary 2.1, so that N must be used. This isolates the square marked with a cross and it must be the hole. Square 3 must now be covered by V and square 4 by F. It is easy to see that square 5 is interior. The same holds for square 6 by Corollary 2.1. By Theorem A, exactly one pentomino is interior, but no pentomino can cover both squares 5 and 6. We have a contradiction to Observation 1.1.

It is much more difficult to prove that V is either interior or spectral. To do so, we have to find out more about how other pentominoes must be placed relative to one another.

In Figure 3-2, the shaded squares are called the "back" of W and the "back" of F, respectively.

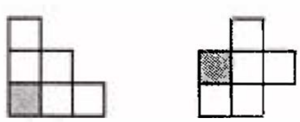


Figure 3-2

☐ Lemma 3.3. *The "back" of W is either interior or covered by L.*

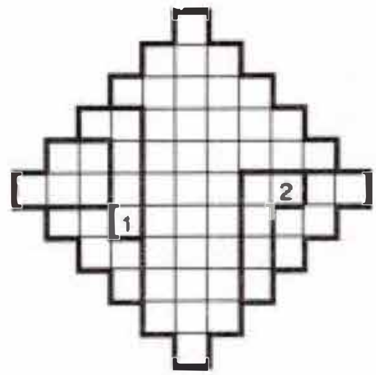


Figure 3-3

Proof: By Corollary 2.1, there are two placings of W, both shown in Figure 3-3. It is easy to see that, if square 1 is not interior, it must be covered by L. There are three possible ways, one of which is shown. On the other hand, Corollary 2.1 shows that, if square 2 is not interior, it must be covered by L as shown.

☐ Lemma 3.4. *The "back" of F must be covered by N.*

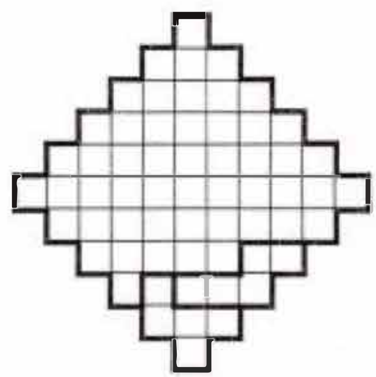


Figure 3-4

Proof: By Corollary 2.1, there is only one placing of F. If its "back" is to be covered by N, it can only be done in one way, as shown in Figure 3-4. On the other hand, if N is placed as shown, the placing of F is forced. To prove our result, we rule out other placings of N, of which there are two.

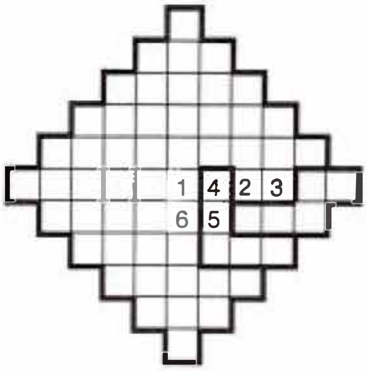


Figure 3-5

Suppose N occupies a corner as shown in Figure 3-5. It is easy to see that squares 1, 2 and 3 are interior. If square 4 is not, it must be covered by V. Note that V cannot fail to cover square 5, as otherwise it forces a placing of P contrary to Corollary 2.1. Now square 6 will also be interior. Moreover, no pentomino can cover three of squares 1, 2, 3 and 6. Hence square 4 is interior. It follows that square 5 is also.

Only Y can cover all five interior squares, 1, 2, 3, 4 and 5. However, this is forbidden by Observation 1.2. Hence the interior pentomino covers four of these five squares and only L, I and T can do so.

If L is interior, then square 5 must be the hole. However, by Lemma 3.3, the "back" of W will be a seventh interior square, contradicting Observation 1.1. If I or T is interior, then squares 5 or 1, respectively, will be the hole. Now square 6 cannot be interior and it is easy to see that it can only be covered by L. Once again, the "back" of W will be an impossible seventh interior square.

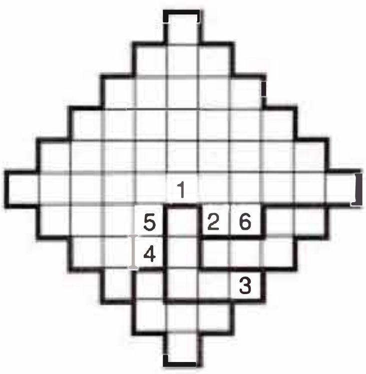


Figure 3-6

Figure 3-6 shows the other placing of N. It is easy to see that squares 1 and 2 are interior. Suppose square 3 is covered by W. By rearranging N and W, we can make them occupy the same region but with N occupying a corner. We have already shown that this is impossible.

Hence square 3 must be covered by V. Now the placing of F is forced and squares 4 and 5 become interior. The hole must either be square 2 or 4 as no pentomino can cover both. The interior pentomino must cover square 5 and cannot cover square 6. Now square 6 must be covered by L, and Lemma 3.3 furnishes a contradiction.

□ Lemma 3.5. *V is either interior or spectral.*

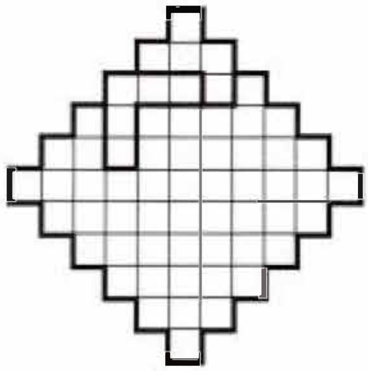


Figure 3-7

Proof: V has two non-interior, non-spectral placings. The one shown in Figure 3-7 can be ruled out as it forces the placing of F contrary to Lemma 3.4.

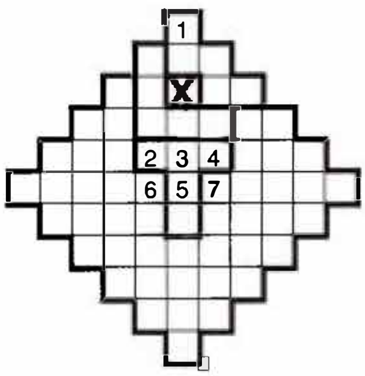


Figure 3-8

Figure 3-8 (previous page) shows the other possibility. Suppose square 1 is covered by W. Then the square marked by a cross must be the hole. Now squares 2, 3, 4 and 5 must all belong to the interior pentomino, and only T can cover all of them, P and Y being ruled out by Observation 1.2. By Observation 3.1 and Lemma 3.2, Z and I must cover squares 6 and 7 collectively. However, neither can cover square 6. We have a contradiction.

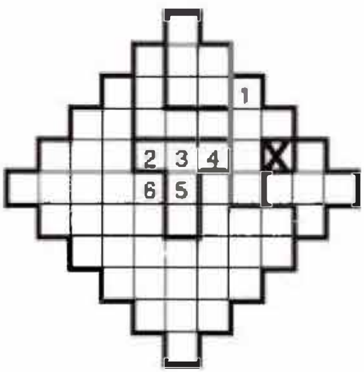


Figure 3-9

The only other pentomino that can be used in place of W is P. By Corollary 2.1, X must occupy a corner. Suppose it takes up its position as shown in Figure 3-9. Then square 1 must be covered by Y, creating a hole at the square marked with a cross. Now squares 2, 3, 4 and 5 are all interior and only T can cover all of them. Thus square 6 is not interior, and it can only be covered by L. By Lemma 3.3, the "back" of W will create an impossible seventh interior square.

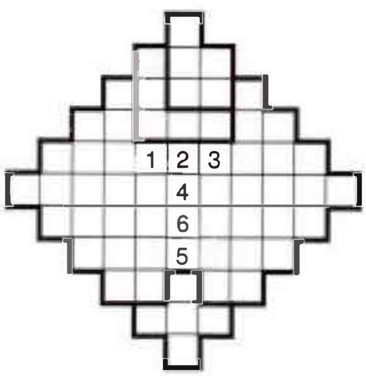


Figure 3-10

Now let X occupy the corner as shown in Figure 3-10. As before, squares 1, 2, 3 and 4 are all interior. Suppose square 5 is also interior. Then the interior pentomino will cover four of these five squares. Only T and L can do that and whichever one is interior will also cover square 6. By Lemma 3.2, I is spectral, but it is easily seen that it now has no spectral placings.

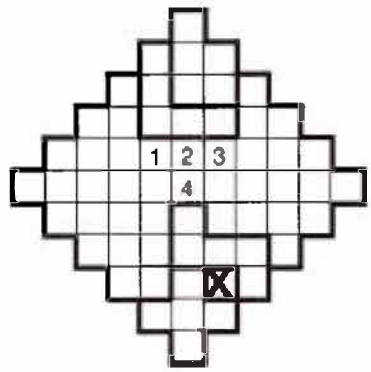


Figure 3-11

It follows that square 5 is not interior and it must be covered by either T or Z. Figure 3-11 shows Z in place, creating a hole at the square marked with a cross. Now no pentomino can cover all of squares 1, 2, 3 and 4 except Y, which cannot be interior. The same contradiction is arrived at if T covers square 5 instead. This completes the proof of Lemma 3.5 and hence of Theorem B.

Conclusion

By Theorem A, there is exactly one interior pentomino. By Theorem B, it is I, V or Z. Hence T is not interior. It has two non-interior placings. The first one, shown in Figure 4-1, can be ruled out since it forces a placing of F contrary to Lemma 3.4.

The second one is shown in Figure 4-2. Now square 1 can only be covered by W. We cannot use V as it again forces a placing of F contrary to Lemma 3.4. By Corollary 2.1, squares 2, 3, 4, 5 and 6 are all interior and the interior pentomino must cover at least four of them. This can only be I, and the hole is at square 4. By Lemma 3.5, V must be spectral, but it is easy to verify that it now has no spectral placings. This completes the proof of our main result.

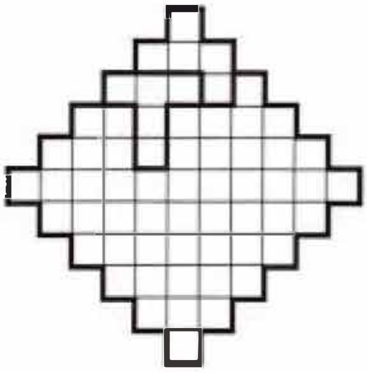


Figure 4-1

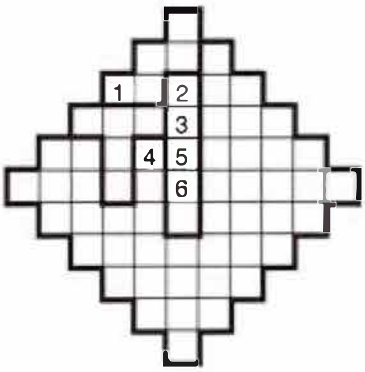


Figure 4-2

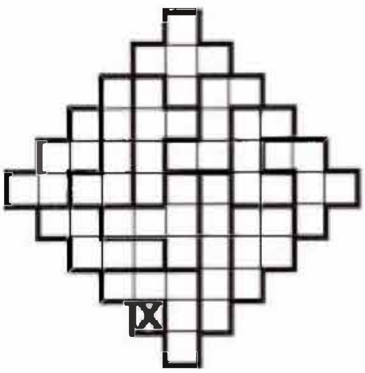


Figure 4-3

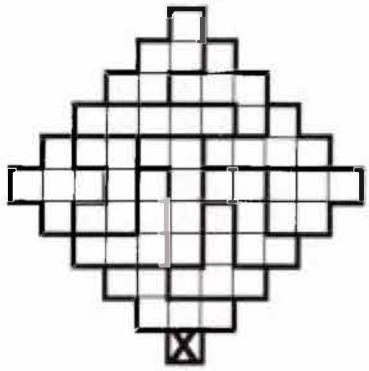


Figure 4-4

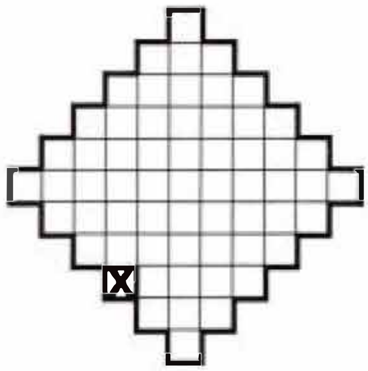


Figure 4-5

Now that we know there are no holes inside the diamond, the natural question is whether the hole can be on the edge. This is indeed possible, as shown in Figure 4-3, attributed to J.A. Lindon (Golomb 1965, 73). Figure 4-4 shows that the hole can be placed more aesthetically at a corner. It is left to the reader to decide whether Figure 4-5 can be constructed with a set of pentominoes and whether eleven pentominoes can fit into the diamond covering all 20 edge squares.

A rough diamond . . . A proverb

References

Golomb, Solomon W. "Checkerboard and Polyominoes," *American Mathematics Monthly* 61(1954):675-682.

Golomb, Solomon W. *Polyominoes,* New York: Charles Scribner's Sons, 1965.

Addendum to There Are No Holes Inside a Diamond

Diamonds are forever.—James Bond movie title

The diamond considered in the main part of this paper is but one member of an infinite family of diamonds. The first five are shown in Figure A-1, and our diamond is D_5, the next in line. If we denote by d_n the number of squares in D_n, it is an easy exercise to show that $d_n = 2n^2 + 2n + 1$.

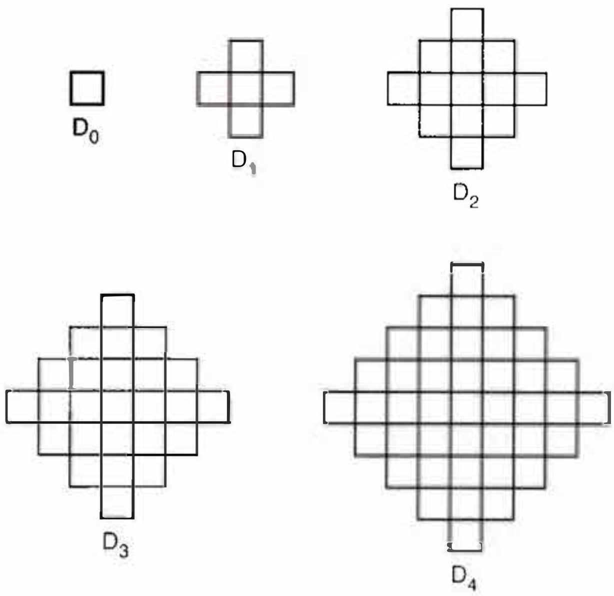


Figure A-1

If $n \equiv 1$ or $3 \pmod 5$, then $d_n \equiv 0 \pmod 5$ and D_n can be constructed with pentominoes. D_1 is trivial since it is just X. Figure A-2 shows a construction of D_3. We pose the following problems.

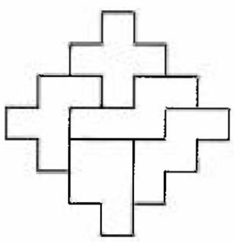


Figure A-2

Problem 1
Construct D_3 using F, W, X, N and Y.

Problem 2
Construct D_6 using a complete set of pentominoes plus F, W, X, P and Y.

Problem 3
Construct D_8 using two complete sets of pentominoes plus F, W, X, N and P.

Note that the best pentominoes for diamonds, F, W and X, are used in each problem. Each problem also uses two of the next best pentominoes, N, P and Y, and a different pair in each case.

If $n \equiv 2 \pmod 5$, then $d_n \equiv 3 \pmod 5$ and D_n cannot be constructed with pentominoes unless we leave three holes. This makes the problem too loose, but the reader may wish to explore for interesting designs.

If $n \equiv 0$ or $4 \pmod 5$, then $d_n \equiv 1 \pmod 5$ and D_n can be constructed with pentominoes if we leave only one hole. D_0 is trivial since no pentominoes are required. Figure A-3 shows a construction of D_4 with the hole inside, and the reader may try to decide if the hole can be at the centre. D_5 having already been dealt with, we pose one final problem.

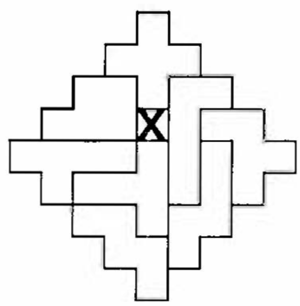
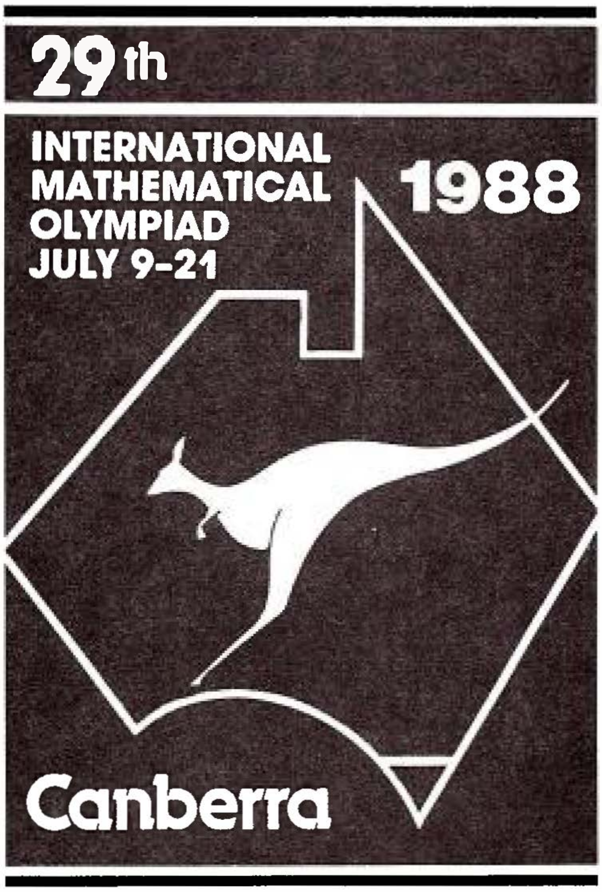


Figure A-3

Problem 4
Construct D_9 using three complete sets of pentominoes, leaving a hole at the centre.

Some Thoughts on Mathematical Olympiads

Jan van de Craats



Mathematics contests for secondary school students are well established and widespread. In some countries they are extremely popular, while in others they are more elitist in character.

A common feature of most competitions is that they are organized stepwise, that is, in two or more rounds, each with an increasing degree of difficulty. Participants obtaining high scores in a lower round are invited to take part in the next one. From the best students in national competitions teams of six are formed to take part in the International Mathematical Olympiad (I.M.O.), a contest of considerable prestige; its problems pose demanding challenges even for the professional mathematician.

As one who served as a member of the jury of the International Mathematical Olympiad from 1973 to 1985 and as one of the organizers of the Dutch Mathematical Olympiad for a somewhat longer period, I would like, on retiring, to express some thoughts on mathematics contests, in particular on those of the highest level: the final rounds of national competitions and the I.M.O. I want to concentrate on the effects of these olympiads on the participating students.

The official goal of all contests is, of course, to enhance interest in mathematics and to discover and stimulate mathematical talent. But there are beneficial side effects that may not be obvious.

By the very nature of these competitions, students obtaining high scores in final rounds have exceptional talents. Usually, they excel not only in mathematics but also in many other areas. Often they are by far the best in their family, their class and their school. For the students themselves, their ability is not always entirely positive. They often find school extremely boring and their brightness may alienate them from their schoolmates. Many of these students are shy, feel lonely and have social problems.

Taking part in an olympiad presents these students with real challenges, often for the first time in their lives, and they very much enjoy it. It also brings them into contact with fellow students with similar interests and abilities. Organizers of olympiads should be alert to this aspect and create ample opportunities for formal and informal contacts among the competitors.

Various "camps for young mathematicians" which have been organized in several countries have taken note of this social need. While competitions, lectures on mathematics by university professors and small research projects are the "official" items on the program of such camps, the informal contacts which are a by-product are probably no less important and contribute greatly to the camps' success.

The same benefits can accrue in the sessions preceding the participation of national teams in the I.M.O. and, of course, at the I.M.O. itself. The Olympiad lasts ten days, with only two days of "examination" and provides a unique opportunity for interchange, both intellectually and socially, among the participants (and, as a matter of fact, also among organizers and members of the jury). Students who have taken part in an I.M.O. usually keep in touch with their team-members when they are at university and these international contacts very often continue for many years.

Thijs Notenboom and Jan Donkers, who currently lead the Dutch team to the I.M.O., pay particular attention to these social aspects and organize formal and informal meetings among members of future and former I.M.O. teams. They plan to arrange a special section during the yearly Dutch Mathematical Congress for former I.M.O. participants.

To summarize, apart from discovering and stimulating mathematical talent, olympiads also provide social benefits, bringing bright students into contact with one another and with professional mathematicians. This fact deserves special recognition by organizers of such competitions and by the mathematics community in general.

Appendix I: Supplementary Problems

Problem 1

There are 1001 pebbles in a heap. The heap is divided into two, the number of pebbles in each is counted and the product of these two numbers is written down. A heap containing at least two pebbles is then chosen, divided into two, the pebbles are counted and the product is written down. This procedure is continued until every heap contains one pebble. Find the maximum value of the sum of the 1000 products written down.

Problem 2

A difficult mathematical competition consisted of a Part I and a Part II with a combined total of 28 problems. Each contestant solved exactly seven problems altogether. For each pair of problems, there were exactly two contestants who solved both of them. Prove that, if every contestant solved at least one problem in Part I, then at least one contestant solved at most three problems in Part II.

Problem 3

Use the hexominoes (see Figure 1a in Kathy Jones' article) to construct a 15 by 15 square with a centrally symmetric 3 by 5 hole.

Problem 4

Use all polyiamonds up to and including the heptiamonds (see Figure 1b in Kathy Jones' article) to construct the star of David with a hole as shown in Figure A. Note that there is a diamond inside the hole!

Problem 5

You have an inexhaustible supply of 5-cent and 8-cent stamps. What amounts of postage cannot be made exactly if only these stamps can be used?

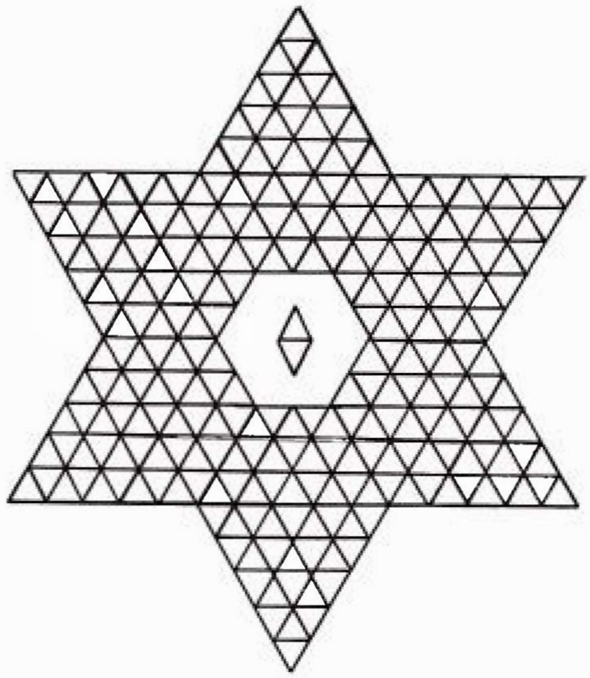


Figure A

Problem 6

A prime triple (x, y, z) consists of three prime numbers x, y and z such that $y - x = z - y$. The common value of $y - x$ and $z - y$ is called the common difference of the prime triple.

(a) Find a prime triple with common difference 2.
(b) Find another prime triple with common difference 2. How many others are there?
(c) Find a prime triple with common difference 3. Are there any?
(d) Find a prime triple with common difference 4.
(e) Find another prime triple with common difference 4. How many others are there?
(f) Find a prime triple with common difference 5. Are there any?
(g) Find a prime triple with common difference 6.
(h) Find another prime triple with common difference 6. How many others are there?

Problem 7

Prove that the shape in Figure B cannot be constructed using a complete set of pentominoes.

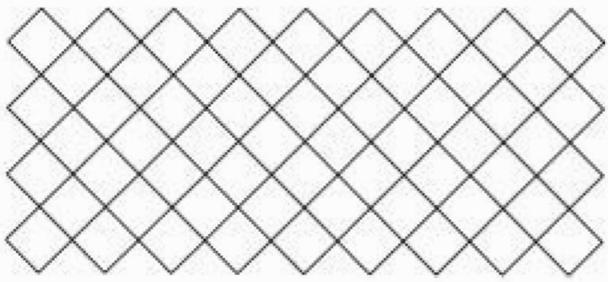


Figure B

Problem 8

Prove that the shape in Figure C cannot be constructed using a complete set of pentominoes.

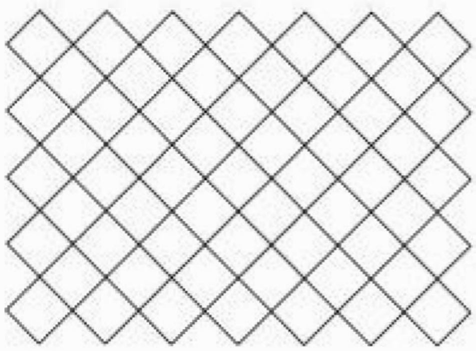


Figure C

Problem 9

Determine the maximum value of $m^2 + n^2$ where m and n are integers satisfying $m, n \in \{1,2,\ldots,1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

Problem 10

Superchess is played on a 12 by 12 superboard and it uses superknights which move between opposite corner cells of any 3 by 4 subboard. Is it possible for a superknight to visit exactly once every other cell of the superboard and return to its starting cell?

Sources

Problem 1 is taken from the Soviet journal *KVANT.*
Problem 3 is taken from the instruction booklet of Sextillions (a registered trademark of Kadon Enterprises).
Problem 5 is taken from Tony Gardiner's *Discovering Mathematics.*
Problem 7 is taken from Solomon W. Golomb's *Polyominoes.*
Problem 9, proposed by Jan van de Craats, appeared in the 1981 International Mathematical Olympiad.
Problems 2, 4, 6, 8 and 10 are the editor's composition.
Problem 2 appeared in the 1984 U.S.A. Mathematical Olympiad.
Problem 10 was proposed for the International Mathematical Olympiad but not used and appeared instead in the problem set of the 1986 International Mathematical Congress.

Appendix II: High School Mathematics Competitions in Alberta

A. Brief History

Although there are older mathematics competitions in Canada for high school students, Alberta was the first to have a province-wide contest. It started in 1957 as the "Alberta Matriculation Prize and Scholarship Examination," sponsored by the Nickle Family Foundation (Calgary), the Canadian Mathematical Congress (now Society), the Mathematics Council of The Alberta Teachers' Association, the Department of Mathematics and Statistics at the University of Calgary and the Department of Mathematics at the University of Alberta.

From the very beginning, the Alberta contest emphasized problem solving questions. Multiple choice questions were introduced in 1967 to accommodate the rising number of participants. That year the contest was renamed the "Alberta High School Mathematics Prize Examination."

In 1969, the Canadian Mathematics Olympiad (C.M.O.) came into being. The Alberta contest acquired an additional role, to be the qualifying round for the national contest. The latter was introduced principally in anticipation of Canada's participation in the International Mathematical Olympiad (I.M.O.), which had been initiated by Romania in 1959. However, it was not until 1981, when the United States hosted the event, that Canada entered a team. It included two members from Alberta, Arthur Baragar and John Bowman.

In 1983, the Alberta High School Mathematics Prize Examination Board was formed to administer the Alberta contest. One of the most pleasant duties of the new board was to welcome two new sponsors, the Peter H. Denham Memorial Fund in Mathematics (Edmonton) and the publishing house of W. H. Freeman (New York).

That same year, 1983, the two parts of the Alberta contest were separated. The muliple choice part retained the title, "Alberta High School Mathematics Prize Examination." Competition among school teams was introduced and the contest was written in the fall. It served as well as the qualifying round for the problem solving part of the contest, which was written in the following spring and renamed the "Alberta High School Mathematics Scholarship Examination."

In 1988, Dover Publications Incorporated (New York) joined the list of sponsors. The board was renamed the "Alberta High School Mathematics Competition Board" (A.H.S.M.C. Board), to emphasize that the contest is not an examination. The two former "examinations" now became the first and second rounds of the "Alberta High School Mathematics Competition."

The late Leo Moser is acknowledged as the father of the Alberta contest. Other individuals have devoted considerable time and effort to this endeavor. The list from the University of Calgary includes Allan Gibbs, Tony Holland, Harold Lampkin, Bill Sands, Jonathan Schaer and Bob Woodrow. (Unfortunately, this part of the record is sadly inadequate and we apologize to those who should be mentioned but are not.) The list from the University of Alberta includes Ken Andersen, Alvin Baragar, Bill Bruce, Graham Chambers, Jim Fisher, Herb Freedman, Murray Klamkin, Ted Lewis, Andy Liu, Jack Macki, Jim Muldowney, Arturo Pianzola, Jim Pounder, Roy Sinclair, Sudarshan Sehgal and Jim Timourian. The late Geoffrey Butler served as the chairman of the Alberta board and of the Canadian Mathematics Olympiad Committee, as well as the leader of the Canadian teams in the International Mathematical Olympiads of 1981 to 1984.

B. Current Information

Contests

The first round of the Alberta High School Mathematics Competition is held in November and the second round in the following February. The top 22 students in the latter competition are nominated to write the Canadian Mathematics Olympiad, which is administered by the Canadian Mathematical Society. All three contests are written in the students' own schools. Top performers in the C.M.O. may earn a spot on the Canadian National Team which competes in the International Mathematical Olympiad, usually held in Europe.

Eligibility

All students enrolled in a high school program in Alberta or the Northwest Territories are eligible to take part in the first round. Those students who qualify through the first round may participate in the second round. A limited number of special applications may be accepted.

Date

The first round is scheduled each year for the morning of the third Tuesday in November. The duration of the competition is one hour and it must start between 8:30 and 9:30. The second round is scheduled from 9 to 12 in the morning of the second Tuesday in February.

Format

The first round is a 60-minute paper which consists of 16 multiple choice questions, to be graded on the computer. Soft pencils (regular pencils will do) *must* be used. Five points are given for each correct answer, two points for each question not attempted and zero points for each incorrect (including multiple) answer. The scores range from 20 to 100 points.

The second round is a three-hour paper consisting of five essay-type questions. Twenty points are given for each complete solution. Partial credits are given for significant progress. The scores range from zero to 100 points.

Pencil, eraser, graph paper, scratch paper, ruler and compass are allowed. Calculators are *not* allowed

All examination materials are the property of the Alberta High School Mathematics Competition Board.

Applications

Applications to write the first round are made through a teacher designated as the contest manager in each school and must *arrive* at the official address at least three weeks before the contest is to be written. Each school entering at least three students is considered to have a team. Team membership is determined after the results are known. The top three students will constitute the team and the team score is the total score of these three members.

The top 50 students will be invited to write the second round. Invitation will also be extended to the top ten students in Grade 11 and the top five students in Grade 10. In exceptional circumstances, contest managers may nominate a number of additional candidates from their schools to write the second round. Special applications must be submitted to the Board and must *arrive* at the official address at least three weeks before the contest is to be written. The number of such candidates from each school is limited and is determined by the Board. It is roughly proportional to the number of participants in the first round from the school.

Fees

The application fee for the first round is \$1 per student. There is no application fee for the second round, except for the special applications for which it is \$5 per student.

All fees are payable to the University of Alberta and must be submitted with the application.

Hotline

Quick consultation with the A.H.S.M.C. Board is possible via telephone. The current contact is Professor Alvin Baragar, 492-3398.

School Prizes

In the first round, the Peter H. Denham Memorial Plaque goes to the first place school.

Book prizes of appropriate values are awarded to: (1) the top three schools; (2) the top school in each zone which does not qualify under (1); (3) the top school which has not won any book prizes from the A.H.S.M.C. Board.

The prize in (3) was created in memory of the former board chairman, Geoffrey Butler.

In addition, certificates go to each school which entered a team and its contest manager.

Individual Prizes

In the first round, the title of W. H. Freeman Scholar goes to the first place student.

Book prizes of appropriate values are awarded to: (1) the top three students; (2) the top student in each of Grade 10 and Grade 11 if not a recipient under (1); (3) the top two students in each zone if not already recipients under (1) or (2).

In addition, a certificate is awarded to each student invited to write the second round, as well as to the top student from each school which entered a team, if none of these was previously a recipient under previous prize categories.

In the second round, five fellowships are awarded on the basis of performance:
(1) The Nickle Family Foundation Fellowship ($500); (2) The Peter H. Denham Memorial Fellowship ($250); (3) The Canadian Mathematical Society Fellowship ($150); (4) The Alberta Teachers' Association Grade 11 Fellowship ($50); (5) The Alberta Teachers' Association Grade 10 Fellowship ($50).

Winners of these fellowships must be Canadian citizens or landed immigrants of Canada. The Nickle Family Foundation Fellowship is awarded on the condition that the winner will attend an Alberta university and is to be credited towards the winner's tuition fees at such an institution.

In normal circumstances, these fellowships are awarded in descending order of scores. In case of ineligibility, they may be withheld, subdivided or offered to candidates with lower scores. No student may win more than one of these fellowships in the competition of any one year. All fellowship winners (plus students who would have been winners but who are ineligible for fellowships) receive a certificate.

The decisions of the Alberta High School Mathematics Competition Board are final.

Geographical Zones

For the purpose of assuring some regional distribution of prizes, Alberta and the Northwest Territories are divided into four zones as follows:
Zone 1: The City of Calgary.
Zone 2: Southern Alberta (north to and including the City of Red Deer and excluding the City of Calgary)
Zone 3: The City of Edmonton.
Zone 4: Northern Alberta (excluding the City of Edmonton) and the Northwest Territories.

C. Further Information

Each of Calgary and Edmonton cities has its own contest for junior high students. Each contest is administered by a group of dedicated teachers. The contests are written in the spring and consist of some multiple choice questions and some problem solving questions.

Alberta students can write two sequences of contests administered from outside the province.

One is the Canadian Mathematics Competition sponsored by the University of Waterloo. It began in 1963 as the Junior Mathematics Contest. It now consists of two "Gauss" contests, one for Grade 7 and one for Grade 8, a "Pascal" contest for Grade 9, a "Cayley" contest for Grade 10, a "Fermat" contest for Grade 11, a "Euclid" contest for Grade 12, and a "Descartes" contest which is open to all students but is primarily for Ontario Grade 13 students. These contests are written in the spring. The papers for the earlier grades consist entirely of multiple choice questions. The regional coordinator is Professor Bob Woodrow of the University of Calgary.

The other sequence is run by the Mathematical Association of America. It consists of an American Junior High School Mathematics Examination for Grades 7 and 8, which is scheduled in December, and an American High School Mathematics Examination for Grades 9, 10, 11 and 12, scheduled in the spring. The latter serves as a qualifying round for the American Invitational Mathematics Examination, which also occurs in the spring and which, in turn, serves as a qualifying round for the U.S.A. Mathematics Olympiad.

The junior high and high school contests consist entirely of multiple choice questions. The questions in the Invitational ask for integral answers between 0 and 999, so that they are really multiple choice with a thousand alternatives. The U.S.A. Mathematics Olympiad consists of five problem solving questions. The regional coordinator is Professor Bill Sands of the University of Calgary.

D. Sample Papers

The Alberta High School Mathematics Competition Board publishes a newsletter, *Postulate,* which contains contest papers of recent years. This section reproduces questions from the papers of the first ten contests, 1957 to 1966. Some questions have been slightly edited to correct typographical errors and remove ambiguities.

Year 1957

Problem 1

Express $(1/(1+x))/(1-1/(1+x)) + (1/(1+x))/(x/(1-x)) + (1/(1-x))/(x/(1+x))$ as a simple fraction.

Problem 2

Solve the equation
$\sqrt{16x+1} - 2(\sqrt[4]{16x+1}) = 3.$

Problem 3

In the centre of the flat rectangular top of a building which is 21 metres long and 16 metres wide, a flagpole is to be erected, 8 metres high. To support the pole, four cables are needed. The cables start from the same point, 2 metres below the top of the pole, and end at the four corners of the top of the building. How long is each of the cables?

Problem 4

Solve the equation $x + 2 = 0$.

Problem 5

Solve the equation $x^2 + x + 1 = 0$.

Problem 6

Solve the equation $x^3 + x = 10$.

Problem 7

Given that $ax^2 + bx + c = 0$ has the roots m and n, prove that
(a) $m + n = -b/a$;
(b) $mn = c/a$.

Problem 8

From a 12 by 18 sheet of tin, we wish to make a box by cutting a square from each corner and turning up the sides. Draw a graph showing how the volume of the box obtained varies with the size of the squares cut out. For what size of squares would the largest box be obtained?

Problem 9

Solve the system of equations $x + y + z = 3$, $x + 2y + 3z = 8$ and $x + 3y + 4z = 11$.

Problem 10

Solve the system of equations $x + y + z = 1$, $2x + 3y + 4z = 2$ and $3x + 5y + 7z = 4$.

Problem 11

A certain sample of radium is decreasing according to the equation $A = 3(2^{-t/1800})$, where t is in years and A is in milligrams.
(a) How much radium was in the sample at $t = 0$?
(b) How much will there be 900 years later?

Problem 12

Two circles, each of radius 1, are such that the centre of each lies on the circumference of the other. Find the area common to both circles.

Problem 13

A man takes a trip from A to B at an average speed of 40 kph and returns at an average speed of 60 kph. What is his average speed for the entire trip?

Problem 14

Four men dined at a hotel. They checked their hats in the cloakroom. Each of the four came away wearing a hat belonging to one of the other three. In how many different ways could this have happened?

Problem 15

Given a cylinder of height 6 cm and base radius 2 cm, prove that a spider can go from any point on the surface to any other point on the surface along a path of total length less than 9 cm.

Problem 16

Prove that the difference between the sum of n terms of $n/n + (n - 1)/n + (n - 2)/n + \cdots$ and the sum of the infinite series $n/(n + 1) + n/(n + 1)^2 + n/(n + 1)^3 + \cdots$ is equal to $(n - 1)/2$.

Problem 17

If two sides of a quadrilateral are parallel to each other, prove that the straight line joining their midpoints passes through the point of intersection of the diagonals.

Problem 18

In a certain school all students study Mathematics, Physics and French. Forty percent prefer both Mathematics and Physics to French. Fifty percent prefer Mathematics to French and sixty percent prefer Physics to French. If all students have definite preferences between subjects, what percentage prefer French to both Mathematics and Physics?

Year 1958

Problem 1

Evaluate $1/7 + 2/3 + 5/8$.

Problem 2

Express $c/(a + (b/c)) + (a + c)/(a - (b/c))$ as a simple fraction.

Problem 3

Evaluate $9!(1/8!1! + 1/7!2! + 1/6!3! + 1/5!4!)$.

Problem 4

Solve the equation $(x - 2)(x - 3) = 1$.

Problem 5

Solve the equation $1/(x - 2) + 1/(x - 3) = 1$.

Problem 6

Solve the system of equations $3x + 2y + z = 1$, $4x - y = 2$ and $x - y + 2z = 3$.

Problem 7

You are travelling along a road at 4 kph parallel to a double track railroad. Two trains meet you, each being of the same length. The first takes 40 seconds to pass you, the second 30 seconds. Prove that it takes 4 minutes for the second train to completely pass the first.

Problem 8

Find the greatest common divisor of 3910, 8551 and 11475.

Problem 9

A quartet is chosen by lot from all the high school students in a certain school. There are 40 students in Grade X, 30 in Grade XI and 20 in Grade XII. In how many ways can the quartet consist of
(a) students from Grade X alone;
(b) at least three students from Grade XII?

Problem 10

Let $f(n + 1) = 5f(n) - 6f(n - 1)$ for a function f defined for all positive integers n. Prove that $f(n) = 3^n - 2^n$ is a solution of this equation.

Problem 11

For $f(n) = 3^n - 2^n$, find the sum $f(1) + f(2) + \cdots + f(9)$.

Problem 12

Use Newton's Binomial Theorem to compute the cube root of 26 to three decimal places.

Problem 13

State the Remainder Theorem (for polynomials).

Problem 14

Solve the equation $4x^3 - 8x^2 - 3x + 9 = 0$.

Problem 15

Let P, R and S be any three points on a circle such that RS extended meets the tangent at P at the point T. Prove that $PS/PR = ST/PT = PT/RT$.

Problem 16

(a) Prove that the sum of the first n positive integers is $n(n + 1)/2$.
(b) Prove that the sum of the cubes of the first n positive integers is $n^2(n + 1)^2/4$.

Problem 17

Sketch the locus corresponding to the equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$ and also the locus corresponding to $y^2 - x^2 - 8xy/3 = 0$.

Problem 18

Solve the system of equations
$x^2 + 4y^2 - 6x - 16y + 21 = 0$ and
$y^2 - x^2 - 8xy/3 = 0$.

Problem 19

At $r\%$ interest compounded semi-annually, how much would p dollars deposited on each of the dates: January 1, 1940, 1941, 1942, 1943 and 1944 be worth altogether on January 1, 1945?

Problem 20

Prove the Sine Law (for triangles).

Problem 21

Prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

Problem 22

From the top of a tower, the angle of depression of a point A, in the same horizontal plane as the base of the tower, is x. On the top of the tower is a flagstaff whose length is equal to one-quarter of the tower. Prove that the tangent of the angle which the flagstaff subtends at A is equal to $\sin x \cos x/(4 + \sin^2 x)$.

Year 1959

Problem 1

Solve the equation $2x - 6 + \sqrt{3x - 2} = 0$.

Problem 2

Solve the equation $x^3 + 3x^2 - 4x - 12 = 0$.

Problem 3

Solve the system of equations $6/x + 15/y = 4$, $18/y - 16/z = 1$ and $14/x + 12/z = 5$.

Problem 4

Given that the roots of $ax^2 + bx + c = 0$ are m and n, find the condition on a, b and c so that
(a) $m = 1/n$;
(b) $m = -n$.

Problem 5

Prove that $n!/(n - 2)!2! + (n + 1)!/(n - 1)!2! = n^2$.

Problem 6

$ABCD$ is a kite-shaped figure with $AB = AD$ and $CB = CD$. Find the point P the sum of whose distances from the four vertices is as small as possible.

Problem 7

Six papers are set in an examination, two of them in mathematics. In how many different orders can the papers be given, provided only that the two mathematics papers are not consecutive?

Problem 8

Prove that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

Problem 9

A vertical flagpole stands on a hillside which makes an angle A with the horizontal. At a distance k down the slope from the pole, it subtends an angle B. Prove that the height of the pole is given by $k \sin B/\cos(A + B)$.

Problem 10

If 1 is added to the product of four consecutive positive integers, is the sum always the square of an integer?

Problem 11

If 41 is added to the sum of two consecutive positive integers, is the sum always a prime number?

Problem 12

What is the coefficient of r^2 in the expansion $(r^2 - 2r\cos\theta + 1)^{-1/2}$ in powers of r?

Problem 13

The first three terms of an arithmetic progression are m, $4m - 1$ and $5m + 3$. What is the sum of the first $4m$ terms?

Problem 14

Explain why and how it is possible to attach definite meanings to zero, negative and fractional powers of a positive number.

Problem 15

A golfer can score 2, 3, 4 or 5 strokes per hole. How many different score sequences yield a score of 25?

Problem 16

The lines OAB and OC intersect at O, with $OA = a$ and $OB = b$. A point P moves along OC, and P_1 and P_2 are two positions of P such that the angles AP_1B and AP_2B are equal.
(a) Prove that $OP_1 \cdot OP_2 = ab$.
(b) At what distance of P from O is the angle APB a maximum?

Problem 17

What is wrong with the following argument? [see Figure 1]

$$
\begin{aligned}
(x + 1)^2 &= x^2 + 2x + 1 \\
(x + 1)^2 - (2x + 1) &= x^2 \\
(x + 1)^2 - (2x + 1) - x(2x + 1) &= x^2 - x(2x + 1) \\
(x + 1)^2 - (x + 1)(2x + 1) + (2x + 1)^2/4 &= x^2 - x(2x + 1) + (2x + 1)^2/4 \\
((x + 1) - (2x + 1)/2)^2 &= (x - (2x + 1)/2)^2 \\
x + 1 - (2x + 1)/2 &= x - (2x + 1)/2 \\
1 &= 0
\end{aligned}
$$

Figure 1

Problem 18

PQ is a chord passing through the focus $(a,0)$ of a parabola with vertex at the origin. The slope of PQ is m.
(a) Prove that the coordinates of its midpoint are $a(1 + 2/m^2)$ and $2a/m$.
(b) Prove that the locus of the midpoints of all chords passing through the focus is again a parabola.
(c) Find the focus and directrix of the parabola in (b).

Problem 19

Given m vertical and n horizontal lines, prove that the number of rectangles which can be formed having segments of these lines as sides is $\binom{m}{2}\binom{n}{2}$.

Year 1960

Problem 1

Find the distance between the points $(-1,-5)$ and $(-13,0)$.

Problem 2

Evaluate $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4$.

Problem 3

Find the area of an equilateral triangle of side 1.

Problem 4

One root of $2hx^2 + (3h - 6)x - 9 = 0$ is the negative of the other.
(a) Find the value of h.
(b) Solve the equation.

Problem 5

Find a cubic equation with integral coefficients which has as roots the numbers $2/3$, -2, -1.

Problem 6

Solve the equation $x^3 - 4x^2 + x + 6 = 0$.

Problem 7

(a) Prove that, for any positive number n that satisfies the equation $x^y = y^x$, $x = (1 + 1/n)^{n+1}$ and $y = (1 + 1/n)^n$.
(b) Do the formulae in (a) give all the positive solutions of $x^y = y^x$?

Problem 8

Given that $f(x) = 1/(x + 1/(x + 1/x))$ and $g(x) = x - 1/x$, find
(a) $f(g(x))$;
(b) $g(f(x))$.

Problem 9

Prove that $\log_a xy = \log_a x + \log_a y$.

Problem 10

Prove that $\log_a x^n = n \log_a x$.

Problem 11

From A, a pilot flies $12\sqrt{2}$ km in the direction N30°W to position B, and then $12\sqrt{2}$ km in the direction S60°E to position C. How far and in what direction must he now fly to again reach A?

Problem 12

Prove that $(1 - \cos x + \sin x)/(1 + \cos x + \sin x) = \tan(x/2)$.

Problem 13

There are ten points A, B, ... in a plane, no three in the same straight line.
(a) How many lines are determined by the points?
(b) How many of the lines pass through A?
(c) How many triangles are determined by the points?
(d) How many of the triangles have A as a vertex?
(e) How many of the triangles have AB as a side?

Problem 14

In how many ways can the word PYRAMID be spelt out, using adjacent letters of the arrangement below?

```
D I M A R Y P Y R A M I D
  D I M A R Y R A M I D
    D I M A R A M I D
      D I M A M I D
        D I M I D
          D I D
            D
```

Problem 15

Prove that if one side of a triangle is greater than another, the angle opposite the greater side exceeds the angle opposite the shorter side.

Problem 16

Prove that if $AC = 2BC$ in triangle ABC, then angle B is more than twice angle A.

Problem 17

A ball is dropped from a height of 6 metres. Each time it strikes the ground after falling from a height of h metres, it rebounds to a height of $2h/3$ metres.
(a) How far has the ball travelled when it hits the floor for the fifth time?
(b) What is the total distance travelled by the ball before it comes to rest?

Problem 18

Three numbers are in geometric progression. If A, G and H are their arithmetic, geometric and harmonic means, respectively, prove that $G^2 = AH$.

Problem 19

Numerical calculation seems to show that the relation $8 - \sqrt{62} = 3\sqrt{2}/10$ is at least approximately true. Find whether the relation is exact.

Year 1961

Problem 1

Express $1 + 2/(x + 3/(x + 4/x)) - 3/(x - 3/(x + 4/x))$ as a simple fraction.

Problem 2

Express $\sqrt{(a - b)/(a + b)} + \sqrt{(a + b)/(a - b)} - \sqrt{c^2a^2 - c^2b^2}$ as a simple fraction.

Problem 3

Solve the equation $x - 7\sqrt{x - 4} - 12 = 0$.

Problem 4

Solve the equation $\sqrt{3x + 9} - \sqrt{x + 5} = \sqrt{2x + 8}$.

Problem 5

Solve the equation $2^x = 10$.

Problem 6

Solve the inequality $(3x - 2)/x > 1$.

Problem 7

In triangle ABC, the longest side BC is of length 20 and the altitude from A to BC is of length 12. A rectangle $DEFG$ is inscribed in ABC, with D on AB, E on AC and both F and G on BC. Find the maximum area of $DEFG$.

Problem 8

Prove that the largest triangle which can be inscribed in a circle is equilateral.

Problem 9

Find the locus of a point P such that $AP^2 - BP^2 = d^2$, where A and B are two fixed points and d is a real number.

Problem 10

Prove that the point of intersection of the lines $2y + x + 1 = 0$ and $y - 3x + 4 = 0$ lies on the line $A(2y + x + 1) + B(y - 3x + 4) = 0$, where A and B are real numbers.

Problem 11

Sketch the graph of $(x^2 + y^2)(x^2 + y^2 - 1) = 0$.

Problem 12

Let x take on a set of values which form a geometric progression. Prove that the corresponding values of $y = \log_d x$ form an arithmetic progression, where d is a positive real number not equal to 1.

Problem 13

Let $f(x) = \sqrt{4 - x^2}$. Consider $(f(x) - f(1))/(x - 1)$.
(a) Interpret this expression geometrically as x takes on values successively nearer to 1.
(b) Determine its limiting value.

Problem 14

Let $S_n = 1 + 1/2 + 1/4 + 1/8 + \cdots + 1/20^n$. What is the least value of n such that $S_n > 127/64$?

Problem 15

Convert the recurring decimal 0.147147147... into a fraction.

Problem 16

The equation $ax^2 + bx + c = 0$ has two real roots. Prove that, if a is very small while b and c are of "moderate" size, then one of the roots is close to $-c/b$ while the other is very large numerically.

Problem 17

Prove that

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

for all non-negative integers n.

Problem 18

Prove that

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^{n-1}\binom{n}{n-1}$
$+(-1)^n\binom{n}{n} = 0$ for all positive integers n.

Problem 19

Nine students are to be assigned to three rooms, three students to a room.
(a) In how many ways can this be done?
(b) What if two particular students refuse to be assigned to the same room?

Problem 20

Fifteen passengers rode on a railway line which leads to 25 towns. If no two persons get off at the same town, what is the total number of ways in which they can get off?

Problem 21

Solve the equation $\sin x = \sin 2x$.

Problem 22

Solve the equation $\tan x = \cot 2x$.

Problem 23

In triangle ABC, $BC = a > b = AC$ and the difference between the angles A and B is x. Given a, b and x, construct ABC with ruler and compass.

Year 1962

Problem 1

Determine d such that when $-1/2 < x < 1/2$,
$((1 - 4x^2)^{1/2} - (x/2)(1 - 4x^2)^{-1/2}(-8x))/(1 - 4x^2)$
$= 1/(1 - 4x^2)^d$.

Problem 2

(a) Find numbers a and b such that
$a(3x + 5) + b(2x + 3) = 12x + 19$ for every x.
(b) Determine numbers A and B so that for every x except $-3/2$ and $-5/3$, $(12x + 19)/(3x + 5)(2x + 3)$
$= A/(2x + 3) + B/(3x + 5)$.

Problem 3

Solve the equation $x^2 - x - 20 = 0$.

Problem 4

Solve the equation
$1/(x - 2) + 1/(x + 2) = 4/(x^2 - 4)$.

Problem 5

Solve the equation $\sqrt{4 - 3x} - x = 12$.

Problem 6

For any two real numbers x and y, each greater than 1, prove that $\log_y x = 1/\log_x y$.

Problem 7

Solve the equation $2^x - 4(2^{-x}) + 3 = 0$.

Problem 8

Using only a T-square, construct the centre of a given circle.

Problem 9

Solve the equation $\cos 2x = \cos x$.

Problem 10

Solve the equation $\sin 5x - \sin x = \cos 3x$.

Problem 11

Solve the equation $\sec x - 2 \cos x - \tan x = 0$.

Problem 12

(a) Let $S_n = 1/1(2) + 1/2(3) + 1/3(4)$
$+ \cdots + 1/n(n + 1)$. Express S_n as a simple fraction in terms of n.
(b) Find the number n such that S_n in (a) is greater than 100/101.

Problem 13

Let m and n be the roots of the equation $ax^2 + bx + c = 0$. Find a quadratic equation with coefficients expressed in terms of a, b and c which has $m + 2$ and $n + 2$ as roots.

Problem 14

Let $T_n = 1 + 2r + 3r^2 + \cdots + nr^{n-1}$ where r is a real number not equal to 1. Prove that
$T_n = (1 - r^n)/(1 - r)^2 - nr^n/(1 - r)$.

Problem 15

An after-dinner speaker anticipates delivering 35 speeches during the next five years. So as not to become bored with his jokes, he decides to tell exactly three jokes in every speech, and in no two speeches to tell exactly the same three jokes.
(a) What is the minimum number of jokes that will accomplish this?
(b) What is the minimum number if he decides never to tell the same joke twice?

Problem 16

In a triangle ABC, side BC and the angles B and C are known. Prove that the length of the altitude from A to BC is $BC/(\cot B + \cot C)$.

Problem 17

Prove that $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$, where A, B and C are the angles of any triangle.

Problem 18

Prove that, for any two positive numbers whose sum is 8, their product is a maximum when they are equal.

Problem 19

Prove that, among all rectangles with a given area, the square has the least perimeter.

Problem 20

The inscribed circle of triangle ABC has centre O and touches BC at P. One of the escribed circles of triangle ABC has centre O' and touches BC at Q.
(a) Prove that $BP = CQ$.
(b) Prove that the points B, C, O and O' lie on a circle.

Problem 21

(a) For the quartic equation $Ax^4 + Bx^3 + Cx^2 + Bx + A = 0$, prove that, if r is a root, so also is $1/r$.
(b) If $t = x + 1/x$, express $x^2 + 1/x^2$ in terms of t.
(c) Solve the equation $x^4 - x^3 - 10x^2 - x + 1 = 0$ and verify that its roots occur in pairs as indicated in (a).

Problem 22

Let ABC be any triangle.
(a) Prove that $\sin A + \sin B + \sin C = 2 \cos(C/2)(\cos((A - B)/2) + \sin (C/2))$.
(b) For a fixed value of C, what is the relation between the angles A and B in order that $\sin A + \sin B + \sin C$ should have its largest value?
(c) Prove that the largest value of $\sin A + \sin B + \sin C$ is $3\sqrt{3}/2$.

Problem 23

Let S be a finite set of points in the plane. There will be a smallest distance d between some pair of them (which may, of course, occur between several pairs). Prove that, for any point P of S, there cannot be more than six other points of S whose distance from P is d.

Year 1963

Problem 1

Express $1/(x - 1/x) - 1/(x + 1/x) - 2x/(x^2 - 1/x^2)$ as a simple fraction.

Problem 2

Express $a - 2ax + 4ax^2 - 8ax^3/(1 + 2x)$ as a simple fraction.

Problem 3

Express $(2x/(1 - x^2))/(2 + 2x^2/(1 - x^2))$ as a simple fraction.

Problem 4

Simplify $(2 + 5x)^2 + (5 - 2x)^2 - 13x^2$.

Problem 5

Express $a/(2 - \sqrt{3})^2 + b/(3 + 2\sqrt{2})^2$ as a simple fraction.

Problem 6

Solve the system of equations $x + y = 80$ and $x^2 + y^2 = 3250$.

Problem 7

If the sum of two numbers is 80, find the largest possible value of their product.

Problem 8

If the sum of two positive numbers is equal to N, what is the smallest value of the sum of their reciprocals?

Problem 9

Two cars, A and B, cover a distance of 200 km, each at constant speed, but with B travelling at a constant speed 25/6 kph faster and hence requiring 12 minutes less time. Find the speeds of the cars.

Problem 10

Solve the equation $\sqrt{8x} - \sqrt{x + 1} = 1$.

Problem 11

Two straight lines $y = 3x + 7$ and $y = 5x - 4$ meet in one point. Prove that all other lines passing through the same point have equations of the form $y = k(3x + 7) + (1 - k)(5x - 4)$.

Problem 12

A hall has dimensions 10 metres by 20 metres. At one end, in the middle, one metre from the floor, is a fly. At the other end, in the middle, one metre from the ceiling, is a spider. The spider, being hungry, wishes to take the shortest route possible to crawl from where it is to where the fly is. What is the length of the shortest route?

Problem 13

(a) Prove that the sum of the first n positive integers is $n(n + 1)/2$.
(b) Prove that the sum of the cubes of the first n positive integers is $n^2(n + 1)^2/4$.
(c) Simplify $n^3(n + 1)^3 - n^3 (n - 1)^3$.
(d) Prove that the sum of the fifth powers of the first n positive integers is $n^3(n + 1)^3/6 - n^2 (n + 1)^2/12$.

Problem 14

Prove that the perpendicular distance of any point (a,b) such that $3a + 4b > 10$ from $3x + 4y = 10$ is $(3a + 4b - 10)/5$.

Problem 15

Let S be the sum of the first n terms of the series $a + 4ax + 9ax^2 + \cdots + n^2ax^{n-1}$.
(a) Prove that $S - xS = a + 3ax + 5ax^2 + \cdots + (2n - 1)ax^{n-1} - n^2ax^n$.
(b) Express $(1 - x)^2S$ as a simple fraction.

Problem 16

From a point P on the circumference of a circle, a distance PT of 10 metres is laid out along the tangent. The shortest distance from T to the circle is 5 metres. A straight line is drawn through T cutting the circle at X and Y. The length of TX is 15/2 metres.
(a) Find the radius of the circle.
(b) Find the length of XY.

Problem 17

The equation $x^4 - 19x^2 + 20x - 4 = 0$ may be rewritten as $(x^2 + sx + p)(x^2 - sx + q) = 0$ for constants s, p and q.
(a) Prove that $(20/s)^2 = (s^2 - 19)^2 + 16$.
(b) Verify that $s = 4$ satisfies the equation in (a).
(c) Solve the original quartic equation.

Problem 18

The sides of a triangle a, b and c are related by $c^2(a + b) = a^3 + b^3$.
(a) Prove that one angle is exactly 60°.
(b) Express the area of the triangle in terms of a and b.

Problem 19

An isosceles triangle has an interior angle of 36° between two sides, each one metre long. One of the angles at the base is bisected by a line from that vertex to the opposite side. This line is x metres long.
(a) Prove that the base is also x metres long.
(b) Prove that one of the segments of the divided side is also x metres long.
(c) Prove that $x + x^2 = 1$.

Problem 20

(a) Prove that $\sin 2x = 2 \sin x \cos x$.
(b) Prove that $4 \sin 18° \sin 54° = 1$.
(c) Prove that $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$.
(d) Prove that $\sin 54° - \sin 18° = 1/2$.
(e) Prove that $\sin 18° = (\sqrt{5} - 1)/4$.

Problem 21

(a) Prove that $2/(\tan 2x) = 1/(\tan x) - \tan x$.
(b) Express $1/\tan x - \tan x - 2 \tan 2x - 4 \tan 4x$ as a simple fraction.

Problem 22

Prove that $2/(\sin 2x) = 1/(\tan x) + \tan x$.

Problem 23

(a) Prove that $(\sin(n + 1)x - \sin nx)/(\cos(n + 1)x + \cos nx) = \tan(x/2)$.
(b) Prove that $1 + 2 \cos x + 2 \cos 2x + 2 \cos 3x + \cos 4x = (\sin 4x)/(\tan(x/2))$.

Year 1964

Problem 1

If the distance s metres that a bomb falls vertically in t seconds is given by the formula $s = 16t^2/(1 + 3t/50)$, how many seconds are required for a bomb released at an altitude of 20000 metres to reach ground level?

Problem 2

(a) Find three consecutive positive even integers such that the square of the largest is equal to the sum of the squares of the other two.
(b) Prove that this is impossible for consecutive positive odd integers.

Problem 3

A man has 15878 equilateral triangular pieces of mosaic, all of side length one cm. He constructs the largest possible mosaic in the shape of an equilateral triangle.
(a) What is the side length of the mosaic?
(b) How many pieces will he have left over?

Problem 4

A, B, C and D are four points in a plane. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively.
(a) Prove that $PQRS$ is a parallelogram.
(b) How is this result modified if the four points A, B, C and D are not all in one plane?

Problem 5

Solve the equation $9(10)^{2x} - 6(10)^x + 1 = 0$.

Problem 6

Explain how logarithms may be used to compute the fifth root of a real number.

Problem 7

Let p, q and r be three positive numbers such that $p + q + r = 12$.
(a) If p is held fixed while q and r are allowed to vary, prove that product pqr is greatest when $q = r$.
(b) What is the greatest possible value of pqr?

Problem 8

Find the value of the constant k so that the equation $kx^2 + 6x - 4 = 0$ has two equal roots.

Problem 9

Find a quadratic equation whose roots are the reciprocals of the roots of $x^2 + x + 4 = 0$.

Problem 10

Prove that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$, where A, B and C are the angles of any triangle.

Problem 11

Divide 100 loaves among five men so that the shares received shall be in arithmetic progression, and so that one-seventh of the sum of the largest three shares shall be equal to the sum of the smallest two shares. Individual loaves may be subdivided if necessary. What are the shares of the five men?

Problem 12

Assume that $\sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \cdots)})})})}$, where the number of 2's and radical signs are infinite, is a meaningful expression and has a definite real value. Prove that this value is 2.

Problem 13

(a) Find the square root of the complex number $4 - 6\sqrt{5}i$.
(b) Represent graphically the given number and also the square roots.

Problem 14

A and B are two points on a circle which is divided into parts by the chord AB. The circle is then folded along AB so that both parts of the circle are on the same side of AB, and a line is drawn from B to cut the two circular arcs at P and Q, respectively. Prove that triangle PAQ is isosceles.

Problem 15

Prove that $1 - 1/2 + 1/4 - \cdots + (-1/2)^{n-1} = 2(1 - (-1/2)^n)/3$.

Problem 16

Two cyclists are 20 km apart on a straight road and, at the same moment, begin cycling towards each other at a speed of 10 kph. At the instant they begin moving, a fly which can travel at 20 kph leaves the nose of one of them and flies towards the other. As soon as it arrives at the second nose, it turns around and flies back to the first, continuing to go backwards and forwards until the cyclists meet. How far has the fly flown?

Problem 17

Use Newton's Binomial Theorem to compute the cube root of 63.9 to four decimal places.

Problem 18

Find the fifth term in the expansion of $(a^2 - 2b^2)^{3/2}$.

Problem 19

Four cards are drawn at random from an ordinary deck of 52. What is the probability that exactly three of these will be clubs?

Year 1965

Problem 1

Let $d = \sqrt{2}$. Find the value of $(d^d)^d$.

Problem 2

Prove that, for any positive number n that satisfies the equation $x^y = y^x$, $x = (1 + 1/n)^{n+1}$ and $y = (1 + 1/n)^n$.

Problem 3

Prove that $1 + 2x/(x^2 - x + 1) = (1 - 2/(x^3 + 1))(1 + 2/(x - 1))$ if $x \neq 1$ and $x^3 + 1 \neq 0$.

Problem 4

Find constants a and b such that the equation $\sqrt{(x + 1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} = 1$ may be rewritten in the form $(x/a)^2 - (y/b)^2 = 1$.

Problem 5

Solve the equation $2x - 5 = \sqrt{2x + 1}$.

Problem 6

Find the sum of the squares of the roots of the equation $x^3 - px = q$ in terms of p and q.

Problem 7

Solve the equation $x^3 - 13x = 12$.

Problem 8

Two parallel walls at some distance apart are perpendicular to the ground level and two ladders are placed one against each wall so that the other ends touch the bases of the opposite walls. The ladders touch each other at some point between the wall, h metres above the ground. The top of the ladder of length m metres is at a height of a metres above the ground. The height of the top of the ladder of length n metres is b metres above the ground.

(a) Prove that $h = ab/(a + b)$.

(b) Find an equation involving a, m, n and h but not b.

Problem 9

(a) Prove that there do not exist positive integers m and n such that $10^m = 2^n$.

(b) Prove that $\log_{10}2$ is not a rational number.

Problem 10

Prove that $\binom{2n}{n}$ is an even number where n is any positive integer.

Problem 11

Prove that $\binom{2n}{n-1} + \binom{2n}{n+1} = 2n\binom{2n}{n}/(n + 1)$.

Problem 12

A motorized column is advancing over flat country at the rate of 15 kph. It is one km long. A dispatch rider is sent from the rear to the front on a motorcycle travelling at a constant speed. He returns immediately at the same speed and his total time is 3 minutes. How fast is he going?

Problem 13

Through a point R outside a circle with centre O and radius r, a line is drawn cutting the circle in two distinct points P and Q. Prove that $RP \cdot RQ = OR^2 - r^2$.

Problem 14

Prove that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

Problem 15

(a) Prove that $1/n^2 - 1/(n + 1)^2 = (2n + 1)/n^2(n + 1)^2$.

(b) Prove that $3/1^2 2^2 + 5/2^2/3^2 + \cdots + (2n - 1)/n^2(n - 1)^2 = 1 - 1/n^2$.

(c) Express $1/n(n^2 - 1)$ as a sum of fractions with simpler denominators.

(d) Express $1/2(2^2 - 1) + 1/3(3^2 - 1) + 1/4(4^2 - 1) + \cdots + 1/n(n^2 - 1)$ as a simple fraction in terms of n.

Problem 16

The radius of the base of a right circular cone is 2 metres and the slant height from the edge of the base to the vertex is 6 metres. Find the total surface area of the cone.

Problem 17

The radius of the base of a right circular cone is 2 metres and the slant height from the edge of the base to the vertex is 6 metres. From a point A on the edge of the base one may proceed to a point B halfway up the cone towards the vertex. Consider the point C directly opposite B, also halfway up the cone. Find the shortest distance from A to C on the surface of the cone.

Year 1966

Problem 1

Factor $x^5 + x^4 + x^3 + x^2 + x + 1$ as far as possible into polynomials with integral coefficients.

Problem 2

Prove that the roots of the equation $bx^3 + a^2x^2 + a^2x + b = 0$ are in geometric progression.

Problem 3

If the product of two positive numbers is 36, prove that their sum is at least 12.

Problem 4

ABC is a triangle with $AB = AC$. D is a point on AB extended, and E is a point on CA (or CA extended) such that the angles BEC and BDC are equal. Prove that $BE = CD$.

Problem 5

Solve the inequality $1/(x - 1) + 1/(x + 1) > 1/2$.

Problem 6

Solve the system of equations $x + y + z = 7$, $3x + 2y - z = 3$ and $x^2 + y^2 + z^2 = 21$.

Problem 7

There are ten guests at a party. Assume that all acquaintances are mutual and that no one is considered an acquaintance of himself or herself. Prove that two of the guests are acquainted with the same number of guests at the party.

Problem 8

Indicate the region in the xy-plane for which $x + y$ takes values between -2 and 2 inclusive.

Problem 9

The line $y = 3x + b$ meets the parabola $2y = x^2 + 2x$ in two distinct points P and Q.
(a) What restriction does this place on b?
(b) Prove that the x coordinate of the midpoint of PQ is independent of b.

Problem 10

Prove that the area of a triangle inscribed in a parallelogram is at most one-half the area of the parallelogram.

Problem 11

Let S_n denote the sum of the first n terms of the series $1 + 2/2 + 3/4 + \cdots + n/2^{n-1} + \cdots$.
(a) Calculate S_5.
(b) Prove that $4 - S_n = (n + 2)(1/2)^{n-1}$.
(c) Find the "sum to infinity" of this series.

Problem 12

P is a point on the side CD of a parallelogram $ABCD$. AP and BC, extended if necessary, meet at Q. AD and BP, extended if necessary, meet at R. Prove that $1/BQ + 1/AR = 1/AD$.

Problem 13

Let $F(x,y)$ be a function of x and y such that for any x and y, (1) $F(x,y) = F(y,x)$;
(2) $F(x,y) = F(x,x - y)$. Prove that $F(x,y) = F(-x,-y)$.

Problem 14

A triangle has sides 20 cm, 20 cm and 5 cm. Find the lengths of its interior angle bisectors.

Problem 15

(a) Prove that two consecutive integers have no common divisors other than ± 1.
(b) Suppose $n + 1$ positive integers are taken, all different and none greater than $2n$. Prove that at least two of them have no common divisors other than ± 1.

Problem 16

(a) Express $x/(1 - x) - x/(1 + x)$ as a simple fraction.
(b) Prove that $x/(1 - x) = x/(1 + x) + 2x^2/(1 + x^2) + 4x^4/(1 + x^4) + \cdots$ for $-1 < x < 1$.

Problem 17

Prove that, however large the positive number N may be, one can always find a number whose logarithm to base 10 is greater than N.

Problem 18

Prove that $\sin A + \sin B + \sin C = 4\cos(A/2)\cos(B/2)\cos(C/2)$, where A, B and C are the angles of any triangle.

Appendix III: A Selected Bibliography on Popular Mathematics

There are so many good books on popular mathematics that it is difficult to list them all. This selected bibliography is limited to books which are in English and are still in print as far as is known, and which have been used in the editor's "Saturday Mathematical Activities, Recreations and Tutorials" (SMART) program of enrichment for students of Grades 4 to 9 in the greater Edmonton area. A typical problem from each book is given.

A. Martin Gardner's Scientific American Series

For over 20 years, Martin Gardner had a monthly column in *Scientific American* called "Mathematical Games." Despite the title, it covered a wide range of topics, from the very elementary to the frontiers of current research but always with a delightful element of play. It was written in a lively style indicative of the strong literary background of Martin Gardner. When the subject under discussion was difficult, he took considerable effort to smooth out the path and guide the reader along gently. The column had and still has enormous influence in the mathematics community of North America and beyond.

Unfortunately, Martin Gardner has retired. His column was replaced briefly by "Metamagical Themas" (an anagram of "Mathematical Games"). A. K. Dewdney's "Computer Recreations" now occupies that distinguished spot in *Scientific American*.

Fortunately, anthologies of Martin Gardner's columns have appeared regularly. To date, 12 volumes are in print, with enough columns left over for at least five more books. If a school library can afford only one set of books on popular mathematics, this is the one!

The Scientific American Book of Mathematical Puzzles and Diversions, 1959, Simon & Schuster.

Topics covered are hexaflexagons, magic with a matrix, ticktacktoe, probability paradoxes, the icosian game and the Tower of Hanoi, curious topological models, the game of hex, Sam Loyd: America's greatest puzzlist, mathematical card tricks, memorizing numbers, polyominoes, fallacies, nim and tac tix, left or right, as well as two collections of short problems. Here's an example.

Problem 1

An old riddle runs as follows. An explorer walks one mile due south, turns and walks one mile due east, turns again and walks one mile due north. He finds himself back where he started. He shoots a bear. What color is the bear? The time-honored answer is: "white," because the explorer must have started at the North Pole. But not long ago someone discovered that the North Pole is not the only starting point that satisfies the given conditions! Can you think of any other spot on the globe from which one can walk a mile south, a mile east, a mile north and find himself back at his original location?

The 2nd Scientific American Book of Mathematical Puzzles and Diversions, 1961, Simon & Schuster.

Topics covered are the five Platonic solids, tetraflexagons, Henry Ernest Dudeney: England's greatest puzzlist, digital roots, the soma cube, recreational topology, phi—the golden ratio, the monkey and the coconuts, mazes, recreational logic, magic squares, James Hugh Rilet Shows Inc., eleusis: the induction game, origami, squaring the square, mechanical puzzles, probability and ambiguity, as well as two collections of short problems.

Problem 2

Two missiles speed directly toward each other, one at 9000 miles per hour and the other at 21000 miles per hour. They start 1317 miles apart. Without using pencil and paper, calculate how far apart they are one minute before they collide.

Martin Gardner's New Mathematical Diversions from Scientific American, 1966, Simon & Schuster.

Topics covered are the binary system, group theory and braids, the games and puzzles of Lewis Carroll, paper cutting, board games, packing spheres, the transcendental number pi, Victor Eigen: mathemagician, the four-color map theorem, Mr. Apollinax visits New York, polyominoes and fault-free rectangles, Euler's spoilers: the discovery of an order-10 Graeco-Latin square, the ellipse, the 24 color-squares and the 30-color cubes, H. S. M. Coxeter, bridg-it and other games, the calculus of finite differences, as well as three collections of short problems.

Problem 3

One morning, exactly at sunrise, a Buddhist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit. The monk ascended the path at varying rates of speed, stopping many times along the way to rest and eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed. Prove that there is a spot along the path that the monk occupies on both trips at precisely the same time of day.

The Magic Numbers of Dr. Matrix, 1985, Prometheus Books.

This is different from any other book in this series. It centres around a mystic character, Dr. Irving Joshua Matrix, a professional numerologist. Dr. Matrix columns appear to be Martin Gardner's own favorite, as he returned to them periodically until the good doctor's untimely demise in the last episode.

Problem 4

```
    FORTY
      TEN
 +    TEN
 ________
    SIXTY
```

"Each letter in that addition problem stands for a different digit," Dr. Matrix explained. "There's only one solution, but it takes a bit of brain work to find it."

The Unexpected Hanging and Other Mathematical Diversions, 1969, Simon & Schuster.

Topics covered are the paradox of the unexpected hanging, knots and Borromean rings, the transcendental number *e*, geometric dissections, Scarne on gambling, the church of the fourth dimension, a matchbox game-learning machine, spirals, rotations and reflections, peg solitaire, flatlands, Chicago Magic Convention, tests of divisibility, the eight queens and other chessboard diversions, a loop of string, curves of constant width, reptiles: replicating figures on the plane, as well as three collections of short problems.

Problem 5

Six Hollywood stars form a social group that has very special characteristics. Every two stars in the group either mutually love each other or mutually hate each other. There is no set of three individuals who mutually love one another. Prove that there is at least one set of three individuals who mutually hate one another.

Martin Gardner's 6th Book of Mathematical Diversions from Scientific American, 1983, University of Chicago Press.

Topics covered are the helix, Klein bottles and other surfaces, combinatorial theory, bouncing balls in polygons are polyhedra, four unusual board games, sliding-block puzzles, parity checks, patterns and primes, graph theory, the ternary system, the cycloid: Helen of geometry, mathematical magic tricks, word play, the Pythagoras Theorem, limits of infinite series, polyiamonds, tetrahedra, the lattice of integers, infinite regress, O'Gara the mathematical mailman, extraterrestrial communication, as well as three collections of short problems.

Problem 6

An oil well being drilled in flat prairie country struck pay sand at an underground spot exactly 21000 feet from one corner of a rectangular plot of farmland, 18000 feet from the opposite corner and 6000 feet from a third corner. How far is the underground spot from the fourth corner?

Mathematical Carnival, 1977, Vintage Books.

Topics covered are sprouts and Brussels sprouts, penny puzzles, aleph-null and aleph-one, hypercubes, magic stars and polyhedra, calculating prodigies, tricks of lightning calculators, the art of M. C. Escher, card shuffles, Mrs. Perkin's quilt and other square-packing problems, the numerology of Dr. Fliess, random numbers, the rising hourglass and other physics puzzles, Pascal's Triangle, jam, hot and other games, cooks and quibble-cooks, Piet Hein's super-ellipse, how to trisect an angle, as well as one collection of short problems.

Problem 7

An infinity of non-touching points lies inside a closed curve. Assume that a million of those points are selected at random. Will it always be possible to place a straight line on the plane so that it cuts across the curve, misses every point in the set of a million and divides the set exactly in half so that 500000 points lie on each side of the line? The answer is yes; prove it.

Mathematical Magic Show, 1978, Vintage Books

Topics covered are nothing, game theory, guess-it and foxholes, factorial oddities, double acrostics, playing cards, finger arithmetic, Mobius bands, polyhexes and polyaboloes, perfect, amicable and sociable, polyominoes and rectification, knights of the square table, colored triangles and cubes, trees, dice, everything, as well as three collections of short problems.

Problem 8

A telephone call interrupts a man after he has dealt about half of the cards in a bridge game. When he returns to the table, no one can remember where he had dealt the last card. Without learning the number of cards in any of the four partly dealt hands, or the number of cards yet to be dealt, how can he continue to deal accurately, everyone getting exactly the same cards he would have had if the deal had not been interrupted?

Mathematical Circus, 1981, Vintage Books

Topics covered are optical illusions, matches, spheres and hyperspheres, patterns of induction, elegant triangles, random walks and gambling, random walks on the plane and in space, Boolean algebra, can machines think?, cyclic numbers, dominoes, Fibonacci and Lucas numbers, simplicity, solar system oddities, Mascheroni constructions, the abacus, palindromes—words and numbers, dollar bills, as well as two collections of short problems.

Problem 9

You have six weights. One pair is red, one pair white, one pair blue. In each pair one weight is a trifle heavier than the other but otherwise appears to be exactly like its mate. The three heavier weights (one of each color) all weigh the same. This is also true of the three lighter weights. In two separate weighings on a balance scale, how can you identify which is the heavier weight of each pair?

Wheels, Life and Other Mathematical Amusements, 1983, W. H. Freeman.

Topics covered are wheels, Diophantine analysis and Fermat's Last Theorem, alephs and supertasks, nontransitive dice and other probability paradoxes, geometrical fallacies, the combinatorics of paper folding, ticktacktoe games, plaiting polyhedra, the game of Halma, advertising premiums, Salmon on Austin's dog, nim and hackenbush, Golomb's graceful graphs, chess tasks, slither, $3x + 1$ and other curious questions, mathematical tricks with cards, the game of life, as well as three collections of short problems.

Problem 10

Make a statement about n that is true for, and only true for, all values of n less than one million.

Knotted Doughnuts and Other Mathematical Entertainments, 1986, W. H. Freeman.

Topics covered are coincidence, the binary Gray code, polycubes, Bacon's cipher, doughnuts: linked and knotted, Napier's bones, Napier's abacus, sim, chomp and racetrack, elevators, crossing numbers, point sets on the sphere, Newcomb's paradox, look-see proofs, worm paths, Waring's problems, cram, bynum and quadraphage, the I Ching, the Laffer curve, as well as two collections of short problems.

Problem 11

My wife and I recently attended a party at which there were four other couples. Various handshakes took place. No one shook hands with himself or herself or with his or her spouse, and no one shook hands with the same person more than once. After all the handshakes were over, I asked each person, including my wife, how many hands he or she had shaken. To my surprise each gave a different answer. How many hands did my wife shake?

Time Travel and Other Mathematical Bewilderments, 1988, W. H. Freeman.

Topics covered are time travel, hexes and stars, tangrams, nontransitive paradoxes, combinatorial card problems, melody-making machines, anamorphic art, six sensational discoveries, the Csaszar polyhedron, dodgem and other simple games, tiling with convex polygons, tiling with polyominoes, polyiamonds and polyhexes, curious maps, magic squares and cubes, block packing, induction and probability, Catalan numbers, fun with a pocket calculator, tree-planting problems, as well as two collections of short problems.

Problem 12

A cake has been baked in the form of a rectangular parallelepiped with a square base. Assume that the square cake is frosted on the top and four sides and that the frosting's thickness is negligible. We want to cut the cake into seven pieces so that each piece has the same volume and the same area of frosting. The slicing is conventional. Seen from above, the cuts are like spokes radiating from the square's centre, and each cutting is perpendicular to the cake's base. How can we locate the required seven points on the perimeter of the cake's top?

B. Raymond Smullyan's Logic Series

While this series deals only with logic, it more than compensates by its tremendous depth. While readers may find parts of the books in this series difficult, they will not find them difficult to read. Each book takes the form of a series of logic puzzles, presented in very attractive settings. The reader can reasonably expect to have success with the earlier ones. As confidence increases, the reader will discover the skill of the author in paving a smooth path towards some very important results in logic and mathematics, particularly those associated with the name Kurt Gödel.

What is the Name of This Book?, 1978, Prentice-Hall.

A distinguishing feature of Raymond Smullyan's logic puzzles is that not all of the information provided is to be taken at its face value. This is exemplified by his favorite characters, the knights who always tell the truth and the knaves who always lie. Before one can utilize a statement made by one of them, it is important to know if it is made by a knight or a knave. Very often the clues are contained in the statement itself.

The knights and knaves made their debut in this wonderful book (what is its name again?), along with other denizens from Alice (of Wonderland fame) to Count Dracula. In solving numerous intriguing logic puzzles, the reader gets an enjoyable lesson in propositional logic, an introduction to some logical paradoxes and curiosities and a proof of a form of Gödel's famous Incompleteness Theorem.

Problem 13

Each of A, B and C is either a knight or a knave. A stranger asks A, "How many knights are among you?" A answers indistinctly. So the stranger asks B, "What did A say?" B replies, "A said that there is exactly one knight among us." Then C says, "Don't believe B; he is lying!" What are B and C?

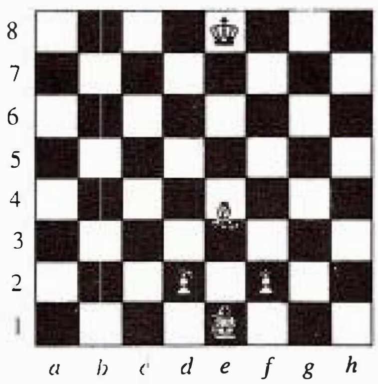
The Chess Mysteries of Sherlock Holmes, 1979, Alfred A. Knopf.

This and the next volume are quite different from the other books in the series in that logic is exercised over the chessboard. The reader needs to know the rules of chess, but being a good player is not essential. In fact, this is more often a handicap, as the moves that are made in the games in these books, though perfectly legal, are hardly what one would describe as good moves. The object is not to win but to deduce the past history of a game, based on the current position and possibly some additional information. In this volume, Watson serves as the narrator, with Sherlock Holmes as the chessboard detective.

Problem 14

A white bishop was placed equally between the squares *e*3 and *e*4. Thinking this was an oversight, I was about to move it when Holmes stopped me. "No, no, Watson. That's precisely the problem! On which square, *e*3 or *e*4, stands the bishop, given that in this game, no piece or pawn has ever moved from a white square to a black square, nor from a black square to a white square?"

Black 1



White 4

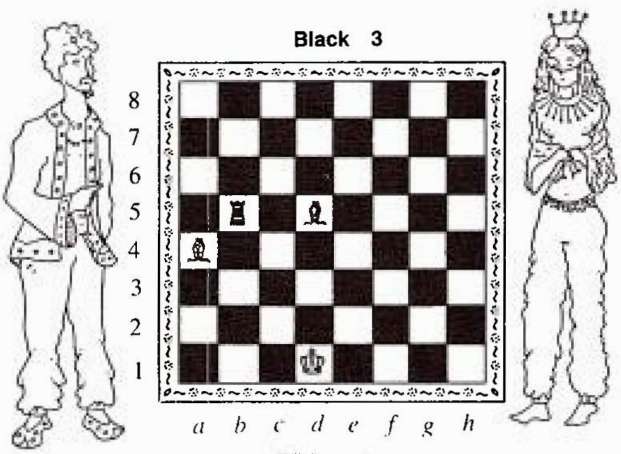
Problem 14

The Chess Mysteries of the Arabian Knights, 1981, Alfred A. Knopf.

In this volume, the characters are the White King Haroun Al Rashid and his entourage and the opposing camp headed by the black King Kazir. The setting is that of the "Tales of the Arabian Nights." The style of the narrative and the charming illustrations by Greer Fitting add to the authenticity.

Problem 15

Haroun Al Rashid—Ruler of the Faithful—had gathered from sorcerers all over the world many secrets of magic. One of his favorite tricks was the art of invisibility. So here Haroun is, standing in broad daylight, on one of the 64 squares of the enchanted chess kingdom. But nobody can see him for the simple reason that he is invisible. On what square does he stand?



White 2
(1 invisible)

Problem 15

The Lady or the Tiger?, 1982, Alfred A. Knopf.

Although the knights and knaves make only a brief appearance in the first part of this book, the other characters are unmistakably knight-like or knave-like, though with fascinating variations. The reader is also introduced to meta-puzzles, or puzzles about puzzles. The second half of the book is a novelty called a "mathematical novel." The step-by-step unveiling of the "Mystery of the Monte Carlo Lock" is an absorbing study in combinatorial logic.

Problem 16

Each of two rooms contained either a lady or a tiger. A sign on the door of the first room read, "In this room there is a lady, and in the other room there is a tiger." A sign on the door of the second room read, "In one of these rooms there is a lady, and in one of these rooms there is a tiger." One of the signs was true but the other one was false. It was possible that both rooms contained ladies or both rooms contained tigers. Which room should be chosen in order to get a lady?

Alice in Puzzle-Land, 1982, William Morrow.

This book celebrates the 150th anniversary of the birth of Lewis Carroll, of whom Raymond Smullyan has been described as a modern version. In this volume, Alice and other Carrollian characters are reunited to entertain the reader with logic and meta-logic puzzles. There are also some elementary mathematical problems. The illustrations by Greer Fitting are simply gorgeous.

Problem 17

Half the creatures are totally mad, meaning that everything true they believe to be false and everything false they believe to be true. The other half are totally sane, meaning that everything true they know to be true and everything false they know to be false. "There's the Cook and the Cheshire Cat," said the Duchess. "The Cook believes that at least one of the two is mad." What can be deduced about the Cook and the Cheshire Cat?

To Mock a Mockingbird, 1985, Alfred A. Knopf.

The first third of this book consists of the basic knight-knave type of puzzles and more meta-puzzles. The remaining part takes the reader on another tour of the realm of combinatorial logic. This second "mathematical novel" shares several common characters with the "Mystery of the

Monte Carlo Lock'' while adding many more, most of which are birds that can talk.

Problem 18

A certain enchanted forest is inhabited by talking birds. Given any birds A and B, if you call out the name of B to A, then A will respond by calling out the name of some bird to you; this bird we designate AB. Given any birds A and B, there is a bird C such that for every bird x, C's response to x is equal to A's response to B's response to x; in other words, $Cx = A(Bx)$. Prove that for any birds A, B and C, there is a bird D such that for every bird x, $Dx = A\ (B(Cx))$.

Forever Undecided, 1987, Alfred A. Knopf.

This is the most challenging yet of Raymond Smullyan's books on logic puzzles. After a review of propositional logic, the reader is gradually introduced to the subject of modal logic, where the principal notions are that of a proposition's being possibly true as opposed to being necessarily true.

Problem 19

Two prizes are offered. If you make a true statement, I will give you at least one of the two prizes and possibly both. If you make a false statement, you get no prizes. Suppose you are ambitious and wish to win both prizes. What statement would you make?

C. Oxford University Press Series on Recreations in Mathematics

This series is edited by David Singmaster of the Polytechnic of the South Bank in London. He is well known for his writing on popular mathematics and is the leading expert in the history of the subject. The four volumes published so far are new, but the series may contain translations and reprints of classic works.

Mathematical Byways: In Ayling, Beeling and Ceiling, by Hugh ApSimon, 1984.

This book contains eleven chapters, each built around a central problem, with solutions and generalizations. Each problem is in the form of an event in the three villages named in the title, though there is occasional wandering off to Dealing and beyond. The titles of the problems are ladder-box, meta-ladder-box, complete quadrilateral, bowling (as in cricket) averages, centre-point, counting sheep, transport, alley ladder, counterfeit coins, wrapping a parcel and sheepdog trials.

Problem 20

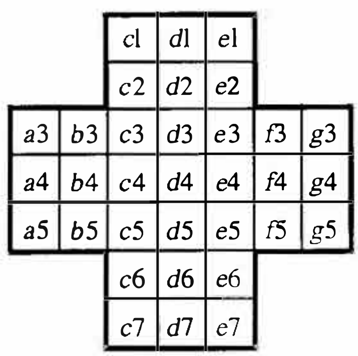
Beeling is rather a quiet village—except on the day of the annual sheep market. On that day, lines of hurdles are erected from the flagpole to each of the corners of the market place (a quadrilateral, not necessarily convex) and along three of its four sides, making three triangular pens of different sizes. Each hurdle is one metre long, there are no overlaps or gaps and no hurdle is bent or broken. Each pen is filled with sheep—one sheep to each square metre. The number of sheep in each pen is equal to the number of hurdles surrounding that pen. What is the area of the Beeling market place?

The Ins and Outs of Peg Solitaire, by John D. Beasley, 1985.

This book deals mainly with peg solitaire on the standard ''English'' board with 33 holes, each of which can hold one man. Each move consists of a jump by one man over an adjacent one onto an empty hole or a continuous sequence of such jumps. The normal objective is to convert a starting configuration into a target position with fewer men (as those jumped over are removed). The book presents a balanced treatment of the mathematical ideas behind the puzzle as well as the actual techniques in solving it.

Problem 21

A marked man is put in hole $d7$. All other holes except $d2$ are occupied by unmarked men. After many moves, only the marked man remains on the board. In which hole will he be?



Problem 21

Rubik's Cube Compendium, by E. Rubik, T. Varga, G. Kéri, G. Marx, T. Vekerdy, 1986.

This book consists of six articles and an informative update by David Singmaster. The lead article is written by the inventor of this mathematical phenomenon, giving some insight into how the idea was originally conceived. The other authors are Rubik's compatriots and their articles deal with the mathematics and the techniques of "cubing." The book has many brightly colored illustrations.

Problem 22

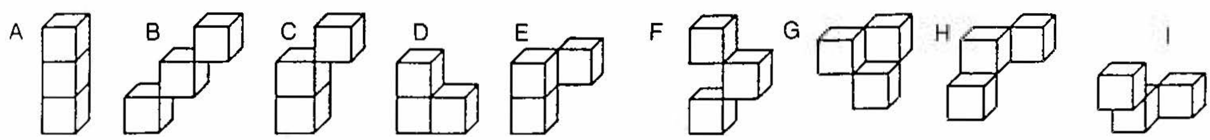
This is Rubik's version of the soma cube. Use the nine pieces (in some of them, certain cubes are only connected along edges) to construct a 3 by 3 by 3 cube. (The pieces are designated by letters for the purpose of the solution in Appendix V.)

Sliding Piece Puzzles, by L. E. Hordern, 1986.

The classic example of a sliding piece puzzle is the 14-15 puzzle of Sam Loyd. This and 271 other puzzles are catalogued by class in this book. In almost all cases, there is enough information for home-made copies to be constructed. There is also a colorful history of the subject.

Problem 23

In a 3 by 3 chessboard, two white knights occupy adjacent corner cells and two black knights occupy the other two corner cells. No captures are allowed. Each knight moves as in a normal game of chess, except that his move is not over until he chooses to stop. The white knights are to trade places with the black knights. Find a seven-move solution.



Problem 22

D. The Dolciani Mathematical Expositions Series of the Mathematical Association of America

The books in this series are selected both for their clear, informal style and stimulating mathematical content. Some are collections of articles and problems while others deal with specific topics. Each has an ample supply of exercises, many with accompanying solutions.

Mathematical Gems I, by Ross Honsberger, 1973.

This book presents 13 articles from elementary combinatorics, number theory and geometry. Topics include combinatorial geometry, recurrence relations, Hamiltonian circuits, perfect numbers, primality testing, Morley's Theorem and a story about a Hungarian prodigy, Louis Posa.

Problem 24

Prove that if you have $n + 1$ distinct positive integers less than or equal to $2n$, some pair of them are relatively prime.

Mathematical Gems II, by Ross Honsberger, 1976.

This book presents 14 articles from elementary combinatorics, number theory and geometry. Topics include combinatorial geometry, box-packing problems, Fibonacci sequence, Hamiltonian circuits, Tutte's Theorem, linear Diophantine equations, the generation of prime numbers, the harmonic series, the isosceles tetrahedron and inversion.

Problem 25

Can 250 copies of a 1 by 1 by 4 block be packed into a 10 by 10 by 10 box?

Mathematical Morsels, by Ross Honsberger, 1978.

This book contains 91 elegant problems and 25 exercises. Most are taken from the problem sections of various journals, in particular, the *American Mathematical Monthly* and the *Mathematics Magazine*.

Problem 26

There are more chess masters in New York City than in the rest of the United States combined. A chess tournament is planned to which all American masters are expected to come. It is agreed that the tournament should be held at the site which minimizes the total inter-city travelling done by the contestants. The New York masters claim that, by this criterion, the site chosen should be their city. The West Coast masters argue that a city at or near the centre of gravity of the players would be better. Where should the tournament be held?

Mathematical Plums, edited by Ross Honsberger, 1979.

This book contains two articles by Honsberger and eight more by others. The respective titles are "Some surprises in probability," "Kepler's conics," "Chromatic graphs," "How to get (at least) a fair share of the cake," "Some remarkable sequences of integers," "Existence out of chaos," "Anomalous cancellation," "A distorted view of geometry," "Convergence, divergence and the computer" and "The Skewes number."

Problem 27

Two red cards and two black cards are shuffled and dealt face down in a row. Two of them are selected at random. What is the probability that they are the same color?

Great Moments in Mathematics (Before 1650), by Howard Eves, 1980.

This is a history of mathematics before 1650 presented in 20 lectures, each highlighted by one of what the author considers a great moment. One such great moment is Euclid's Elements. Although the text is a condensed version of the author's presentation, it retains much of the vitality and smoothness of a gifted lecturer.

Problem 28

Let p and q be positive numbers with $q \leq p/2$. Segments of lengths p and q respectively are given. Construct by Euclidean means segments of lengths r and s respectively where r and s are the roots of the equation $x^2 - px + q^2 = 0$.

Maxima and Minima Without Calculus, by Ivan Niven, 1981.

The central result considered in this book is the Isoperimetric Theorem, which states that, among all figures of fixed perimeter, the circle has the greatest area. It is a problem which is not easy to handle using standard techniques in calculus. After an introduction to inequalities, in particular, the Arithmetic-Mean-Geometric-Mean Inequality and Jensen's Inequality, these elementary tools are applied to various maxima and minima problems. Several related topics are also discussed.

Problem 29

Describe the shortest path across an equilateral triangle to bisect the area.

Great Moments in Mathematics (After 1650), by Howard Eves, 1981.

This is a history of mathematics after 1650 presented in 20 lectures. As in its companion volume, each lecture is centred around a great moment in mathematics. Here, the choice of topics, by the author's own admission, is more difficult. One of the great moments chosen is the invention of differential calculus. Another is the impact of computers and the resolution of the four-color conjecture.

Problem 30

Given a point F and a line l, the locus of a point P equidistant from F and l is a parabola. Construct by Euclidean means the tangent to this parabola at a point P on it.

Map Coloring, Polyhedra and the Four-Color Problem, by D. Barnette, 1983.

The central result considered in this book is the Four-Color Theorem, but only one of eight chapters is devoted to the computer-assisted proof by Appel and Haken. After giving an early history of the problem, the book discusses many related concepts and results, including Euler's Formula, Hamiltonian circuits and convex polyhedra. Map coloring on other surfaces is also considered.

Problem 31

A convex polyhedron is said to be combinatorially regular if each face has the same number of sides and each vertex is the endpoint of the same number of sides. These two numbers may be different. Prove that there are only five types of such polyhedra.

Mathematical Gems III, by Ross Honsberger, 1985.

This book presents 18 articles from elementary combinatorics, number theory and geometry. Topics include combinatorial geometry, generating functions, Fibonacci and Lucas numbers, probability, Ramsay's Theorem, cryptography, Helly's Theorem and a selection of problems from various olympiads.

Problem 32

Given a square grid S containing 49 points in seven rows and seven columns, a subset T consisting of k points is selected. What is the maximum value of k such that no four points of T determine a rectangle with sides parallel to the sides of S?

E. The New Mathematical Library Series of the Mathematical Association of America

This series is written by professional mathematicians with the high school student in mind. The books cover topics which are not usually included in high school curricula, but are nevertheless not too far removed from classroom mathematics. Each volume contains numerous exercises with answers.

Numbers: Rational and Irrational, by Ivan Niven, 1961.

The book begins with a review of elementary number theory and the basic properties of rational numbers. It then goes on to prove that irrational numbers exist, with explicit examples. Algebraic and transcendental numbers are then introduced. There is a discussion of the impossibility of the three classical problems in geometric construction, as well as the problem of approximating irrational numbers by rational numbers.

Problem 33

Prove that sin 10° is irrational.

What Is Calculus About, by W. W. Sawyer, 1961.

The book uses a practical example to introduce the reader to the subject of calculus. From the consideration of distance, velocity and acceleration as functions of time, the concepts and techniques of differential calculus emerge. There is a brief discussion of integral calculus towards the end of the book.

Problem 34

Prove that the volume of a sphere of radius r is given by $4\pi r^3/3$.

An Introduction to Inequalities, by E. F. Beckenbach and R. Bellman, 1961.

The heart of this book lies in the fourth chapter where classical inequalities are discussed. These include the Arithmetic-Mean-Geometric-Mean Inequality, Cauchy's Inequality, Hölder's Inequality, the Triangle Inequality and Minkowski's Inequality. There is a brief discussion of basic properties of inequalities and absolute values in the earlier chapters. The classical inequalities are later applied to maxima and minima problems.

Problem 35

Prove that among all triangles with fixed perimeter, the equilateral triangle has the greatest area.

Geometric Inequalities, by N. D. Kazarinoff, 1961.

The book covers the Arithmetic-Mean-Geometric-Mean Inequality and the Isoperimetric Theorem. It emphasizes the method of reflection in solving maxima and minima problems.

Problem 36

Let P be any point inside triangle ABC. Let P_a, P_b and P_c denote the respective distances from P to BC, CA and AB. Prove that $PA + PB + PC \geq 2(p_a + p_b + p_c)$, with equality if and only if ABC is an equilateral triangle and P is its centre.

The Contest Problem Book I, by C. T. Salkind, 1961.

The book contains a reprint of the American High School Mathematics Examinations from 1950 to 1960. Each paper consists of 50 multiple choice questions except for 1960 where there are only 40 questions. An answer key is provided, followed by complete solutions.

Problem 37

For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, the values of p and q must be, respectively—
(a) $-2, 5$, (b) 5, 25, (c) 10, 20, (d) 6, 25, (e) 14, 25.

The Lore of Large Numbers, by P. J. Davis, 1961.

The book presents a fascinating account of the notations and techniques in computation and approximation. The large numbers serve as a binding theme in the discussion. It starts with fairly basic material and a historical background, which later leads to glimpses of more advanced mathematics.

Problem 38

Prove that 12078521834 is not a square.

Uses of Infinity, by Leo Zippin, 1962.

Starting from an account of the popular notion of infinity, the book leads the reader on to mathematical treatments of sequences and series, limit and convergence, irrational numbers and their approximation, as well as countability and cardinal numbers. The golden ratio is featured as a detailed example.

Problem 39

Prove that the sum of any finite number of terms in the series $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$ is strictly less than 2.

Geometric Transformations I, by I. M. Yaglom, 1962.

The book discusses Euclidean geometry from the transformation point of view. Only distance-preserving transformations or isometries are considered, and these are classified into translations, rotations, reflections and glide reflections.

Problem 40

Suppose that two chords AB and CD are given in a circle together with a point J on the chord CD. Find a point X on the circle such that the chords AX and BX cut off on the chord CD a segment whose midpoint is J.

Continued Fractions, by C. D. Olds, 1963.

The book begins with a definition of continued fractions and shows that finite continued fractions are equivalent to rational numbers. These continued fractions are used to solve linear Diophantine equations. To represent irrational numbers, infinite continued fractions are introduced. It is then shown that periodic continued fractions are equivalent to quadratic irrationals. These continued fractions are then used to solve Pell's equation.

Problem 41

Express 68/77 as the sum of two fractions whose denominators are 7 and 11, respectively.

Graphs and Their Uses, by Oystein Ore, 1963.

This is an excellent introduction to the theory of graphs, covering connected graphs, trees, directed graphs, planar graphs, map coloring and matchings.

Problem 42

Each of four neighbors has connected his house with the other three houses by paths which do not cross. A fifth man builds a house nearby. Prove that he cannot connect his house with all the others by non-intersecting paths.

Hungarian Problem Book I, edited by G. Hajós, G. Neukomm and J. Surányi, 1963.

Hungary has probably the longest and strongest tradition in mathematics contests for high school students. This volume collects the papers from 1894 to 1905 inclusive, with detailed solutions.

Problem 43

Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y.

Hungarian Problem Book II, edited by G. Hajós, G. Neukomm and J. Surányi, 1963.

This volume collects the papers of the Hungarian contests from 1906 to 1928. Contest activities were interrupted in 1919, 1920 and 1921 as an aftermath of World War I.

Problem 44

Prove that the product of four consecutive positive integers cannot be the square of an integer.

Episodes from the Early History of Mathematics, by A. Aaboe, 1964.

The book consists of four chapters, one on Babylonian mathematics and three on Greek mathematics, the latter centred around Euclid, Archimedes and Ptolemy.

Problem 45

Let AOB be any given angle. Let $OA = OB = 1$. Extend OA to C with $OC = OA$ and draw a semicircle with diameter AC and passing through B. Draw a line through B intersecting the semicircle at D and the extension of OC at E such that $DE = 1$. Prove that $\measuredangle BEA$ is equal to one-third of $\measuredangle BOA$.

Groups and Their Graphs, by I. Grossman, 1964.
The book introduces the reader to the abstract algebraic concept of groups via many concrete examples. The graphs of the groups come into the picture when the groups are defined by generators and relations. There is a discussion of the result that there are 17 essentially different wallpaper patterns.

Problem 46

Consider the set $\{1,2,3,\ldots,p-1\}$, p a prime number. Prove that for any element x of the set, there is an element y of the set such that $xy \equiv 1 (\text{mod } p)$.

Mathematics of Choice, by Ivan Niven, 1965.
As its subtitle "How to Count without Counting" suggests, this book deals with counting techniques. Starting from the basic ideas of permutations, combinations and the Binomial Theorem, it leads the reader on to more sophisticated topics such as the Principle of Inclusion-Exclusion, generating functions, recurrence relations, along with many applications. There is a brief discussion of mathematical induction and the Pigeonhole Principle.

Problem 47

At formal conferences of the Supreme Court, each of the nine judges shakes hands with each of the others at the beginning of the session. How many handshakes initiate such a session?

From Pythagoras to Einstein, by K. O. Friedrichs, 1965.
The book discusses Pythagoras' Theorem and the concept of vectors in various mathematical and mechanical settings, leading eventually to the roles they play in the theory of relativity. Unlike other titles in this series, there are no exercises.

Problem 48

Let a and b be the lengths of the legs of a right triangle and let c be the length of its hypotenuse. Find a tiling of the plane using squares of sides a and b and a tiling of the plane using squares of side c. Superimpose the two tilings to obtain a proof of Pythagoras' Theorem: $a^2 + b^2 = c^2$.

The Contest Problem Book II, by C. T. Salkind, 1966.
The book contains a reprint of the American High School Mathematics Examinations from 1961 to 1965. Each paper consists of 40 multiple choice questions. An answer key is provided, followed by complete solutions.

Problem 49

If $5x + 12y = 60$, then the minimum value of $\sqrt{x^2 + y^2}$ is—
(a) 60/13, (b) 13/5, (c) 13/12, (d) 1, (e) 0.

First Concepts of Topology, by W. G. Chinn and N. E. Steenrod, 1966.
The first chapter in this book introduces the reader to point-set topology, leading to the important result in analysis: the Bolzano-Weierstrass Theorem. The second chapter involves concepts of algebraic topology from which the fundamental theorem of algebra is deduced.

Problem 50

Given a circular pancake and a second pancake which is irregular in shape, prove that there is a straight line which simultaneously cuts each pancake in half.

Geometry Revisited, by H. S. M. Coxeter and Samuel Greitzer, 1967.
This is an excellent review of Euclidean geometry, dealing with points and lines connected with a triangle, some properties of circles, collinearity and concurrence and transformations. There is also an introduction to inversive geometry and to projective geometry.

Problem 51

Prove that, if a quadrilateral is inscribed in a circle, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.

Invitation to Number Theory, by Oystein Ore, 1967.

This is an excellent introduction to number theory, covering concepts such as divisibility, primes, greatest common divisors, congruence, Diophantine equations and numeration systems.

Problem 52

Find the smallest positive integer with exactly 100 positive divisors.

Geometric Transformations II, by I. M. Yaglom, 1968.

The book adds similarity transformations to isometries and gives many applications of the transformation approach to geometric problems.

Problem 53

Inscribe a square in a given triangle ABC so that two vertices lie on the base AB, and the other two lie on the sides AC and BC, respectively.

Elementary Cryptanalysis: A Mathematical Approach, by Abraham Sinkov, 1968.

The book gives a mathematical approach to the popular topic of secret codes. There are discussions of various systems of codes, mostly based on modular arithmetic. Methods of cracking these codes without the knowledge of the key are also given.

Problem 54

The secret message "FRZDUGV GLH PDQB WLPHV EHIRUH WKHLU GHDWKV" is obtained from the original message by shifting each letter a fixed number of places. What is the original message?

Ingenuity in Mathematics, by Ross Honsberger, 1970.

The book consists of 19 essays from elementary combinatorics, number theory and geometry. Topics include Sylvester's problem, the Isoperimetric problem, the Theorem of Barbier, probability and π, the Farey series, complementary sequences, abundant numbers, squaring the square and the construction problems of Mascheroni and Steiner.

Problem 55

A positive integer n is said to be abundant if the sum of all its positive divisors exceeds $2n$. Prove that all multiples of abundant numbers are abundant.

Geometric Transformations III, by I. M. Yaglom, 1973.

The book takes the reader beyond Euclidean geometry to affine and projective geometries, again discussed from the transformational point of view. There is a supplement on hyperbolic geometry.

Problem 56

Given a segment AB and another line parallel to AB, use a straight-edge only to construct the midpoint of AB.

The Contest Problem Book III, by C. T. Salkind and J. M. Earl, 1973.

The book contains a reprint of the American High School Mathematics Examinations from 1966 to 1972. Each of the first two papers consists of 40 multiple choice questions, the number dropping to 35 for subsequent papers. An answer key is provided, followed by complete solutions.

Problem 57

Three times Dick's age plus Tom's age equals twice Harry's age. Double the cube of Harry's age is equal to three times the cube of Dick's age added to the cube of Tom's age. Their respective ages are relatively prime to each other. The sum of the squares of their ages is—
(a) 42, (b) 46, (c) 122, (d) 290, (e) 326.

Mathematical Methods in Science, by G. Pólya, 1976.

A distinguished mathematician and scientist illustrates the applications of various mathematical concepts such as measurement, successive approximation, vectors, and differential equations in astronomy, statics and dynamics.

Problem 58

Newton's method of finding approximate values for $\sqrt{a}$ may be described as follows. If $x = \sqrt{a}$, then $x^2 = a$. If $x \neq \sqrt{a}$, then $xy = a$ for some y. One of x and y will be greater than $\sqrt{a}$ while the other is less. Thus $(x + y)/2$ will be a better approximation of $\sqrt{a}$. Use Newton's method and $x = 2$ as an initial guess to find an approximation of $\sqrt{2}$ to two decimal places.

International Mathematical Olympiads, 1959-1977, by Samuel Grietzer, 1978.

The International Mathematical Olympiad began in 1959 in Romania as an all East-European affair. It has since grown to truly international proportions. This book contains the papers of the first 19 Olympiads and their solutions.

Problem 59

Prove that the fraction $(21n + 4)/(14n + 3)$ is irreducible for every positive integer n.

The Mathematics of Games and Gambling, by E. W. Packel, 1981.

The book treats the subject of probability in a gambling setting, discussing various games of dice and cards such as backgammon, craps, poker and bridge. Basic concepts such as permutations, combinations, the binomial distribution and mathematical expectation are covered. There is also a chapter on elementary game theory.

Problem 60

Does one have an even chance of getting at least one 6 in three rolls of an honest die?

The Contest Problem Book IV, by R. A. Artino, A. M. Gaglione and N. Shell, 1982.

The book contains a reprint of the American High School Mathematics Examinations from 1973 to 1982. Each paper consists of 30 multiple choice questions except for 1973 where there are 35 questions. An answer key is provided, followed by complete solutions.

Problem 61

The units digit in the decimal expansion of $(15 + \sqrt{220})^{19} + (15 + \sqrt{220})^{82}$ is—
(a) 0, (b) 2, (c) 5, (d) 9, (e) none of these.

The Role of Mathematics in Science, by M. M. Shiffer and Leon Bowden, 1984.

The book contains seven chapters, dealing with the beginnings of mechanics, growth functions, the role of mathematics in optics, mathematics with matrices, transformations, Einstein's space-time transformation problem, relativistic addition of velocities and energy.

Problem 62

A hyperbolic mirror is in the shape of one branch of a hyperbola, with the reflecting surface on the convex side. Prove that if a light source is placed at the focus behind the other branch of the hyperbola, the reflected rays will appear to have originated from the focus behind the mirror.

International Mathematical Olympiads, 1978-1985, by M. S. Klamkin, 1986.

This book contains the International Mathematical Olympiad papers for the years indicated, except for 1980 when the Olympiad was not held. In addition to complete solutions, there are also 40 supplementary problems. Professor Klamkin of the University of Alberta is the acknowledged world authority on problem-solving.

Problem 63

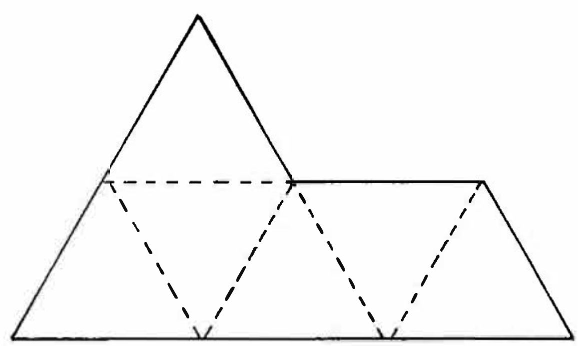
Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1,2,\ldots,n\}$. Each of these subsets has a smallest member. Prove that the arithmetic mean of these smallest numbers is equal to $(n + 1)/(r + 1)$.

Riddles of the Sphinx, by Martin Gardner, 1987.

This is an anthology of Martin Gardner's contributions to Isaac Asimov's *Science Fiction Magazine*. There are 36 puzzles in science fictional settings. Often, when the "first" answers are given, further questions arise, to be answered in a "second" section. "Third" answers and "fourth" answers pursue the matter even further.

Problem 64

Divide the sphinx into four smaller copies of the sphinx that are congruent to one another.



Problem 64

This superb series on popular mathematics is translated from Russian by Mir Publishers of Moscow. The books are paperbacks ranging from 25 to 150 pages, and may be obtained from Progress Books of Toronto at $2.50 a copy! The series is written for high school students. Each volume is a stimulating study of a particular topic, and the mathematics is of the highest quality.

Algebraic Equations of Arbitrary Degrees, by A. G. Kurosh, 1977.

High school students are familiar with polynomial equations of degrees one and two, the latter usually solved by the Quadratic Formula. This book gives a brief description of the Cubic Formula and goes on to discuss the general theory of polynomial equations.

Problem 65

Prove that the equation $x^3 - 5x^2 + 2x + 1 = 0$ has three real roots.

Areas and Logarithms, by A. I. Markushevich, 1981.

This book presents a geometric theory of logarithm, in which logarithms are introduced as various areas. Properties of logarithms are then derived from those of areas. The reader is introduced to rudimentary integral calculus without first going through differential calculus.

Problem 66

Prove that the area of the region enclosed by the graphs of $y = 1/x$, $y = 0$, $x = 1$ and $x = 3$ is equal to that enclosed by the graphs of $y = 1/x$, $y = 0$, $x = 3$ and $x = 9$.

Calculus of Rational Functions, by G. E. Shilov, 1982.

This book is an informal introduction to differential and integral calculus with sufficient rigor when attention is restricted to the rational functions, that is, functions expressible as a quotient of two polynomials. There is an illuminating preamble on graph sketching.

Problem 67

Sketch the graphs of $y = 3x^2$, $y = 3x^2 - 1$, $y = (3x^2 - 1)^2$ and $y = 1/(3x^2 - 1)^2$.

Complex Numbers and Conformal Mappings, by A. I. Markushevich, 1982.

Not assuming prior acquaintance with complex numbers, this book introduces them to the reader in geometric form as directed line segments. Functions of a complex variable are considered as geometric transformations. Of particular interest is the class of conformal mappings or angle-preserving transformations.

Problem 68

What geometric transformation corresponds to the function $f(z) = (1+i)z/\sqrt{2}$? What will be the image of the triangle with vertices at 0, $1 - i$ and $1 + i$?

Differentiation Explained, by V. G. Boltyansky, 1977.

This book introduces the reader to differential calculus by considering problems in physics, such as the problem of a free-falling body. This is followed by an informal discussion of differential equations, which are applied to tackle the problem of harmonic oscillations. Other applications of differential calculus are also given.

Problem 69

A steam boiler in the shape of a cylinder is to be built so that it will have the required volume V. It is desirable to keep the total surface area of the boiler down to a minimum. Find the best dimensions of the boiler.

Dividing a Segment in a Given Ratio, by N. M. Beskin, 1975.

This book begins with an analysis of the very elementary problem of how to divide a line segment in a given ratio. From this, the reader is led to concepts such as parallel projections, ideal points, separation, cross ratio and complete quadrilaterals. It is an excellent introduction to projective geometry.

Problem 70

Given a segment AB, construct by Euclidean means a point C on AB and a point D on AB extended such that $AC/BC = AD/BD = 4$.

Elements of Game Theory, by Ye. S. Venttsel, 1980.

Game theory deals with conflict scenarios which are resolved according to definite rules. Each party in the conflict has a finite number of options known to all others. After presenting the basic concepts, the book discusses pure and mixed strategies as well as general and approximate methods for solving games.

Problem 71

We have three kinds of weapon at our disposal, A_1, A_2 and A_3. The enemy has three kinds of aircraft, B_1, B_2 and B_3. Our goal is to hit an aircraft and our enemy's goal is to avoid that. When armament A_1 is used, aircraft B_1, B_2 and B_3 are hit with probabilities 0.4, 0.5 and 0.9, respectively. When armament A_2 is used, they are hit with probabilities 0.8, 0.4 and 0.3, respectively. When armament A_3 is used, the respective probabilities are 0.7, 0.6 and 0.8. What is our optimal strategy and what is the optimal strategy of the enemy?

Fascinating Fractions, by N. M. Beskin, 1986.

The fascinating fractions are the continued fractions. After two introductory problems, the book gives an algorithm for converting a real number into a continued fraction. Further applications follow, including the solution to Diophantine equations and the approximation of real numbers by rational numbers.

Problem 72

What real number is represented by the continued fraction $1/(1 + 1/(1 + 1/(1 + \cdots)))$?

The Fundamental Theorem of Arithmetic, by L. A. Kaluzhnin, 1979.

The Fundamental Theorem of Arithmetic states that prime factorization is unique. This is usually taken for granted, but the book argues that its importance deserves a rigorous proof. Examples of number systems in which prime factorization is not unique are given. The proof of the theorem also leads to a method of solving linear Diophantine equations.

Problem 73

Prove that every integer greater than 1 can be expressed as a product of primes in at least one way.

Geometrical Constructions with Compass Only, by A. Kostovskii, 1986.

This book presents a proof of the Mascheroni-Mohr Theorem that the ruler is a redundant tool in Euclidean constructions. Of course, the compass alone cannot draw straight lines, but we may consider a straight line constructed if at least two points on it are obtained. However, if further use of this line is made to intersect other lines and circles, those points of intersections have to be constructed explicitly. The main idea behind the proof is inversion. Later, further constraints are placed on the compass.

Problem 74

Given three points, A,B,C, and a positive number r, construct the points of intersection of the line AB and the circle with centre C and radius r, using only a compass.

Gödel's Incompleteness Theorem, by V. A. Uspensky, 1987.

Gödel's Incompleteness Theorem says roughly that, under certain very reasonable conditions in a mathematical system, there exist true but unprovable statements. This book provides the necessary background in mathematical logic and gives a formal proof of this important result.

Problem 75

A set is said to be countable if its elements can be matched one-to-one onto the set of positive integers. Prove that the union of two countable sets is countable.

Images of Geometric Solids, by N. M. Beskin, 1985.

The subject of this book is descriptive geometry, that is, the two-dimensional representation of three-dimensional objects. The concept of projection plays a central role. Numerous practical exercises are included.

Problem 76

Prove that the image of a straight line under parallel projection is either a point or a straight line.

Induction in Geometry, by L. I. Golovina and I. M. Yaglom, 1979.

This book gives an excellent account on the power of mathematical induction, applied to problems in geometry. There are problems of computation, proof, construction and locus. Induction is also used to define concepts and to generalize results to higher dimensions.

Problem 77

Prove that it is possible to cut up each of n given squares into finitely many pieces and use all pieces thus obtained to reassemble a single square.

Inequalities, by P. P. Korovkin, 1975.

This book begins with the basic Arithmetic-Mean-Geometric-Mean Inequality and uses the Bernoulli Inequality to generalize it to the Power-Means Inequality. Various applications are then given.

Problem 78

Prove that for integers n greater than 2, we have $n! < ((n + 1)/2)^n$.

The Kinematic Method in Geometrical Problems, by Yu. I. Lyubich and L. A. Shor, 1980.

When solving a geometrical problem, it is helpful to imagine what would happen to the elements of the figure under consideration if some of its points start moving. After a review of vector algebra, this book shows how kinematics, or the theory of velocities, can be applied to tackle geometric problems.

Problem 79

A treasure map gives the following directions. Go to the gallows. From there, walk in a straight line to the pine tree, turn through a right angle to the left, walk the same distance in a straight line and mark the spot. Return to the gallows. From there, walk in a straight line to the oak tree, turn through a right angle to the right, walk the same distance in a straight line and mark the spot. The treasure is buried at the point halfway between the two marked spots. The treasure hunter finds that the gallows has disappeared without a trace. Fortunately, the trees are still there. Can the treasure hunter find the treasure?

Lobachevskian Geometry, by A. S. Smogorzhevsky, 1976.

This is an introduction to hyperbolic geometry. The principal tool is the inversive transformation in Euclidean geometry. This is used to construct a model of the hyperbolic plane and various theorems in hyperbolic geometry are proved. The book ends with a discussion of the hyperbolic functions and their uses for computation in the hyperbolic plane.

Problem 80

Given a circle with centre O and radius r, the inversive image of a point $A \neq O$ with respect to this circle is the point B such that O, A and B lie on a straight line and $OA \cdot OB = r^2$. What is the inversive image of a circle passing through O?

Method of Coordinates, by A. S. Smogorzhevsky, 1980.

This is an introduction to analytic geometry, covering the most basic concepts. There is also a brief discussion of polar coordinates.

Problem 81

Find a single equation in x and y which is satisfied by every point (x, y) inside the first quadrant but no others.

The Method of Mathematical Induction, by I. S. Sominsky, 1975.

This is an excellent introduction to the important method of mathematical induction. After a detailed discussion of the basic idea, numerous examples from arithmetic, algebra and trigonometry are provided, including many proofs of identities and inequalities.

Problem 82

Prove that any amount of postage over 7¢ may be made up exactly using only 3¢ and 5¢ stamps.

Method of Successive Approximation, by N. Ya. Vilenkin, 1979.

This is an introduction to numerical analysis. Starting with the simplest form of successive approximation, the reader is led to the method of iteration, the method of chords and Newton's method which uses differential calculus.

Problem 83

The method of chords uses a straight line to approximate a small portion of a curve. Use the method of chords to find an approximate solution of the equation $x^3 + 3x - 1 = 0$.

The Monte Carlo Method, by I. M. Sobol, 1975.

This is an introduction to the theory of statistical sampling. The Monte Carlo method calculates a certain parameter by running a sequence of simulations, taking the average value and estimating the error. It is based on the probabilistic concept of a random variable. In the case of a continuous variable, calculus is used.

Problem 84

A plane figure is enclosed within a unit square. Forty points inside the square are chosen at random and the number of those inside the figure counted. The experiment is repeated ten times. If the counts are 15, 15, 14, 12, 12, 15, 14, 14, 13 and 15, respectively, what would be a good estimate of the area of the figure?

Pascal's Triangle and *Certain Applications of Mechanics to Mathematics*, by V. A. Uspensky, 1976.

This volume consists of two independent booklets by the same author. The first is an introduction to the binomial coefficients and Pascal's Formula which lead to the construction of Pascal's triangle. The second uses the mechanical principle that, in a position of equilibrium, the potential energy of a weight attains its lowest value, to solve a number of problems in geometry and number theory.

Problem 85

Give a proof based on mechanical considerations that a tangent to a circle is perpendicular to the radius at the point of tangency.

Post's Machine, by V. A. Uspensky, 1983.

The book deals with a certain abstract computing machine. Though it does not exist physically, calculations on it involve many important features inherent in the computations on computers. This machine is also known as Turing's machine.

Problem 86

Post's machine consists of an infinite tape divided into cells and a tapehead which is positioned over one cell at any time. A cell may contain either nothing or a marking. A program consists of a sequence of instructions, of which there are seven types: L instructs the tapehead to move one cell to the left, R instructs the tapehead to move one cell to the right, M instructs the tapehead to mark the cell it is over, U instructs the tapehead to unmark the cell it is over, S instructs the machine to stop, G *n* instructs the machine to execute statement *n* next, and B *m n* instructs the machine to execute statement *m* next if the tapehead is over a marked cell, and statement *n* if the tapehead is over an unmarked cell. The statements in a program are numbered consecutively and are executed in order unless directed otherwise by a G or B statement. Consider the following program: (1) M, (2) R, (3) B 4 2, (4) L, (5) B 6 4, (6) U, (7) S. The tapehead is initially at cell 0 and the tape is unmarked except for cell 2. At the end of this program, where would the tapehead be and what changes have been made on the tape?

Proof in Geometry, by A. I. Fetisov, 1978.

This book raises and answers the following questions. What is proof? Why is proof a necessity? What conditions should a proof satisfy for us to call it a correct one? What propositions may be accepted without proof? The discussion is illustrated with numerous examples from geometry.

Problem 87

Prove that, if two interior angle bisectors of a triangle are equal, the triangle is isosceles.

Recursion Sequences, by A. I. Markushevich, 1975.

This book deals with the counting technique based on recurrence relations. After a review of geometric progressions, the method of characteristic equations is introduced to solve recurrence relations.

Problem 88

A sequence $\{s_n\}$ is defined by $s_o = 1$ and $s_n = s_{n-1} + 3^n$. Find an explicit formula for s_n in terms of n.

Remarkable Curves, by A. I. Markushevich, 1980.

This book presents a collection of very attractive curves and their interesting properties. Starting with the basic conic sections, the reader is led on to the lemniscate of Bernoulli, the cycloid, the spiral of Archimedes, the catenary and the logarithmic spiral.

Problem 89

Find the locus of a point P such that $PA = 3PB$ where A and B are fixed points.

Shortest Lines, by L. A. Lyusternik, 1976.

This may be considered as a non-technical introduction to differential geometry. The concepts of space curve, curvature and geodesics are discussed. However, the emphasis is on the shortest-line problem on special surfaces which can be solved by elementary methods. Plenty of applications are considered.

Problem 90

A and B are diametrically opposite points on the base of a cylinder and C is the point on the top directly above B. The cylinder has height 8 and base circumference 12. Find the shortest path from A to C on the surface of the cylinder.

Solving Equations in Integers, by A. O. Gelfond, 1981.

The subject is Diophantine equations. Detailed study of the linear Diophantine equation, Pythagoras' equations and Pell's equation are presented, with continued fractions playing a central role. There is a brief discussion of other Diophantine equations.

Problem 91

Prove that the equation $x^2 - dy^2 = 1$ has no solutions in integers other than $(\pm 1, 0)$ if d is a perfect square.

Stereographic Projection, by B. A. Rosenfeld and N. D. Sergeeva, 1977.

The stereographic projection is a projection of a sphere from one of its points onto the plane tangent to the sphere at the diametrically opposite point. Applications to astronomy, geography and hyperbolic geometry are included.

Problem 92

Let S, M and N be three points on a great circle of a sphere. Let M' and N' be the respective images of M and N under the stereographic projection from S. Prove that $\measuredangle SMN = \measuredangle SN'M'$ and $\measuredangle SNM = \measuredangle SM'N'$.

Systems of Linear Inequalities, by A. S. Solodovnikov, 1979.

This is an introduction to linear programming. Systems of linear inequalities in two or three unknowns are visualized geometrically followed by a brief discussion of convexity. The simplex method is then presented, with a proof of the Duality Theorem and an application to a transportation problem.

Problem 93

Find the minimum value of the function $2x + y$ provided that $x \geq 0$, $y \geq 0$, $x - y \leq 0$, $x + y \geq 2$ and $x + 4y \geq 12$.

An Unusual Algebra, by I. M. Yaglom, 1978.

This is an introduction to Boolean algebra or the algebra of sets. Comparison with the algebra of numbers is made. Applications to propositional logic and switching circuits are discussed.

Problem 94

Let A and B be subsets of a universal set U. Prove that $A \cap (A \cup B) = A$.

G. Books from W. H. Freeman & Company, Publishers

W. H. Freeman has a relatively small selection of titles for a major publisher, but what a selection! Among the more advanced books are the acclaimed *The Fractal Geometry of Nature* by B. B. Mandelbrot and the long-awaited *Tilings and Patterns* by B. Grünbaum and G. C. Shephard. We will concentrate on books at the high school and beginning college levels. We point out that three of the books in Martin Gardner's *Scientific American* series are published by Freeman.

aha! Insight, by Martin Gardner, 1978.

This is the book form of six filmstrips titled *Combinatorial aha! Geometry aha! Number aha! Logic aha! Procedural aha!* and *Word aha!* Selected sequences of frames featuring problems and paradoxes are shown with accompanying text.

Problem 95

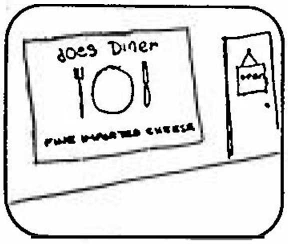
See the sequence of frames (next page).

aha! Gotcha, by Martin Gardner, 1982.

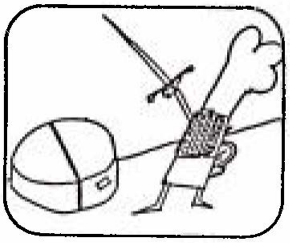
This is the book form of six filmstrips titled *aha Logic! aha Number! aha Geometry! aha Probability! aha Statistics!* and *aha Time!* Selected sequences of frames featuring problems and paradoxes are shown with accompanying text.

Problem 96

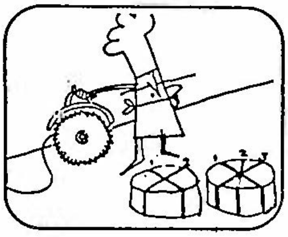
See the sequence of frames (next page).



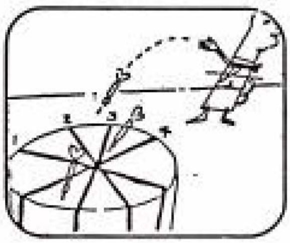
The food at Joe's Diner may not be the best, but the place is famous for its delicious cheese.



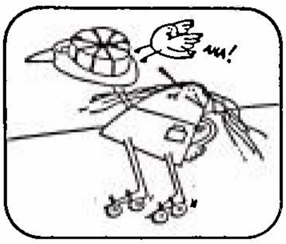
You can have a lot of fun with the cylindrical pieces of cheese. With one straight cut it's easy to divide one piece into two identical pieces.



With two straight cuts it's easy to cut it into four identical pieces. And three cuts will make six identical pieces.

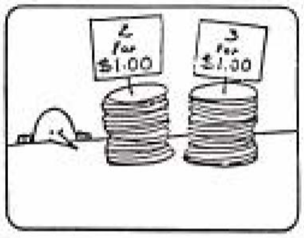


One day, Rosie, the waitress, asked Joe to slice the cheese into eight identical pieces.
Joe: Okay, Rosie. That's simple enough. I can do it with four straight cuts like this.

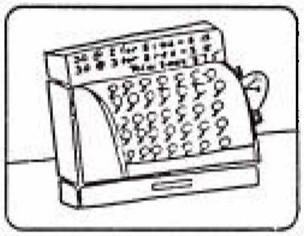


While Rosie was carrying the slices to the table, she suddenly realized that Joe could have gotten the eight identical pieces with only three straight cuts. What insight did Rosie have?

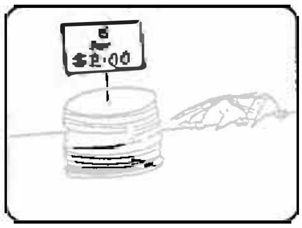
Problem 95



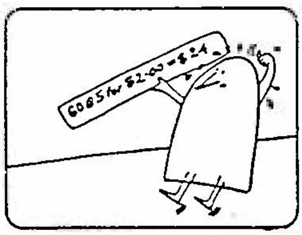
A record store put 30 old rock records on sale at two for a dollar, and another 30 on sale at three for a dollar. All 60 were gone by the end of the day.



The 30 two-for-a-dollar disks brought in $15. The 30 three-for-a-dollar disks brought in $10. Altogether—$25.



The next day the store manager put another 60 records on the counter.
Clerk: Why bother to sort them? If 30 sell at two for a dollar, and 30 at three for a dollar, why not put all 60 in one pile and sell that at five for $2? It's the same thing.



When the store closed, all 60 records had been sold at five for $2. But when the manager checked the cash, he was surprised to find that proceeds from the sale were only $24, not $25.

What do you think happened to that missing dollar? Did the clerk steal it? Did a customer get the wrong change?

Problem 96

Geometry, by H. R. Jacobs, 1987.
The second edition of this outstanding textbook covers standard Euclidean plane and solid geometry in a way which students will find enjoyable. The serious mathematics is interlaced with cartoons, anecdotes and practical problems. A Teacher's Guide provides specific lesson plans.

Problem 97

Shipwrecked on an island which is in the shape of an equilateral triangle, a sailor builds a hut so that the total of its distances to the three sides of the triangle is a minimum. Where is the best place on the island for the hut?

Elementary Algebra, by H. R. Jacobs, 1979.

This textbook covers functions and graphs, number systems, equations, polynomials, exponents and radicals, inequalities and number sequences. It is done in a lucid and refreshing manner, much in the style of its companion volume *Geometry*. The Teacher's Guide provides specific lesson plans.

Problem 98

Solve the system of simultaneous equations $55x + 45y = 520$ and $45x + 55y = 480$.

Mathematics: A Human Endeavor, by H. R. Jacobs, 1982.

The subtitle of this book is "A Book for Those Who Think They Don't Like the Subject." It is a stimulating survey of number theory, algebra, geometry, combinatorics, probability, statistics and topology.

Problem 99

There are three boxes, one containing two red marbles, one containing two white marbles and one containing a red marble and a white marble. The labels telling the contents of the boxes have been switched, however, so that the label on each box is wrong. You are allowed to choose one of the three boxes, draw out at random one marble from inside and deduce from its color the contents of each box. How can this be done?

Mathematics: A Man-Made Universe, by S. K. Stein, 1976

The third edition of this classic contains 19 chapters, covering number theory, geometry, graph theory, modern algebra, number systems, constructibility problems and infinite sets. Its outstanding feature is a large collection of exercises that urge the reader to explore and discover. Despite the ease with which various topics are handled, the book has tremendous depth.

Problem 100

We outline a way of multiplying any two positive integers that uses only multiplication and division by 2. We illustrate it by computing 35×56. First, find the quotient when 2 is divided into 35, namely 17. Repeat the same process on 17, obtaining 8. Continue till you reach 1. Pair off with these numbers those that you obtain by repeatedly multiplying by 2, in this manner:

35	56
17	112
8	224(X)
4	448(X)
2	896(X)
1	1792
	1960

Next cross out all entries in the right-hand column that correspond to even entries in the left-hand column. Add the remaining numbers, 56, 112 and 1792, in the right-hand column. Their sum, 1960, is the product 35×56. Prove that this method always works.

Mathematics: Problem Solving Through Recreational Mathematics, by B. Averbach and O. Chein, 1980.

This book presents a minimum of theory and plenty of problems. After a general discussion on problem solving techniques, problems in logic, algebra, number theory, graph theory, games and puzzles are posed and solved.

Problem 101

A college student sent a postcard to her parents with the message

$$\begin{array}{r} \text{SEND} \\ +\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

If each letter represents a digit, with different letters representing different digits and the same letter representing the same digit each time it occurs, how much money is being requested?

Mathematics: An Introduction to Its Spirit and Use, edited by M. Kline, 1979.

This book contains 40 articles reprinted from the *Scientific American*, with 14 from Martin Gardner's "Mathematical Games." They are classified into six categories: history, number and algebra, geometry, statistics and probability, symbolic logic and computers and applications. Note that W. H. Freeman also handles offprints of other *Scientific American* articles.

Problem 102

A cone of light casts the shadow of a ball onto the level surface on which the ball is resting. Prove that the shadow is elliptical.

H. Books from Dover Publications, Inc.

The majority of Dover's publications are reprints of excellent (otherwise, why do it?) books that are no longer available in other formats. The new editions are usually paperbound and inexpensive (averaging about $5 U.S. each). Often, errors in the original versions are corrected and new material is appended. While Dover has a large selection of titles in mainstream mathematics (as well as in many other areas, academic or otherwise), we will focus on the best of its line on popular mathematics.

Challenging Mathematical Problems with Elementary Solutions I, by A. M. Yaglom and I. M. Yaglom, 1987.

This is one of the finest collections of problems in elementary mathematics. The 100 problems in combinatorial analysis and probability theory are all easy to understand, but some are not easy to solve, even though no advanced mathematics is required.

Problem 103

Evaluate $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \cdots$, where n is a positive integer and $\binom{n}{k}$ denotes the binomial coefficient, with $\binom{n}{k} = 0$ if $n<k$.

Challenging Mathematical Problems with Elementary Solutions II, by A. M. Yaglom and I. M. Yaglom, 1987.

In this second volume, 74 problems are selected from various branches of mathematics, in particular, number theory and combinatorial geometry. This book and the earlier volume are a must for every school library.

Problem 104

A certain city has 10 bus routes. Is it possible to arrange the routes and the bus stops so that, if one route is closed, it is still possible to get from any one stop to any other (possibly changing buses along the way) but, if any two routes are closed, there are at least two stops such that it is impossible to get from one to the other?

Mathematical Bafflers, edited by A. F. Dunn, 1980.

The bafflers in this book originally appeared as a most successful weekly corporate advertisement in technical publications. They are contributed by the readers, with a consequent diversity in their levels of sophistication. Some require almost no mathematics while others are quite demanding. However, there is a beautiful idea behind each baffler, which is compactly stated and accompanied by a cartoon.

Problem 105

Two similar triangles with integral sides have two of their sides the same. The third sides differ by 387. What are the lengths of the sides?

Second Book of Mathematical Bafflers, edited by A. F. Dunn, 1983.

This second collection of bafflers, like the earlier volume, is organized by chapters, each dealing with one area of mathematics. These include algebra, geometry, Diophantine problems and other number theory problems, logic, probability and "insight."

Problem 106

Only two polygons can have a smallest interior angle of 120° with each successive angle 5° greater than its predecessor. One is a nonagon with angles 120°, 125°, 130°, 135°, 140°, 145°, 150°, 155° and 160°. What is the other?

Ingenious Mathematical Problems and Methods, by L. A. Graham, 1959.

The 100 problems in the book originally appeared in the *Graham Dial*, a publication circulated among engineers and production executives. They are selected from areas not commonly included in school curricula and have new and unusual twists that call for ingenious solutions.

Problem 107

Given three unequal disjoint circles, prove that the three points of intersection of the external common tangents of the three pairs of circles lie on a straight line.

The Surprise Attack in Mathematical Problems, by L. A. Graham, 1968.

These 52 problems are selected from the *Graham Dial* on the criterion that the best solutions are not the ones the original contributors had in mind. The reader will enjoy the elegance of the unexpected approach. Like the earlier volume, the book includes a number of illustrated Mathematical Nursery Rhymes.

Problem 108

Construct an isosceles triangle given the base and the bisector of one of the base angles.

One Hundred Problems in Elementary Mathematics, by H Steinhaus, 1979.

The 100 problems cover the more traditional areas of number theory, algebra, plane and solid geometry, as well as a host of practical and non-practical problems. There are also 13 problems without solutions; some but not all of these actually have known solutions. The unsolved problems are not identified in the hope that the reader will not be discouraged from attempting them.

Problem 109

We construct a sequence of numbers as follows. The first term is 2 and the next is 3. Since $2 \times 3 = 6$, the third term is 6. Since $3 \times 6 = 18$, the fourth term is 1 and the fifth is 8. The product of each pair of consecutive terms is computed in turn and appended digit by digit to the sequence. Prove that the number 5 never appears in this sequence.

Fifty Challenging Problems in Probability with Solutions, by F. Mosteller, 1987.

This book actually contains 56 problems, each with an interesting story line. There are the familiar ''gambler's ruin'' and ''birthday surprises'' scenarios, but with new twists. Others are unconventional, including one which turns out tc be a restatement of Fermat's Last Theorem.

Problem 110

A three-person jury has two members each of whom independently has probability p of making the correct decision and a third member who flips a coin for each decision. A one-person jury has probability p of making the correct decision. Which jury has the better probability of making the correct decision if majority rules in the three-person jury?

Mathematical Quickies, by C. Trigg, 1985.

This book contains 270 problems. Each is chosen because there is an elegant solution. Classification by subject is deliberately avoided, nor are the problems graduated in increasing level of difficulty. This encourages the reader to explore each problem with no preconceived idea of how it should be approached.

Problem 111

How many negative roots does the equation $x^4 - 5x^3 - 4x^2 - 7x + 4 = 0$ have?

Entertaining Mathematical Puzzles, by Martin Gardner, 1986.

The master entertains with 39 problems and 28 quickies, covering arithmetic, geometry, topology, probability and mathematical games. There is a brief introduction to the basic ideas and techniques in each section.

Problem 112

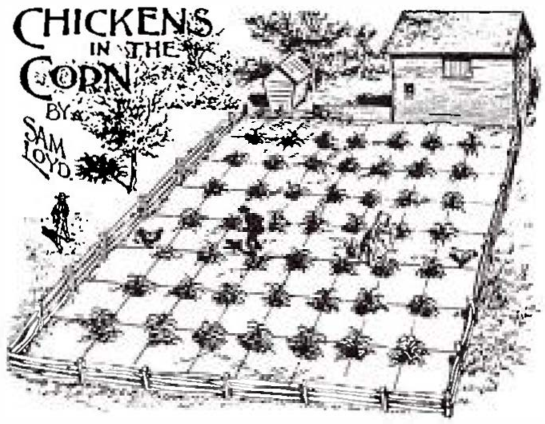
A straight line is called self-congruent because any portion of the line can be exactly fitted to any other portion of the same length. The same is true of the circle. There is a third type of curve that is self-congruent. What is it?

Mathematical Puzzles of Sam Loyd I, edited by Martin Gardner, 1959.

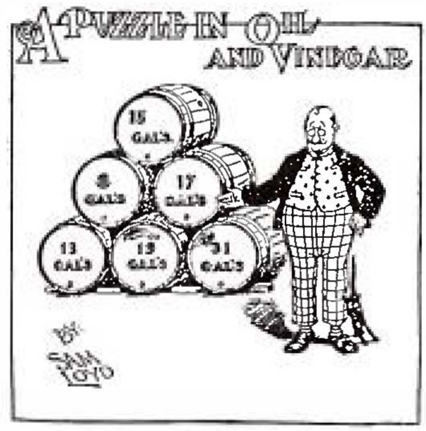
Sam Loyd is generally considered the greatest American puzzlist. He had a knack of posing problems in a way which attracted the eye of the public. Many of the 117 problems in this book, the first of two volumes, had been used as novelty advertising give-aways.

Problem 113

The game of chicken-catching is played in a garden divided into 64 square patches as shown in the illustration (next page). There are the farmer and his wife, as well as a rooster and a hen. The humans move first, followed by the chickens, and alternately thereafter. In a move, each of the two members of the team must move to an adjacent square (not diagonally). A chicken is caught if a human moves into the square it occupies. Can the humans catch the rooster and the hen?



Problem 113



Problem 114

Mathematical Puzzles of Sam Loyd II, edited by Martin Gardner, 1960

This book contains 166 problems, most of which are accompanied by Loyd's own illustrations, as is the case with the earlier volume. The two books represent the majority of the mathematical problems in the mammoth *Cyclopedia* by Sam Loyd, published after his death.

Problem 114

Each barrel in the illustration contains either oil or vinegar. The oil sells for twice as much per gallon as the vinegar. A customer buys $14 worth of each, leaving one barrel. Which barrel did he leave?

Amusements in Mathematics, by H. E. Dudeney, 1970.

Henry Ernest Dudeney, a contemporary of Sam Loyd, is generally considered the greatest English puzzlist and a better mathematician than Loyd. This book contains 430 problems, representing only part of Dudeney's output. There are plenty of illustrations in the book.

Problem 115

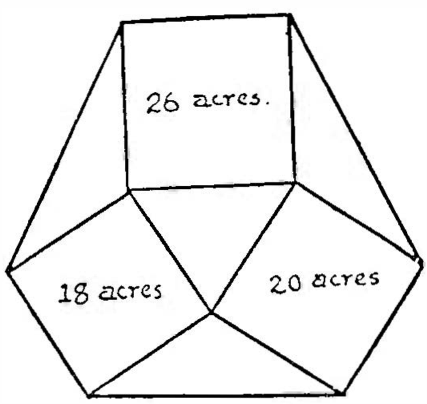
Farmer Wurzel owned the three square fields shown in the illustration. In order to get a ring-fence round his property he bought the four intervening triangular fields. What is the total area of his estate?

Mathematical Puzzles for Beginners and Enthusiasts, by G. Mott-Smith, 1954.

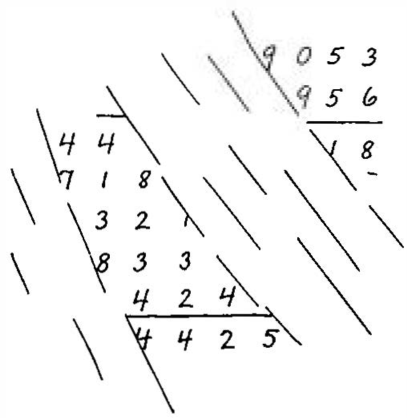
This book contains 189 problems in arithmetic, logic, algebra, geometry, combinatorics, probability and mathematical games. They are both instructive and entertaining.

Problem 116

The janitor inadvertently wiped out a good share of Miss Gates' multiplication. What are the erased digits?



Problem 115



Problem 116

Mathematical Recreations and Essays, by W. R. Ball and H. S. M. Coxeter, 1988.

This is the foremost single-volume classic of popular mathematics. Written by two distinguished mathematicians, it covers a variety of topics in great detail. After arithmetical and geometrical recreations, it moves on to polyhedra, chessboard recreations, magic squares, map-coloring problems, unicursal problems, Kirkman's schoolgirls problem, the three classical geometric construction problems, calculating prodigies, cryptography and cryptanalysis.

Problem 117

A schoolmistress was in the habit of taking her girls for a daily walk. The girls were 15 in number and were arranged in five rows of three each, so that each girl might have two companions. How is it possible that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once?

Mathematical Recreations, by M. Kraitchik, 1953.

This is a revision of the author's original work in French. It covers more or less the same topics as *Mathematical Recreations and Essays*. There is a chapter on ancient and curious problems from various sources.

Problem 118

For their common meal, Caius provided seven dishes and Sempronius eight. But Titus arrived unexpectedly, so all shared the food equally. Titus paid Caius 14 denarii and Sempronius 16. The latter cried out against this division and the matter was referred to a judge. What should his decision be, granting that the 30 denarii is the correct total amount?

The Master Book of Mathematical Recreation, by F. Schuh, 1968.

This is a translation of the author's original work in German. Four of the fifteen chapters are devoted to the analysis of mathematical games. The remaining ones deal with puzzles of various kinds. General hints for solving puzzles are given in the introductory chapter. The last chapter is on puzzles in mechanics.

Problem 119

Two players start out with two piles of matches, say, with 19 and 89, respectively. They take turns removing 1, 2, 3, 4 or 5 matches, but all from one pile. Taking the last match means a win. Which player wins this game and what is a winning strategy?

Puzzles and Paradoxes, by T. H. O'Beirne, 1984.

Like Martin Gardner's series, this book is an anthology of the author's column in *New Scientist*. It consists of 12 largely independent articles.

Problem 120

There are twelve coins that look identical, but there may be one which has a different weight from the others. It may be either be slightly heavier or slightly lighter. Using a beam balance three times, how can one determine if there is a "false" coin and, if so, which coin is "false" and whether it is heavier or lighter?

I. Individual Titles

In this penultimate section, we present some outstanding work that is not in a continuing series. Regrettably, we have to be content with two dozen titles. There is simply not enough room to mention many other worthwhile books.

How to Solve It, by G. Pólya, Princeton University Press, 1973.

This is the second edition of the first of five books by an outstanding scientist and educator on his theory and methods of problem solving. Here, numerous examples illustrate the Pólya method which divides the task into four phases, understanding the problem, devising a plan, carrying out the plan and looking back.

Problem 121

Construct a triangle by Euclidean means, given one angle, the altitude from the vertex of the given angle and the perimeter of the triangle.

Mathematics and Plausible Reasoning I, by G. Pólya, Princeton University Press, 1954.

The subtitle of this volume is "Induction and

Analogy in Mathematics.'' It is a continuation and elaboration of the ideas propounded in *How To Solve It*. It discusses inductive reasoning and making conjectures, with examples mainly from number theory and geometry. The transition to deductive reasoning is via mathematical induction.

Problem 122

Guess an expression for $(1 - 1/4)(1 - 1/9)(1 - 1/16) \cdots (1 - 1/n^2)$ valid for $n \geq 2$ and prove it by mathematical induction.

Mathematics and Plausible Reasoning II, by G. Pólya, Princeton University Press, 1954.

The subtitle of this volume is ''Patterns of Plausible Inference.'' It is primarily concerned with the role of plausible reasoning in the discovery of mathematical facts. Two chapters in probability provide most of the illustrative examples.

Problem 123

Of nine patients treated the old way, six died. Of eleven patients treated the new way, two died. If there is really no difference between the two treatments, what is the probability that the observed results are as favorable as, or more favorable than, the above for the new way?

Mathematical Discovery I, by G. Pólya, Wiley, 1965.

This volume contains part one of the work, titled ''Patterns'' and the first two chapters of the second part of the work, titled ''Toward a General Method.'' The first part gives practices for pattern recognition in geometric loci, analytic geometry, recurrence relations and interpolation. The two chapters in the second part discuss general philosophy in problem solving.

Problem 124

A farmer has chickens and rabbits. These animals have 50 heads and 140 feet. How many chickens and how many rabbits has the farmer?

Mathematical Discovery II, by G. Pólya, Wiley, 1965.

This volume contains the remaining nine chapters of the second part of this work by Pólya. More specific advice on the art and science of problem solving is offered. There is a chapter on learning, teaching and learning teaching.

Problem 125

Three circles have the same radius r and pass through the same point O. Let A, B and C be the other points of intersection of the three pairs of these circles, respectively. Prove that A, B and C lie on a circle of radius r.

Discovering Mathematics, by A. Gardiner, Oxford University Press, 1987.

This is a do-it-yourself package through which the reader can learn and develop methods of problem solving. The first part of the book contains four short investigations and the second part two extended ones. Each investigation is conducted via a structured sequence of questions.

Problem 126

Find all five-digit numbers such that, when multiplied by 9, the product is given by writing the five digits of the number in reverse order.

Mathematical Puzzling, by A. Gardiner, Oxford University Press, 1987.

This book contains 31 chapters. The first 29 are groups of related puzzles. The chapters are independent except for three on counting and three on primes. In each chapter, commentaries follow the statements of the puzzles. Answers are given at the end of the book. Chapter 30 re-examines four of the earlier puzzles while Chapter 31 discusses the role of puzzles in mathematics.

Problem 127

Every digit in a multiplication has been copied down wrongly, but each digit is only one out. It now reads $16 \times 4 = 64$. What should it have been?

Selected Problems and Theorems in Elementary Mathematics, by D. O. Shklyarsky, N. N. Chentsov and I. M. Yaglom, Mir Publications, 1979.

This excellent book contains 350 problems in arithmetic and algebra, many from papers of the U.S.S.R. Olympiads. It is the first of three volumes but, unfortunately, the volumes on plane geometry and solid geometry have not been translated into English.

Problem 128

Find all three-digit numbers such that, when each is raised to any integral power, the result is a number whose last three digits form the original number.

All the Best from the Australian Mathematics Competition, edited by J. D. Edwards, D. J. King and P. J. O'Halloran, Australian Mathematics Competition, 1986.

This book contains 463 multiple choice questions taken from one of the world's most successful mathematics competitions. They are grouped by subject area to facilitate the study of specific topics.

Problem 129

For all numbers a and b, the operation $a \cdot b$ is defined by $a \cdot b = ab - a + b$. The solution of the equation $5 \cdot x = 17$ is—
(a) 17/5, (b) 2, (c) −2, (d) 3, (e) 11/3.

The First Ten Canadian Mathematics Olympiads, 1969-1978, edited by E. Barbeau and W. Moser, Canadian Mathematical Society, 1978.

This booklet contains the questions and solutions of the first ten Canadian Mathematics Olympiads. Each of the first five papers consists of ten questions. The number of questions in the remaining five varies between six and eight.

Problem 130

Let n be an integer. If the tens digit of n^2 is 7, what is the units digit of n^2?

The Canadian Mathematics Olympiads, 1979-1985, edited by C. M. Reis and S. Z. Ditor, Canadian Mathematical Society, 1987.

This booklet contains the questions and solutions of the Canadian Mathematics Olympiads from 1979 to 1985. Each paper consists of five questions.

Problem 131

$P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P(Q(x)) = Q(P(x))$ for all real numbers x. If the equation $P(x) = Q(x)$ has no real solutions, show that the equation $P(P(x)) = Q(Q(x))$ also has no real solutions.

1001 Problems in High School Mathematics, edited by E. Barbeau, M. S. Klamkin and W. Moser. Canadian Mathematical Society, 1985.

To date, half of this work has appeared in a preliminary version in the form of five booklets. In addition to problems and solutions, a mathematical "tool chest" is appended to each booklet.

Problem 132

The number 3 can be expressed as an ordered sum of one or more positive integers in four ways, namely, as 3, 1 + 2, 2 + 1 and 1 + 1 + 1. Show that the positive integer n can be so expressed in 2^{n-1} ways.

Winning Ways I, by E. R. Berlekamp, J. H. Conway and R. K. Guy, Academic Press, 1982.

This is the definitive treatise on mathematical games, As the subtitle "Games in General" suggests, the general theory of mathematical games is presented in this first volume, but there are also plenty of specific games to be analyzed, played and enjoyed. The book is written with a great sense of humor and is profusely illustrated, often in bright colors.

Problem 133

The army has been in disarray and the general has reduced all officers to the ranks and made everyone directly responsible to him. He now intends, on the alternate recommendation of his two military advisers, to recruit from outside the army a new hierarchy of officers. Each adviser, in turn, recommends that some officer (possibly the general himself) currently in direct charge of four or more officers and men should recruit a new subordinate. The new officer will be directly responsible to the one who appointed him and will, until further notice, take over direct responsibility for three or more, but not all, of those officers and men presently directly responsible to his appointer. Of course, no further appointment will be possible when every officer has either two or three direct subordinates. Whichever adviser makes the last recommendation retains the confidence of the general. In a seven-man army, does the first adviser or the second adviser win the confidence game?

Winning Ways II, by E. R. Berlekamp, J. H. Conway and R. K. Guy, Academic Press, 1982.

The subtitle of this volume is "Games in Particular." Here, all kinds of mathematical games, classical as well as brand new, are presented attractively. Most of them are two-player games. There are two chapters devoted to one-player games or solitaire puzzles and the book concludes with a chapter on a zero-player game!

Problem 134

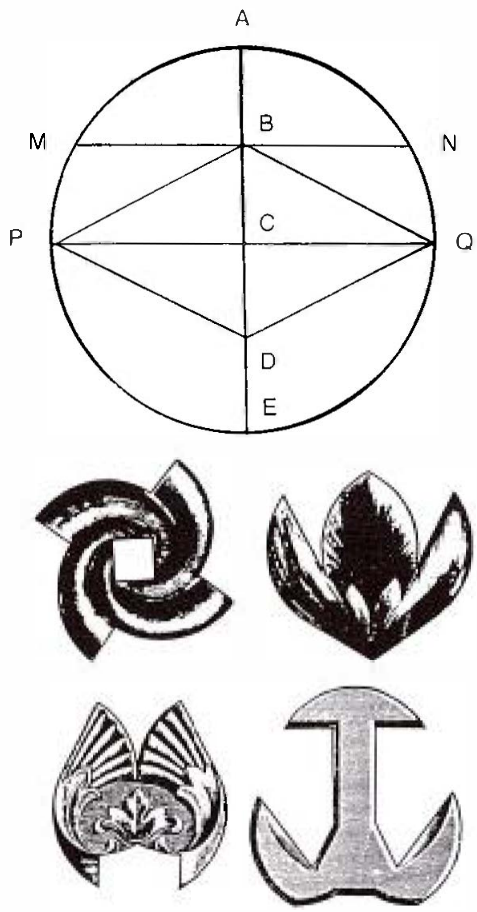
Pack one 2 by 2 by 2 block, one 2 by 2 by 1 block, three 3 by 1 by 1 blocks and thirteen 4 by 2 by 1 blocks into a 5 by 5 by 5 box.

Puzzles Old and New, by J. Slocum and J. Botermans, distributed by University of Washington Press, 1986.

Jerry Slocum has probably the largest puzzle collection in the world. This book features a small subset of his mechanical puzzles, that is, puzzles made of solid pieces that must be manipulated by hand to obtain a solution. They are classified into ten broad categories, with enough information to make most of them and to solve some of them. The book is full of striking full-color plates.

Problem 135

The Circular Puzzle, which dates back to 1891, consists of ten pieces of a circle as follows. *AE* is a diameter of a circle with centre *C*. *B* and *D* are the midpoints of *AC* and *CE*, respectively. *MN* and *PQ* are perpendicular to *AE* and passing through *B* and *C*, respectively. *P* and *Q* are then joined to *B* and *D*. Use the resulting ten pieces to construct each of the following four figures.



Problem 135

The Mathematical Gardner, edited by D. A. Klarner, Wadsworth International, 1981.

This book contains 30 articles dedicated to Martin Gardner for his sixty-fifth birthday. They reflect part of his mathematical interest and are classified under the headings, Games, Geometry, Two-Dimensional Tiling, Three-Dimensional Tiling, Fun and Problems, and Numbers and Coding Theory.

Problem 136

For what values of n can an n by n square be tiled using 2 by 2 squares and 3 by 3 squares?

Mathematical Snapshots, by H. Steinhaus, Oxford University Press, 1983.

This is an outstanding book on significant mathematics presented in puzzle form. Topics include dissection theory, the golden ratio, numeration systems, tessellations, geodesics, projective geometry, polyhedra, Platonic solids, mathematical cartography, spirals, ruled surfaces, graph theory and statistics.

Problem 137

An 8 by 8 square is to be divided into three rectangles with equal diagonal. What is the minimum common value of the length of the diagonal?

Mathematics Can Be Fun, by Y. I. Perelman, Mir Publishers, 1979.

This is a translation of two books in Russian, *Figures for Fun* and *Algebra Can Be Fun*. The former is an excellent collection of simple puzzles. The latter is a general discourse of algebra with quite a few digressions into number theory.

Problem 138

It is claimed that a tripod always stands firmly on a level surface, even when its three legs are of different length. Is that right?

Fun with Maths and Physics, by Y. I. Perelman, Mir Publishers, 1984.

The first half of this beautiful book describes a large number of interesting experiments in physics. The second half consists of a large collection of mathematical puzzles.

Problem 139

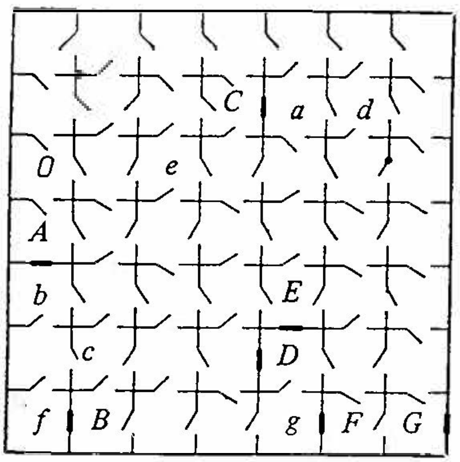
A hundred nuts are to be divided (not necessarily equally) among 25 people. Is it possible to arrange it so that each gets an odd number of nuts?

The Moscow Puzzles, by B. A. Kordemsky, Charles Scribner's Sons, 1972.

This is the translation of the outstanding single-volume puzzle collection in the history of Soviet mathematics. Many of the 359 problems are presented in amusing and charming story form, often with illustrations.

Problem 140

The dungeon has 49 cells. In each of cells *A* to *G*, there is a locked door (black bar). The keys are in cells *a* to *g*, respectively. The other doors open only from one side. How does a prisoner in cell *O* escape? The doors need not be unlocked in any special order and an open door can be passed through any number of times.



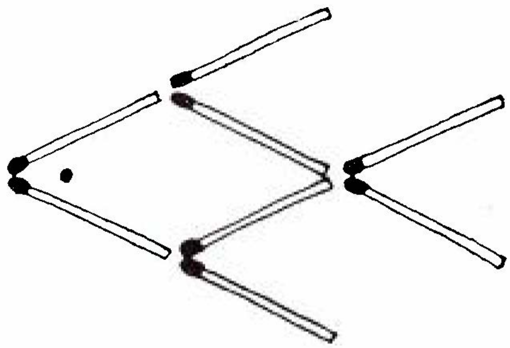
Problem 140

The Tokyo Puzzles, by K. Fujimura, Charles Scribner's Sons, 1978.

This is the translation of one of many books from the leading puzzlist of modern-day Japan. It contains 98 problems, most of them previously unfamiliar to the western world.

Problem 141

A tropical fish is swimming in a tank. The fish faces west. Make it face east by changing the positions of only three matches, in addition to the dot that represents the eye.



Problem 141

536 Puzzles and Curious Problems, by H. E. Dudeney, Charles Scribner's Sons, 1967.

This book is a combination of two out-of-print works of the author, *Modern Puzzles* and *Puzzles and Curious Problems*. Together with Dover's *Amusements in Mathematics*, they constitute a substantial portion of Dudeney's mathematical problems. Those in this book are classified under three broad headings, arithmetic and algebra, geometry, and combinatorics and topology.

Problem 142

A man entered a store and spent one-half of the money on him. When he came out, he found that he had just as many cents as he had dollars when he went in and half as many dollars as he had cents when he went in. How much money did he have on him when he entered?

Science Fiction Puzzle Tales, by Martin Gardner, Clarkson N. Potter, 1981.

This is the first of three collections of Martin Gardner's contribution to Isaac Asimov's *Science Fiction Magazine*. The book contains 36 mathematical puzzles in science fiction settings. When solutions are presented, related questions are often raised.

Problem 143

Dr. Moreau III produced a new type of microbe which multiplies at an alarming rate. Every hour, a single microbe replicates into seven microbes. One day, he put a single microbe, just "born," into a large and empty glass container. Fifty hours later, the container was completely filled. How many hours had elapsed when the container was 1/7 full?

Puzzles From Other Worlds, by Martin Gardner, Vintage Books, 1984.

This is the sequel to *Science Fiction Tales* and the predecessor of New Mathematical Library's *Riddles of the Sphinx*. It contains 37 mathematical puzzles plus further questions raised in the answer sections.

Problem **144**

VOZ, the computer on the spaceship Bagel, was getting bored. So it began to search for palindromic primes, that is, prime numbers which read the same in either direction. The first few VOZ found (not counting single-digit primes) are 11, 101, 131, 151, 181, 191, 313, 353, 373, 383, 727, 757, 787, 797, 919, 929 and 10301. There are no four-digit palindromic primes. Apart from 11, can there be a palindromic prime with an even number of digits?

J. Addresses of Publishers

Academic Press Inc.
465 South Lincoln Drive, Troy, MO 63379

Alfred A. Knopf
(a division of Random House)
see Random House

Australian Mathematics Competition
Canberra College of Advanced Education, P.O. Box 1, Belconnen, A.C.T. Australia 2616

Canadian Mathematical Society
577 King Edward Avenue, Ottawa, ON K1N 6N5

Charles Scribner's Sons
866 Third Avenue South, New York, NY 10022

Clarkson N. Potter
225 Park Avenue, New York, NY 10003

Dover Publications Inc.
31 East Second Street, Mineola, NY 11501

Mathematical Association of America
1529 Eighteenth Street NW, Washington, DC 20036

Oxford University Press
70 Wynford Drive, Don Mills, ON M3C 1J9

Prentice-Hall Canada Inc.
1870 Birchmount Road, Scarborough, ON M1P 2J7

Princeton University Press
41 William Street, Princeton, NJ 08540

Progress Books
71 Bathurst Street, Third Floor, Toronto ON M5V 2P6

Note—The addresses given are those to be used for orders.

Prometheus Books
700 East Amherst Street, Buffalo, NY 14215

Random House
1265 Aerowood Drive, Mississauga, ON L4W 1B9

Simon & Schuster
1230 Avenue of the Americas, New York, NY 10020

University of Chicago Press
11030 South Langley Avenue, Chicago, IL 60628

University of Washington Press
P.O. Box 50096, Seattle, WA 98145-5096

Vintage Books
(a division of Random House)
see Random House

W. H. Freeman & Company
4419 W. 1980 S., Salt Lake City, UT 84104

Wadsworth Publishing Company
7625 Empire Drive, Florence KY 41042

John Wiley & Sons Canada Ltd
22 Worcester Road, Rexdale, ON M9W 1L1

William Morrow & Company Inc.
39 Plymouth Street, Fairfield, NJ 07006

We have not reviewed any books from the National Council of Teachers of Mathematics or Creative Publications, Inc., since they are likely to be familiar to members of the teaching profession. Those addresses are—

National Council of Teachers of Mathematics
1906 Association Drive, Reston, VA 22091

Creative Publications, Inc.
P.O. Box 10328, Palo Alto, CA 94303

Appendix IV: A Selection of Resource Material

As in Appendix III, this list is restricted to material which is still available as far as is known and which has been used in the "SMART" program. More extensive listings and intensive treatments may be found in the publication *World Game Review.*

A. Mathematical Games

Many of the games described here also provide the milieu for countless puzzles, but we will concentrate on their gaming aspects.

Kadon's "Lemma" is the brainchild of Kathy Jones. It can best be described as a meta-game, in which the specific rules change from game to game and evolve gradually within each game in a logically consistent manner. It challenges the creativity of the players.

Avalon Hill's "Sleuth" is a multi-player game in which a deck of special cards is divided among the players, except for one card whose identity is to be deduced. Each player can examine the cards in hand and there are rules which allow for the examination of other players' cards.

Parker Brothers' "Black Box" is a game in which two players assume distinct roles. One player sets up on the playing board a secret configuration of four or five counters, the locations of which the other player tries to discover. The information-gathering mechanism utilizes geometric reflexion in an ingenious way.

In Kadon's "Colormaze," each player has a secret configuration which is to be constructed on the playing board, utilizing counters contributed by all players. The counters can be manipulated in various ways as players inadvertently aid or hinder one another while trying to attain their own objectives.

One of the most popular configuration games is undoubtedly Tic-Tac-Toe. Mag-Nif's "Re" is a non-rectangular, three-dimensional variation in which each stack of counters must be built from the ground up.

The classic game Hex exemplifies another form of configuration game, which requires the building of connected chains linking a pair of opposite edges of the playing board. Here, two players work at cross purposes, since the only way to prevent the opponent from forming a chain is to complete one's own. Avalon Hill's "Twixt" is another game in this category, where the chains are formed along knight moves.

Kadon's "Octiles" has an unusual playing board formed of 17 octagonal tiles that are continually being rotated or replaced by an eighteenth tile. The object of the game is to advance counters from one edge of the board to the opposite one along paths on the tiles, while opposing players get in the way either by blocking the paths with their counters or by altering the board.

The oriental game "Go" utilizes connectivity to enclose and control space on the playing board. Mattel's "Cathedral" is a variation with polyomino-shaped pieces in the form of attractive medieval buildings.

As its name implies, the theme of Lakeside's "Isolation" is the destruction of connectivity. As the game progresses, the squares of the playing board disappear, until one player's counter can make no legal move.

Gabriel's "Point Blank" is another last-move-wins type of game. Each of two players builds a connected path until one player runs out of room. Jeremy Jackson, a Grade 6 student in Garneau Elementary School in Edmonton, discovered a winning strategy for the second player; so a special rule must be added to make that strategy inoperative.

In the game Nim, players take turns removing counters, with the last to move being the winner. Perfect strategies can be formulated from an analysis based on the binary system. Mag-Nif's "Psych Out" is a compact physical version of the game.

Milton Bradley's "Domination" is also a last-move-wins type of game, but here the winning way consists of preventing the opposing players from moving by dominating their counters or, better still, by capturing them.

Lakeside's "Shogun" is a magnetic version of chess. Each square of the playing board is magnetized, as is each of the counters. The mobility of each counter is determined by a number which depends on both the counter and its location. The objective is to checkmate the opponent's king.

"Quantum," available from Kadon, is a fantastic combination of chess and checkers. The initial set-up is randomized by shaking the specially constructed playing board until the weighted counters fall in place. Each counter begins as a checker but turns into a chess king or queen after its first move. The objective is to occupy the centre of the board.

The "Game of Solomon" is the invention of Martin Gardner. It is a version of checkers on a compact board which forces immediate interaction. It can also be played as a Nim-type game. It is available from Kadon.

B. Two-dimensional Polyforms

The polyominoes, polyiamonds and polyhexes, discussed in Kathy Jones' article, provide an endless number of puzzles, both entertaining and aesthetically pleasing. They are by far the most versatile of all puzzle sets.

Kadon's "Poly-5," "Sextillions" and "Heptominoes" are the finest series of the polyominoes in the market. They are made of high-quality acrylic, are laser-cut for perfect fit and are compatible with one another. The first consists of all polyominoes up to and including the pentominoes. The second consists of the hexominoes together with an instructive booklet. The third series, available by special order only, consists of the 108 heptominoes which are unavailable elsewhere.

Tenyo has put out a series of plastic puzzles under the collective title "Beat the Computer." It includes a double set of the tetrominoes, a set of the pentominoes, a set of the hexominoes, a set of the hexiamonds, a set of the heptiamonds and a set of the pentahexes. They are attractively made but are too small for easy manipulation. The pentominoes are also available from Gabriel in a set named "Hexed."

The polyominoes, polyiamonds and polyhexes are based on the regular tilings of the plane using squares, equilateral triangles and regular hexagons, respectively. If only one kind of regular polygon is used and the polygons meet edge-to-edge, these are the only possible tilings. However, one can relax the conditions in a number of ways.

If two or more kinds of regular polygons are allowed, one can obtain semi-regular tilings. One of them has two octagons and a square meeting at each corner. This forms the basis of Kadon's "Super Roundominoes," a wonderful set in six splendid colors.

On the other hand, we may restore the condition that only one kind of polygon be used, but allow the prototype to be non-regular, such as an isosceles right triangle. The popular Tangram is derived from this tiling. A nice wooden set is available from Pentangle while an inexpensive set is available from Setsco. There are several variations of this puzzle, including Kadon's "Grand Tans" and Milton Bradley's "Boxed In."

C. Three-dimensional Polyforms

There are two broad categories of these puzzles, the polycubes and the polyspheres. Of the former, there is the classic Soma Cube. While this is available in many forms, a nice wooden set may be ordered from Sivy Farhi or from Dale Seymour Publications. Lakeside has put out six variations collectively called the "Impuzzables." Pentangle has one called "Question Mark."

Kadon's flagship is the set "Quintillions" which consists of the pentominoes with the appropriate thickness, so that each piece may be considered as being formed of five unit cubes. This allows for three-dimensional constructions. Pentangle has a colored set called "Super Pentacubes."

Kadon has a companion set, "Super Quintillions," containing all non-planar pentacubes. A set called "Polycube Supplement," which consists of all polycubes up to and including the tetracubes, may be obtained by special order. Finally, there is the magnificent "Hexacubes," all 166 of them, which together with four unit cubes fill a 10 by 10 by 10 box! The Kadon sets are all made of fine hardwood and are laser-cut and compatible with one another.

Mag-Nif's "Tut's Tomb" is a simple but elegantly intriguing pyramid puzzle, consisting of 20 spheres in four pieces. "Perplexing Pyramid," distributed by Kadon, is the same size but more difficult. Even more difficult polysphere puzzles from Kadon are "Big Pyramid," "Giant Pyramid" and the excellent four-in-one puzzle "Warp 30."

D. Mathematical Jigsaw Puzzles

Jigsaw puzzles are characterized by the jagged edges of the pieces which interlock with one another. This may be considered as a physical way of imposing conditions on what pieces may be adjacent and in what combination.

Kadon's "Stockdale Super Square" is a 6 by 6 configuration in the standard mode, but the jagged edges are beautiful, heart-shaped patterns. The set is much more than a jigsaw puzzle. Its instruction booklet is full of suggestions for games and puzzles.

Milton Bradley's "It's Knot Easy" consists of 16 square pieces each containing the picture of part of a piece of rope. They are to be put together in a 4 by 4 configuration so that the pieces link up into a knot.

Pentangle's "Perplexing Python" is the same idea extended to the third dimension. Eight cubes are to be assembled into a 2 by 2 by 2 configuration, showing a python from head to tail.

Heye has put out a series of "Crazy" puzzles. "Crazy Dogs" consists of nine squares each showing four halves of dogs terminating at the edges of the squares. The squares are to be put into a 3 by 3 configuration and two squares may be adjacent if the two halves on either side of the common edge form a complete dog. "Crazy Cats" consists of nine equilateral triangles to be assembled into a larger equilateral triangle. There are a dozen or so puzzles in this series.

Mag-Nif's "On the Level" consists of nine square pillars. The cross-section of each is divided by the diagonals into four quadrants. Each individual quadrant may be at one of three levels. The object is to form a 3 by 3 configuration where adjacent quadrants along a common edge must be on the level with each other.

Kadon's "Hexmozaix" consists of twelve hexagons. Each is subdivided into six sections in three colors, with two sections of each color. They are used for many games and puzzles, all calling for the hexagons to be assembled in such a way that the colors match along common edges.

The same principle applies in Kadon's "Triangoes" which uses tangram-type pieces in five distinctive colors to fill up a beautiful playing board so that the colors on the pieces match those on the board. On the reverse side is an uncolored playing board which is used for many games and puzzles.

E. Rubik-type Puzzles

Rubik's Cube and related puzzles are an outstanding invention of all time in recreational mathematics. Unfortunately, the fad has faded and devotees find it increasingly difficult to obtain these puzzles. Very few new ones are being made.

Fortunately, some are still available from Cubes International, including the "Cube Ultimate" (5 by 5 by 5) which never made it into the mass market in North America. There are also "Rubik's Revenge" (4 by 4 by 4) and the original classic (3 by 3 by 3), as well as the wonderful "Skewb," a cube with an unusual turning motion. Others include "Impossiball," "Tower of Babylon," "Alexander's Star," "Pyraminx," "Magic Dodecahedron," "Magic Domino" and "Ten Billion Barrels." Some of these are also available from Pentangle.

F. Take-apart Puzzles

There are two broad categories of these puzzles. The first category may also be described as disentanglement puzzles. Outstanding in this field are four puzzles from Binary Arts. As the name of the Corporation suggests, the binary system has a lot to do with the solution of these puzzles.

Both the "Cat" and the "Horse" are fine sculptures of wood and wire and the object is to remove a loop from the wire form. "Spinout" is a splendid variation of the classic Tower of Hanoi puzzle. Here a tray of seven turnable discs is to be disengaged from the base. "Hexadecimal" is a more elaborate sixteen-in-one puzzle. The "Brain" from Mag-Nif is based on the same principle but, unlike the products of Binary Arts, it does not give the solver a full view of the inner structure of the puzzle.

The second group of take-apart puzzles consists of those in which the whole structure decomposes into its component parts. A wide selection is available from Stewart Coffin.

"Mayer's Cube" from Pentangle is a 4 by 4 by 4 cube with a 1 by 1 by 1 interior hole. It is made of six interlocking pieces. It is not easy to take it apart and harder to reassemble it.

Mag-Nif has a series of these puzzles. The simpler ones are the "Magic Box" (two pieces), "Curious Cross" (three pieces), "Four Squares" (four pieces) and "Astrologic" (six pieces).

Kadon's "Icosatriad" is a great dodecahedron composed of twelve wing-shaped pieces with three in each of four colors. They are held together by six frog-shaped pieces that form a hidden cube. The puzzle comes in a cage consisting of four interlocking hexagonal rings that form the skeleton of a cuboctahedron.

G. Mathematical Models

''Rubik's Snake'' from Ideal is different from the Rubik-type puzzles in that it is less of a puzzle and more of construction set. Jan van de Craats' ''De Slang van Rubik'' (in Dutch) is a beautiful picture book of 101 symmetric designs for this puzzle.

Mag-Nif's ''Tri-logic'' consists of 24 triangular plates that can be snapped together along their edges to form various polyhedra. Several sets can be combined for larger constructions.

"M. C. Escher Kaleidocycles," published by Pomegranate Artbooks, is a wonderful collaboration of Doris Schattschneider and Wallace Walker. It consists of a set of cardboard models (glue not supplied) and a book. The models fall into two sets. The first consists of the five Platonic Solids plus one of the Archimedean Solids. They are attractively decorated with colored versions of Escher's graphic designs. The mathematics of the Escher tilings and their association with both sets of the models are discussed in the book.

The second set of models consists of the mathematical objects in the title role, the kaleidocycles. Each is a loop of tetrahedra linked only along common edges. The loop can be twisted continuously, with different groups of faces coming into view in a forever changing kaleidocycle.

H. Addresses

We have not listed addresses of companies such as Mattel whose products are widely available. We do not have the addresses of Tenyo or Heye, their products being handled by importers.

Avalon Hill Game Company
4517 Harford Road, Baltimore, MD 21214

Binary Arts Corporation
703 Timber Branch Drive, Alexandria, VA 22302

Cubes International
Haarholzer Strasse 13, D4630, Bochum 1, West Germany

Dale Seymour Publications
P.O. Box 10888, Palo Alto, CA 94303

Kadon Enterprises Inc.
1227 Lorene Drive #16, Pasadena, MD 21122

Mag-Nif Inc.
8820 East Avenue, Mentor, OH 44060

Pentangle
Over Wallop, Hampshire, United Kingdom
SO20 8HT

Pomegranate Artbooks Inc.
P.O. Box 980, Corte Madera, CA 94925

Setsco Educational Ltd.
567 Clarke Road, Coquitlam, BC V3J 3X4

Sivy Farhi
815 South California Avenue #B, Monrovia, CA 91016

Stewart Coffin
79 Old Sudbury Road, Lincoln, MA 01773

World Game Review
3367-I North Chatham Road, Ellicott City, MD 21043

Appendix V: Answers and Solutions

A. Supplementary Problems in Appendix I

Problem 1

Pretend that the pebbles have hands and that every two shake hands when they are separated. When a heap of pebbles is divided into two, the product generated simply counts the number of handshakes that are caused by that division. At the end, every two pebbles shake hands exactly once. Since there are 1,000 pebbles, the total number of handshakes is $1000 \cdot 1001/2 = 500500$. This is the desired sum, regardless of what sequence of divisions is used.

Problem 2

Consider an arbitrary problem and let it be solved by exactly r contestants. These contestants solve $6r$ other problems (counting multiplicities). Each of the remaining 27 problems is counted twice in $6r$, so that $r = 2 \cdot 27/6 = 9$. It follows that each problem is solved by exactly 9 contestants, and the number of contestants is $9 \cdot 28/7 = 36$.

Suppose every contestant solves one, two or three problems in Part I. Let n be the number of problems in Part I, and x, y and z be the respective numbers of contestants who solve one, two and three of these problems. Then

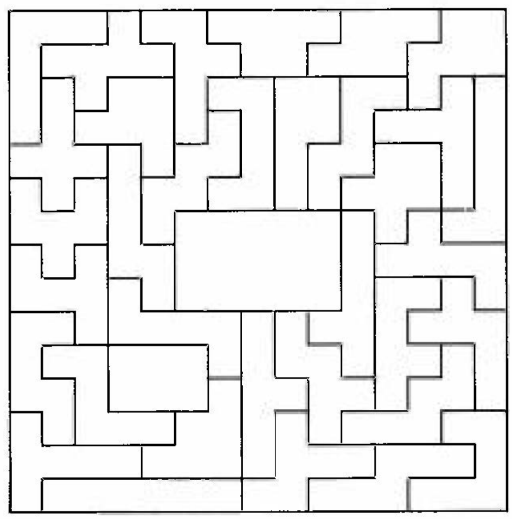
(1) $x + y + z = 36$,

(2) $x + 2y + 3z = 9n$,

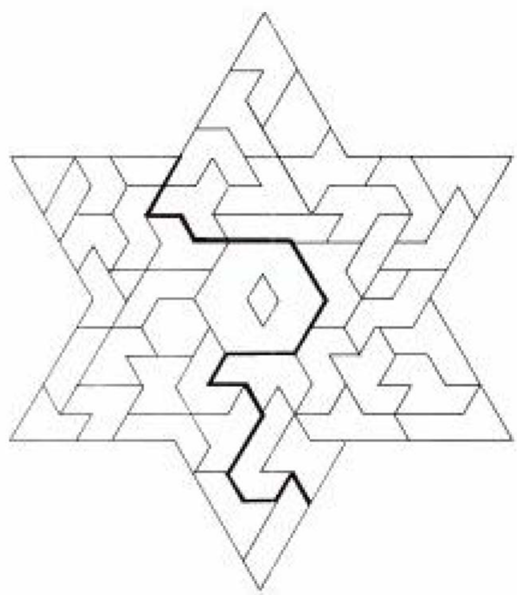
(3) $y + 3z = 2\binom{n}{2}$.

The last equation arises from the observation that, for every pair of problems in Part I, exactly two contestants solve both of them. Multiplying (1), (2) and (3), respectively, by -3, 3 and -2 and adding the resulting equations, we have $y = -2n^2 + 29n - 108 = -2(n - 29/4)^2 - 23/8 < 0$, an impossible situation. Hence at least one contestant solves four or more problems in Part I and consequently three or fewer problems in Part II.

Problem 3



Problem 4



Problem 5

0 5 10 15 20 25 30 35 . . .
1 6 11 16 21 26 31 36 . . .
2 7 12 17 22 27 32 37 . . .
3 8 13 18 23 28 33 38 . . .
4 9 14 19 24 29 34 39 . . .

We write the non-negative integers in five rows as shown in the above table. The first five multiples of 8 are underlined.

Note that since gcd(5,8) = 1, there is one in each row. Clearly, a multiple of 8 is a sum of 8's. Moreover, any number to the right of a multiple of 8 can be expressed as a sum of 8's and 5's, since the numbers go up by 5 at a time in each row. In particular, all numbers in the first row can be so expressed. In each of the remaining rows, we claim that no number to the left of the first multiple of 8 can be expressed as a sum of 8's and 5's. Otherwise, there will be a smallest such number. The sum for this number cannot consist only of 8's, as we have not yet hit a multiple of 8. Hence there is at least one 5 in the sum. Knocking off a 5 yields a valid expression for the previous number in the row and our number could not have been the smallest in the first place. This contradiction justifies our claim.

Hence the numbers that cannot be expressed as a sum of 8's and 5's are 1, 2, 3, 4, 6, 7, 9, 11, 12, 14, 17, 19, 22 and 27.

Problem 6

(a) A prime triple with common difference 2 is (3,5,7).

(b) There are no others. If (x,y,z) is a prime triple with common difference d and d is not divisible by 3, then one of x, y and z must be divisible by 3.

(c) There are none. If (x,y,z) is a prime triple with common difference d and d is not divisible by 2, then one of x, y and z must be divisible by 2.

(d) A prime triple with common difference 4 is (3,7,11).

(e) There are no others. See (b).

(f) There are none. See (c).

(g) A prime triple with common difference 6 is (5,11,17).

(h) Another prime triple with common difference 6 is (7,13,19). There are lots of others.

Problem 7

The figure has 22 edge squares, but the pentominoes can cover at most 21 of them. Hence the construction is impossible.

Problem 8

The construction is impossible since the figure has only 59 squares.

Problem 9

We call an ordered pair (n,m) admissible if $m,n\in[1,2,\cdots,1981]$ and $(n^2-nm-m^2)^2=1$. If $m=1$, then (1,1) and (2,1) are the only admissible pairs. Now for any admissible pair (n_1,n_2) with $n_2>1$, we have $n_1(n_1-n_2)$ $=n_2^2\pm 1>0$, so that $n_1>n_2$. Define $n_3=n_1-n_2$; then $n_1=n_2+n_3$, and substituting this into the original equation, we have $1=(n_1^2-n_1n_2-n_2^2)^2$ $=((n_2+n_3)^2-(n_2+n_3)n_2-n_2^2)^2$ $=(-n_2^2+n_2n_3+n_3^2)^2=(n_2^2-n_2n_3-n_3^2)^2$. Thus (n_2,n_3) is also an admissible pair. If $n_3>1$, then, in the same way, we conclude that $n_2>n_3$; and, letting $n_2-n_3=n_4$, we find that (n_3,n_4) is an admissible pair. Thus we have a sequence $n_1>n_2>n_3>\cdots$ (necessarily finite) such that $n_{i+1}=n_{i-1}+n_i$, and where (n_i,n_{i+1}) is admissible for all i. The sequence terminates if $n_i=1$. Since $(n_{i-1},1)$ is admissible and $n_{i-1}>1$, we must have $n_{i-1}=2$. Therefore, (n_i,n_{i-1}) consists of consecutive terms of the truncated Fibonacci sequence 1597, 987,..., 13, 8, 5, 3, 2, 1. Conversely, any such pair is admissible.

Every step of this construction is reversible; so, running it backwards from (2,1), determines uniquely the Fibonacci sequence 1, 2, 3, 5, 8, 13, ..., 987, 1597, which contains, as adjacent members, all admissible pairs. The largest such pair not exceeding 1981 is (1597,987); so the maximum value of m^2+n^2 is 1597^2+987^2.

Problem 10

We make the following two claims—

(1) There are at least three ways of placing 72 superknights on a superboard such that no two attack each other.

(2) If there exists a re-entrant superknight tour on a superboard, then there are at most two ways of placing 72 superknights on a superboard such that no two attack each other.

It will then follow that there are no re-entrant Superknight tours on a Superboard.

Proof of (1)—
Color the superboard in the usual checkerboard fashion.

We can then place the 72 superknights either on all the white squares or on all the black squares. In addition, we can place them on all the squares in the 1st, 2nd, 6th, 7th, 11th and 12th rows.

Proof of (2)—
Let 72 superknights be placed on the superboard such that no two attack each other. Then along the re-entrant superknight tour, there must be at least one vacant square between two occupied squares. Since there are 72 of each, vacant and occupied squares must alternate along the tour. Hence there are at most two possible placements of the superknights.

B. Contest papers in Appendix II

Year 1957

1, $(3 + x^2)/x(1 - x)(1 + x)$
2, 5
3, 29/2 metres
4, -2
5, $(-1 \pm \sqrt{3}i)/2$
6, 2, $-1 \pm 2i$
8, 2.4 by 2.4
9, $x = 0$, $y = 1$, $z = 2$
10, no solutions
11, (a) 3 milligrams
(b) $3/\sqrt{2}$ milligrams
12, $(4\pi - 3\sqrt{3})/6$
13, 48 kph
14, 9
18, 36%

Year 1958

1, 241/168
2, $ac(2c^2 + ac + b)/(ac + b)(ac - b)$
3, 255
4, $(5 \pm \sqrt{5})/2$
5, $(7 \pm \sqrt{5})/2$
6, $x = 9/25$, $y = -14/25$, $z = 26/25$
8, 17
9, (a) 91390
(b) 84645
11, $(3^{10} + 1)/2 + 2^{10}$
12, 2.962
14, -1, 3/2(repeated)
18, $x = 3(-1 \pm 4\sqrt{17}i)/13$,
$y = (1 \pm 4\sqrt{17}i)/13$, or
$x = (27 \pm 4\sqrt{3}i)/37$,
$y = 3(27 \pm 4\sqrt{3}i)/37$
19, $pr(r^5 - 1)/(r - 1)$ dollars

The omission of a problem number or subsection signifies that the question requires a proof.

Year 1959

1, 2
2, 2, -2, -3
3, $x = 4$, $y = 6$, $z = 8$
4, (a) $a = c$
(b) $b = 0$
6, intersection of AC and BD
7, 480
10, yes
11, no
12, $(\sin^2\theta)/2$
13, 635/2
15, 216
16, (b) $\sqrt{ab}$
18, (c) $(2 + a/2, 0)$, $x = 2 - a/2$

Year 1960

1, 13
2, 194
3, $\sqrt{3}/4$
4, (a) 2
(b) $\pm 3/2$
5, $3x^3 + 7x^2 - 4 = 0$
6, -1, $(5 \pm \sqrt{21})/2$
7, (b) no
8, (a) $x(x^4 - x^2 + 1)/(x^4 + 1)(x + 1)(x - 1)$
(b) $(1 - 2x^2 - 3x^4 - x^6)/x(x^2 + 1)(x^2 + 2)$
11, $12(\sqrt{3} - 1)$ km, S45°W
13, (a) 45
(b) 9
(c) 120
(d) 36
(e) 8
14, 127
17, (a) 682/27 metres
(b) 30 metres
19, no

Year 1961

1, $(x^5 - x^4 + 8x^3 - 24x^2 + 7x - 76)$
$/x(x^2 + 1)(x^2 + 7)$
2, $(2a - a^2c + b^2c)/\sqrt{(a + b)(a - b)}$
3, 68
4, no solutions
5, $-\log_{10}2$
6, $x > 1$ or $x < 0$
7, 36
9, a line perpendicular to AB
13, (b) $-1/\sqrt{3}$
14, 9
15, 49/333
19, (a) 1680
(b) 1280
20, 25!/10!
21, 0°, 60°, 180°, 300°
22, 30°, 150°, 210°, 330°

Year 1962

1, 3/2
2, (a) $a = 2$, $b = 3$
(b) $A = 2$, $B = 3$
3, −4, 5
4, no solutions
5, −7
7, 0
9, 0°, 120°, 240°
10, 15°, 30°, 75°, 90°, 150°, 195°, 210°, 255°, 270°, 330°
11, 90°, 210°, 270°, 330°
12, (a) $1 - 1/n$
(b) 102
13, $ax^2 - (4a - b)x + (4a - 2b + c) = 0$
15, (a) 7
(b) 105
21, (b) $t^2 - 2$
(c) $2 \pm \sqrt{3}$, $(-3 \pm \sqrt{5})/2$
22, (b) $A = B$

Year 1963

1, $-2x/(x^2 + 1)$
2, $a/(1 + 2x)$
3, x
4, $16x^2 + 29$
5, $(a(17 + 12\sqrt{2}) + b(7 - 4\sqrt{3}))/(7 - 4\sqrt{3})$
$(17 + 12\sqrt{2})$
6, $x = 35$, $y = 45$ or $x = 45$, $y = 35$
7, 1600
8, $4/N$
9, 125/2 kph, 200/3 kph
10, 8
12, $4\sqrt{52}$ metres
13, (c) $6n^5 + 2n^3$
15, (b) $a + ax(1 - x^{n-1})/(1 - x)$
$-(n^2 + 2n - 1)\,ax^n + n^2ax^{n+1}$
16, (a) 15/2 metres
(b) 35/6 metres
17, (c) $2 \pm \sqrt{3}$, $-2 \pm 2\sqrt{2}$
18, $\sqrt{3}ab/4$
21, (b) $8/\tan 8x$

Year 1964

1, $25(3 + \sqrt{17})/2$
2, (a) 6, 8, 10
3, (a) 126 cm
(b) 2
4, (b) no modification
5, $-\log_{10}3$
7, (b) 64
8, −9/4
9, $4x^2 + x + 1 = 0$
11, 5/3, 65/6, 20, 175/6, 115/3
13, (a) $\pm(3 - \sqrt{5}i)$
16, 20 km
17, 3.9979
18, $3b^8/8a^5$
19, 858/20825

Year 1965

1, 2
4, $a = 1/2$, $b = \sqrt{3}/2$
5, 4
6, $2p$
7, −1, −3, 4
8, (b) $n^2 - (ah/(a - h))^2 = m^2 - a^2$
12, 45 kph
15, (c) $1/2(n - 1) - 1/n + 1/2(n + 1)$
(d) $(n - 1)(n + 2)/4n(n + 1)$
16, $24\pi^2 + 4\pi$ metres^2
17, $3\sqrt{3}$ metres

Year 1966

1, $(x + 1)(x^2 - x + 1)(x^2 + x + 1)$
5, $1 < x < 2 + \sqrt{3}$ or $-1 < x < 2 - \sqrt{3}$
6, $x = 1$, $y = 2$, $z = 4$ or $x = 16/13$,
$y = 22/13$, $z = 53/13$
9, (a) $b > -2$
11, (a) 57/16
(c) 4
14, 6 cm, 6 cm, $15\sqrt{7}/2$ cm
16, (a) $2x^2/(1 - x)(1 + x)$

C. Sample Problems in Appendix III

Problem 1

Just above the South Pole, there is a circle C_1 with circumference one mile. If the explorer starts from any point X one mile north of C_1, walking one mile south will take him to a point Y on C_1. Walking one mile east will take him once around C_1 back to Y, and walking one mile north will return him to X. We can use the same argument with the circle C_2 with circumference 1/2, the circle C_3 with circumference 1/3, and so on.

Problem 2

The distance between the two missiles is diminishing at a rate of 30000 miles per hour or 500 miles per minute. Thus one minute before they collide, they must have been 500 miles apart. The information that they were 1317 miles apart initially is redundant.

Problem 3

Pretend that there is a second monk who ascends the mountain on the day the first monk comes down and that his ascent duplicates exactly that of the first monk. Somewhere along the path, the two monks must pass each other. This is the spot we seek.

Problem 4

(Answer) 29786 + 850 + 850 = 31486.

Problem 5

Let the six stars be A, B, C, D, E and F. Suppose A loves at least three of the others, say B, C and D. If any pair of B, C and D loves each other, these two with A will form a love triangle. If not, then B, C and D form a hate triangle. If A loves at most two of the others, then A hates at least three of the others, and the same argument yields the desired result.

Problem 6

Let the distances be as indicated in the diagram (not drawn to scale). By Pythagoras' Theorem, $18000^2 + 21000^2 = (x^2 + y^2) + (w^2 + z^2)$ $= (w^2 + y^2) + (x^2 + z^2) = 6000^2 + d^2$. From this, we obtain $d = 27000$.

Problem 7

Since the number of points selected is finite, we can find a direction such that no two of the selected points lie on a line in that direction. Draw a line in that direction. If more than 500000 of the selected points lie on one side of it, move the line toward that side by parallel displacement. Since the moving line passes over the selected points one at a time, it eventually arrives at a position with exactly 500000 of the selected points on each side.

Problem 8

Deal from the bottom of the deck, first card to himself and continue counterclockwise.

Problem 9

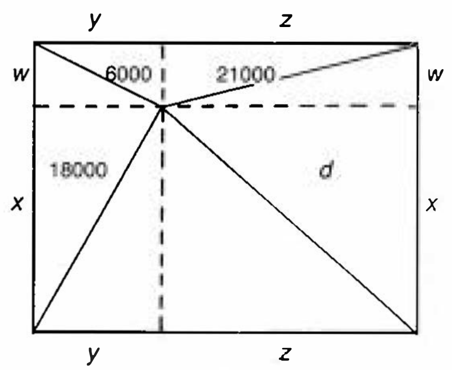
Let the weights be R_1, R_2, W_1, W_2, B_1 and B_2. Weigh R_1 and W_1 against R_2 and B_1. If they balance, weigh R_1 against R_2. If R_1 is heavier, then the heavy ones are R_1, W_2 and B_1. Otherwise, they are R_2, W_1 and B_2. If R_1 and W_1 are heavier than R_2 and B_1 in the first weighing, weigh W_1 against B_2. If they balance, the heavy ones are R_1, W_1 and B_2. If W_1 is heavier than B_2, the heavy ones are R_1, W_1 and B_1. If W_1 is lighter than B_2, the heavy ones are R_1, W_2 and B_2. The case where R_1 and W_1 are lighter than R_2 and B_1 in the first weighing can be dealt with similarly.

Problem 10

(Answer) $n < 1000000$.

Problem 11

Since there are 10 people at the party and nobody shakes hands with himself or herself or with his or her spouse, the number of handshakes is at most 8. Since each of nine people gives a different answer,



Problem 6

the answers are, collectively, 0, 1, 2, 3, 4, 5, 6, 7 and 8. Clearly, the person who shakes 8 hands is married to the person who shakes 0 hands. Eliminating this couple, we can see that the one who shakes 7 hands is married to the one who shakes 1 hand, and so on. Thus the narrator's wife shakes 4 hands.

Problem 12

(Answer) The seven points divide the perimeter into seven equal parts.

Problem 13

Suppose A does say that there is exactly one knight among them. Then B's statement is true and C's is false, making the former a knight and the latter a knave. However, we now see that this leads to a contradiction. If A is a knight, then there will be two knights and A's statement will be false. If A is a knave, then there will be one knight and A's statement will be true. It follows that B is a knave and C is a knight. We cannot determine whether A is a knight or a knave.

Problem 14

Consider all the pieces (on both sides) that move on black squares. It is possible for a pawn to be captured *en passant* by a pawn moving on white squares, but the captured pawn itself could not have made any captures. Hence the last piece moving on black squares that has captured another piece moving on black squares is still on the board. It cannot be either of the white pawns as they obviously have not moved. If the white king has moved, it must be by castling to *g*1, but then it can never get back to *e*1. Thus the white king also has not moved. Therefore, the white bishop must be on a black square since the black king is not. Hence the white bishop is on square *e*3.

Problem 15

(Answer) The white king is on square *c*3. Two moves ago, the white king was on *b*3, a white pawn was on *c*2 and a black pawn was on *b*4. As the black bishop gave check, the white pawn blocked and was captured *en passant*.

Problem 16

If the first sign is true, then the second must also be true. Hence the first sign is false and the lady is in the second room.

Problem 17

If the Cook is mad, then one of the two is indeed mad and the mad Cook will be believing something true. Hence the Cook is sane and at least one of the two is mad. The mad one must be the Cheshire Cat.

Problem 18

For B and C, there is a bird E such that $B(C(x)) = E(x)$ for every x. For A and E, there is a bird D such that $A(E(x)) = D(x)$ for every x. Hence $A(B(C(x))) = A(E(x)) = D(x)$.

Problem 19

(Answer) "I will not get exactly one prize."

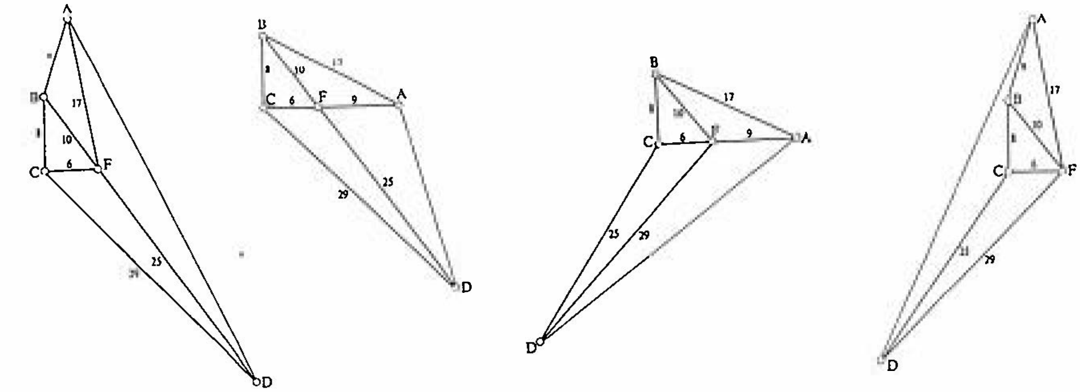
Problem 20

Let a, b and c be the sides of a triangle whose area is numerically equal to its perimeter. If s denotes the semi-perimeter, then the area is given by $\sqrt{s(s-a)(s-b)(s-c)}$ and the perimeter by $2s$. Setting $x = s-a$, $y = s-b$ and $z = s-c$, we have $xyz = 4(x + y + z)$. We may assume that $x \le y \le z$. If $x \ge 4$, then $xyz \ge 16z$ while $4(x + y + z) \le 12z$. Hence $x \le 3$. If $x = 3$, we have $3xy - 4y - 4z = 12$ or $(3y - 4)(3z - 4) = 52$. We cannot have $3y - 4 = 1$ and $3z = 52$, nor can we have $3y - 4 = 4$ and $3z - 4 = 13$. If $3y - 4 = 2$ and $3z - 4 = 26$, then $y = 2$ and this contradicts $x \le y$. Thus there are no integral solutions in this case.

If $x = 2$, we have $(y - 2)(z - 2) = 8$ and (y,z) is one of (3,10) and (4,6). If $x = 1$, then $(y - 4)(z - 4) = 20$ and (y,z) is one of (5,24), (6,14) and (8,9). Now the corresponding values for (a,b,c) are (13,12,5), (10,8,6), (29,25,6), (20,15,7) and (17,10,9).

We must use three of these five triangles for forming the Beeling market place. Since we need two pairs of common sides, the triangles must be (17,10,9), (10,8,6) and (29,25,6).

Let $ABCD$ be the marketplace and F be the flagpole. There are four possible configurations as shown (next page), but the last is inadmissible. In each of the remaining cases, either C, F and A are collinear or B, F and D are collinear. The area of the remaining triangle is easily found to be 90 square metres. Hence the overall area is $36 + 24 + 60 + 90 = 210$ square metres.



Problem 20

Problem 21

We label the holes A, B or C in diagonal fashion as shown in the first illustration. Let a, b and c denote the numbers of men in holes with labels A, B and C, respectively. Initially, $a = b = 11$ and $c = 10$. In each jump, two of these numbers go down by 1 while the third goes up by 1, so that each goes from odd to even or vice versa. After 31 jumps, a and b will be even and c will be odd. Since by then only the marked man is left, $a = b = 0$ and $c = 1$, and it stands on a C-hole. If we label the holes D, E or F as shown in the second illustration, the same argument indicates that the marked man ends on a D-hole. Now the only CD-holes are $d2$, $a5$, $d5$ and $g5$. Since the marked man starts on $d7$, the only one of these holes it can get to is $d5$, and that is where it ends.

		A	B	C		
		B	C	A		
A	B	C	A	B	C	A
B	C	A	B	C	A	B
C	A	B	C	A	B	C
		C	A	B		
		A	B	C		

(1)

		D	E	F		
		F	D	E		
F	D	E	F	D	E	F
E	F	D	E	F	D	E
D	E	F	D	E	F	D
		E	F	D		
		D	E	F		

(2)

Problem 21

Problem 22

(Answer)

Top layer	*Middle layer*	*Bottom layer*
F E C	I H B	A A A
E F C	E B G	I H D
F C G	B G D	H I D

Problem 23

(Answer) Let the rows be labelled a, b and c and the columns 1, 2 and 3. Let the white knights be on $a1$ and $c1$ and the black knights on $a3$ and $c3$ initially. A seven-move solution is $a1-b3$, $a3-c2-a1$, $c3-b1-a3-c2$, $c1-a2-c3-b1-a3$, $b3-c1-a2-c3$, $a1-b3-c1$ and $c2-a1$.

Problem 24

Two of the numbers will be consecutive and therefore relatively prime.

Problem 25

Divide the 10 by 10 by 10 box into 125 2 by 2 by 2 portions; color each portion black or white so that the whole box resembles a three-dimensional chessboard. Now 63 portions are of one color while the remaining 62 are of the other color. Since each 1 by 1 by 4 block occupies two unit cubes of each color, there will be eight unit cubes of the same color left over. Thus the task is impossible.

Problem 26

The tournament should be held at New York. Pair each non-New York master with a New York master. For this pair, the optimal place is anywhere on the straight line between the first master's home town and New York. There are more masters in New York than elsewhere and, for those who have not been paired, their optimal place is obviously New York.

Problem 27

It does not matter what color the first card is. Once drawn, only one of the remaining three cards will match it in color; thus the desired probability is 1/3.

Problem 28

(Answer) Draw $AB = p$ and on AB draw a semicircle. Draw a line at a distance q from AB, cutting the semicircle at C. Draw a line through C perpendicular to AB, cutting AB at D. Then AD and BD are the desired line segments.

Problem 29

(Answer) It is one-sixth of a circle with one of the vertices of the equilateral triangle as centre.

Problem 30

Draw a line through P perpendicular to l, intersecting l at a point D. Draw the bisector of $\measuredangle DPF$. Now every point on this bisector is equidistant from D and F. Hence apart from P, every point is closer to l than to F. Hence this bisector has only one point P in common with the parabola and is therefore the tangent to the parabola at P.

Problem 31

Suppose in a combinatorially regular polyhedron, each face has x sides and each vertex is the endpoint of y sides. Let there be V vertices, E sides and F faces.

By counting the sides of each face, we obtain a total count of xF. Since each side is counted exactly twice, we have $xF = 2E$. Similarly, $yV = 2E$. Substituting into Euler's Formula $V - E + F = 2$, we have $E(2/y - 1 + 2/x) = 2$.

Since each of x and y is at least 3, this equation shows that each is at most 5. If (x,y) is (4,4), the equation becomes $0 = 2$. If (x,y) is (4,5) or (5,4), the equation becomes $E = -20$. If (x,y) is (5,5), the equation becomes $E = -10$. Ruling out these four cases, we have five types of combinatorially regular polyhedra. If the faces are regular polygons, these polyhedra correspond to the tetrahedron (3,3), the cube (4,3), the dodecahedron (5,3), the octahedron (3,4) and the icosahedron (3,5).

Problem 32

(Answer) Let the rows be labelled a, b, c, d, e, f and g and the columns 1, 2, 3, 4, 5, 6 and 7. We can take the 21 points $a1$, $a2$, $a3$, $b1$, $b4$, $b5$, $c1$, $c6$, $c7$, $d2$, $d4$, $d6$, $e2$, $e5$, $e7$, $f3$, $f4$, $f7$, $g3$, $g5$, and $g6$.

Problem 33

Let $x = \sin 10°$. Then $1/2 = \sin 30°$ $= \sin 10° \cos 20° + \cos 10° \sin 20° = x(1 - 2x^2) + 2x(1 - x^2) = 3x - 4x^3$ or $8x^3 - 6x + 1 = 0$. The only rational numbers that can possibly be roots of this equation are ± 1, $\pm 1/2$, $\pm 1/4$ and $\pm 1/8$, but none checks out. Hence $\sin 10°$ is irrational.

Problem 34

We divide a hemisphere with horizontal base into n horizontal slices with uniform thickness. The i-th slice from the base is roughly a circular cylinder of height r/n and radius $\sqrt{r^2-(ir/n)^2}$. Hence its volume is $\pi r^3(1/n - i^2/n^3)$. Summing from $i = 1$ to n, we have $\pi r^3(1 - n(n + 1)(2n + 1)/6n^3)$ which tends to $2\pi r^3/3$ as n tends to infinity. Thus the volume of the sphere is $4\pi r^3/3$.

Problem 35

Let a, b and c be the sides of a triangle and let s be the semi-perimeter. Now the area A of the triangle is maximum if and only if $A^2/s = (s - a)(s - b)(s - c)$ is maximum. Since $(s - a) + (s - b) + (s - c) = s$ is constant, the product is maximum if and only if $s - a = s - b = s - c$. Thus the equilateral triangle has the greatest area among all triangles with fixed perimeter.

Problem 36

Let Q be the reflection of P across the bisector of $\measuredangle BAC$. Then Q is at a distance p_c from AC and a distance p_b from AB, so that the total area of the triangles QAB and QAC is $(p_bAB + p_cAC)/2$. Now these two triangles have a common base $AQ = AP$ and a combined height of at most BC. Hence $p_bAB + p_cAC \le PA \cdot BC$. Similarly, $p_aAB + p_cBC \le PB \cdot AC$ and $p_aAC + p_bBC \le PC \cdot AB$. Hence
$PA + PB + PC \ge p_a\,(AB/AC + AC/AB)$
$+ p_b(AB/BC + BC/AB)$
$+ p_c(AC/BC + BC/AC) \ge 2(p_a + p_b + p_c)$
since $x + 1/x \ge 2$ for all positive numbers x.

Problem 37

(Answer) (d).

Problem 38

Since the last digit is 4, the number is divisible by 2. Since the last digits are 3 and 4, the number is not divisible by 4. Hence it cannot be a square.

Problem 39

For $k > 1$, $k > k - 1$ so that $1/k^2 < 1/k(k-1) = 1/(k-1) - 1/k$. It follows that $1/1^2 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 < 1 + (1 - 1/2) + (1/2 - 1/3) + \cdots + (1/(n-1) - 1/n) = 2 - 1/n$, which is strictly less than 2.

Problem 40

(Answer) Extend AJ to K so that $AJ = AK$. On the same side as A of BK, draw a circular arc subtending an angle supplementary to ACB. Let this arc cut CD at F and let the extension of BF cut the original circle at X. This is the desired point.

Problem 41

(Answer) $68/77 = 3/7 + 5/11$.

Problem 42

Let A, B, C and D be the first four houses. We may assume that they are connected in such a way that ABC is a triangle and D is inside. If the fifth house E is outside ABC, it cannot be connected to D. If it is inside ABD, it cannot be connected to C. If it is inside ACD, it cannot be connected to B. If it is inside BCD, it cannot be connected to A.

Problem 43

Suppose $2x + 3y$ is divisible by 17. Then so is $9(2x + 3y) - 17y = 2(9x + 5y)$. Since 17 and 2 are relatively prime, $9x + 5y$ is divisible by 17. Conversely, suppose $9x + 5y$ is divisible by 17. Then so is $2(9x + 5y) + 17y = 9(2x + 3y)$. Since 17 and 8 are relatively prime, $2x + 3y$ is divisible by 17.

Problem 44

Let n, $n+1$, $n+2$ and $n+3$ be the four consecutive positive integers. Since $n(n+1)(n+2)(n+3) = (n^2 + 3n + 1)^2 - 1$, and the only two consecutive squares are 0 and 1, $n(n+1)(n+2)(n+3)$ is not a square.

Problem 45

We have $DE = OD = OB$. Hence, $\measuredangle AOB = \measuredangle OBD + \measuredangle BEA = \measuredangle ODB + \measuredangle BEA = 2\measuredangle BEA + \measuredangle DOE = 3\measuredangle BEA$.

Problem 46

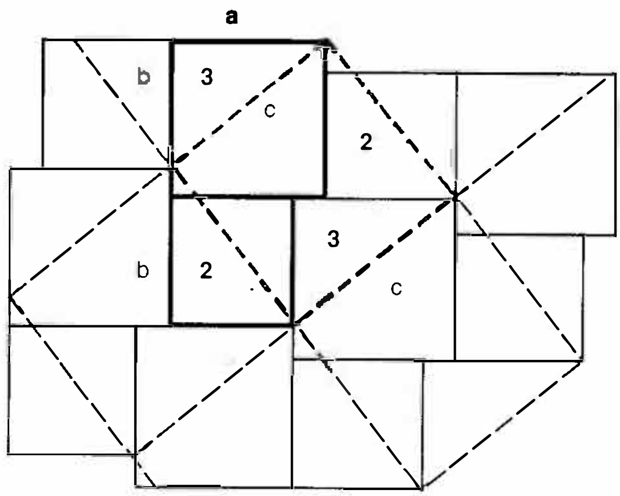
Consider the elements $x, x^2, \ldots, x^p$. Since there are p elements and only $p-1$ modulo classes, we must have $x^i \equiv x^j \pmod p$ for some $1 \leq i < j \leq p$. Hence $x^{j-i} \equiv 1 \pmod p$ and we can take y to be the element in the set congruent to x^{j-i-1}.

Problem 47

(Answer) 45 handshakes.

Problem 48

(Answer) See illustration.



Problem 48

Problem 49

(Answer) (a).

Problem 50

Any line passing through the centre O of the circular pancake bisects its area. Draw a directed line through O. If it also bisects the area of the other pancake, all is well. Suppose there is more of the other pancake on the right side of this directed line. Rotate it about O. After a 180° rotation, it will return to its original position but in the opposite direction. Now there is more of the other pancake on its left. Hence somewhere during the rotation, the amount of the other pancake on each side of the line must equalize.

Problem 51

Let $ABCD$ be the quadrilateral, E be the point of intersection of AC and BD, and F be a point on AC such that $ABF = EBC$. Then the triangles ABF and DBC are similar, as are BFC and BAD. Hence $AB/AF = BD/DC$ and $BC/FC = BD/AD$, and $AB \cdot CD + BC \cdot AD = BD \cdot AF + BD \cdot FC = BD \cdot AC$.

Problem 52

(Answer) 45360.

Problem 53

(Answer) We use a transformation called homothety. Take any point P on AC and draw the square $PQRS$ with Q and R on AB. Draw the line AS intersecting BC at Z. Draw a line through Z parallel to AB, cutting AC at W. The rectangle $WXYZ$ with X and Y on AB is easily seen to be the desired square.

Problem 54

We consider all possible shifting of the first word:

FGHIJKLMNOPQRSTUVWXYZABCDE
RSTUVWXYZABCDEFGHIJKLMNOPQ
ZABCDEFGHIJKLMNOPQRSTUVWXY
DEFGHIJKLMNOPQRSTUVWXYZABC
UVWXYZABCDEFGHIJKLMNOPQRST
GHIJKLMNOPQRSTUVWXYZABCDEF
VWXYZABCDEFGHIJKLMNOPQRSTU

The only meaningful word is COWARDS. Thus each letter has been shifted three places forward and the original message is COWARDS DIE MANY TIMES BEFORE THEIR DEATHS.

Problem 55

Let $d_1, d_2, \ldots, d_m$ be the positive divisors of an abundant number n. Then $kd_1, kd_2, \ldots, kd_m$ are among the positive divisors of kn. Since $d_1 + d_2 + \cdots + d_m > 2n$, $kd_1 + kd_2 + \cdots + kd_m > 2kn$ and kn is abundant.

Problem 56

Take a point C on the opposite side of AB of the line parallel to AB. Join AC and BC, cutting the line at E and D, respectively. Join AD and BE, intersecting at G. Join CG and extend it to intersect AB at F. We claim that F is the midpoint of AB. By Ceva's Theorem, we have $BD \cdot CE \cdot AF/DC \cdot EA \cdot FB = 1$. Since ED and AB are parallel, we have $CE/EA = DC/BD$. Hence $AF/FB = 1$ or $AF = FB$ as claimed.

Problem 57

(Answer) (a).

Problem 58

Let $x_0 = 2$. Define $y_0 = 2/x_0 = 1$.
Take $x_1 = (x_0 + y_0)/2 = 1.5$.
Define $y_1 = 2/x_1 \approx 1.333$ to three decimal places.
Take $x_2 = (x_1 + y_1)/2 \approx 1.416$.
Define $y_2 = 2/x_2 \approx 1.412$.
Take $x_3 = (x_2 + y_2)/2 \approx 1.414$.
Define $y_3 = 2/x \approx 1.414$.
Since x_3 and y_3 agree to three decimal places, we have $\sqrt{2} \approx 1.41$ rounded off to the required two decimal places.

Problem 59

We have $2(21n + 4) + 1 = 3(14n + 3)$. Hence any number dividing both $21n + 4$ and $14n + 3$ must also divide 1. It follows that $21n + 4$ and $14n + 3$ are relatively prime so that there is no reduction for $(21n + 4)/(14n + 3)$.

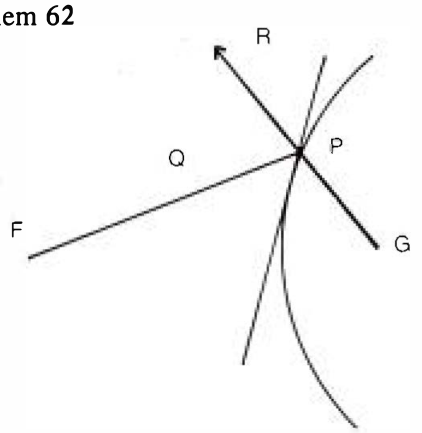
Problem 60

The chance of not getting any 6's in three rolls of an honest die is $(5/6)^3 = 125/216 > 1/2$. Thus the chance of getting at least one 6 is less than 1/2.

Problem 61

(Answer) (d).

Problem 62



Problem 62

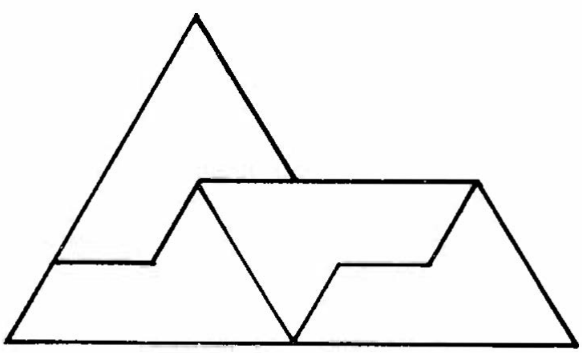
Let F and G be the foci of the hyperbola. Let the branch closer to G be the mirror, so that the light source is placed at F. For any point P on the mirror, $PF - PG$ is a constant k. We claim that the bisector l of $\angle FPG$ is the tangent to the hyperbola at P. Let Q be the reflection of G across l. Then Q lies on PF. For any point T other than P on l, $FT - GT = FT - QT < FQ = k$, so that it is not on the hyperbola. It follows that l is indeed a tangent as claimed. Now the light ray from F falling on P will reflect along the ray PR which is equally inclined to l as FP. Since l bisects $\measuredangle FPG$, G, P and R are collinear so that the reflected ray will appear to originate from G.

Problem 63

Take a circle and divide it into $n + 1$ arcs, numbered $0, 1, \ldots, n$. Take 1 red and r green markers. Consider the ways of placing the markers on the arcs so that no arc has no more than one marker. For each possible position of the red marker, each way of placing the green marker corresponds to a subset of r elements of $1, 2, \ldots, n$, namely, the distances of the green markers from the red one, say, measured in the clockwise direction. We want to find the average distance from the red marker to the next marker in the clockwise direction. We consider the configurations in which a given set of arcs is occupied. There are $r + 1$ possible places for the red marker when the occupied arcs are given. The sum of the distances from the red marker to the next marker, taken over all these possibilities, is just the sum of the distances from one occupied arc to the next taken around the whole circle, which is $n + 1$. Hence the average distance over each of these groups is $(n + 1)/(r + 1)$, which must also be the overall average.

Problem 64

(Answer) See illustration.



Problem 64

Problem 65

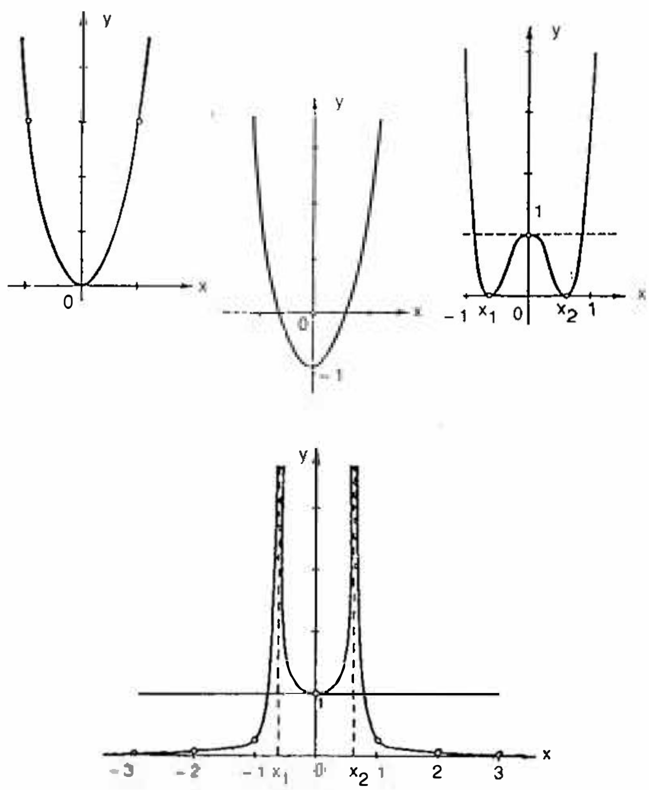
Let $f(x) = x - 5x + 2x + 1$. Then $f(-1) = -7$, $f(0) = 1, f(1) = -1, f(2) = -7, f(4) = -7$ and $f(5) = 11$. Hence the equation $f(x) = 0$ has three real roots, one between -1 and 0, one between 0 and 1 and one between 4 and 5.

Problem 66

Divide each area into n vertical strips of uniform width. The i-th strip from the left in the first area is roughly a rectangle of width $2/n$ and height $1/(1 + 2i/n)$, while that in the second area is roughly a rectangle of width $6/n$ and height $1/(3 + 6i/n)$. Thus the corresponding rectangles have the same area and so will their sums and the limiting values.

Problem 67

(Answer) See illustrations.



Problem 67

Problem 68

(Answer) The transformation is a 45° counterclockwise rotation. The image is a triangle with vertices at 0, $\sqrt{2}$ and $\sqrt{2}i$.

Problem 69

Let r and h, respectively, be the radius and height of the cylinder. The $V = \pi r^2 h$ and the surface area is given by $2\pi r^2 + 2\pi rh = 2\pi r^2 + V/r + V/r$. The product of the three terms is $2\pi V^2$, a constant. Hence the minimum sum occurs when $2\pi r^2 = V/r$ or $h = 2r$.

Problem 70

(Answer) Draw any ray from A and mark off on it points P, Q, R, S and T so that $AP = PQ = QR = RS = ST$. Draw a line through S parallel to BT, intersecting AB at C. Draw a line through S parallel to BR, intersecting the extension of AB at D.

Problem 71

(Answer) We should always use weapon A_3 and the enemy should always use aircraft B_2.

Problem 72

Let $x = 1/(1 + 1/(1 + 1/(1 + \cdots)))$. Then $x = 1/(1 + x)$ or $x^2 + x - 1 = 0$. By the Quadratic Formula, $x = (-1 \pm \sqrt{1+4})/2$. Since $x > 0$, $x = (\sqrt{5}-1)/2$.

Problem 73

If there are numbers which have no prime factorizations, then there must be a smallest such number x. Now x cannot be a prime, as otherwise x itself constitutes a prime factorization. Hence $x = ab$ for smaller positive integers a and b. Being smaller than x means that each of a and b has a prime factorization, but then the concatenation of the prime factorizations of a and b will be a prime factorization, contradicting the assumption that x has no prime factorizations. Hence every integer greater than 1 has a prime factorization.

Problem 74

(Answer) With A as centre, draw an arc with AC as radius. With B as centre, draw an arc with BC as radius, intersecting the first arc in a second point D. With C and D as respective centres, draw two circles with radius r. Their points of intersection are the desired points.

Problem 75

Let the elements of the first set be $f(1)$, $f(2)$, $f(3)$, . . . and the elements of the second set be $g(1)$, $g(2)$, $g(3)$, . . ., where f and g are the respective counting functions. A counting function h for the union of these two sets may be defined by $h(2n-1) = f(n)$ and $h(2n) = g(n)$ for $n = 1, 2, 3, \ldots$.

Problem 76

Let A and B be two points on a line projected from a point V onto a plane Π. If V, A and B are collinear, the image of the line AB is clearly the point of intersection of AB with Π. Otherwise, V, A and B determine a plane Π' which intersects Π in a line l. It is easy to see that l is the image of AB on Π.

Problem 77

We use induction on n. For $n = 1$, the result is trivial. Suppose the task is accomplished for a particular value of n. Consider now an $(n+1)$st square. This can be combined with the composite square obtained from the first n squares using the construction given in Problem 48. Since only a finite number of steps are involved, the number of pieces generated is clearly finite.

Problem 78

By the Arithmetic-Mean-Geometric-Mean Inequality, we have $n! = (1 \cdot n)(2(n-1))(3(n-2)) \cdots < ((n+1)/2)^n$.

Problem 79

(Answer) The treasure can be found. The hunter can take any spot as the location of the missing gallows.

Problem 80

(Answer) The image is a straight line perpendicular to the line joining O and the centre of the circle passing through O.

Problem 81

(Answer) $x/|x| + y/|y| = 2$.

Problem 82

Postage amounting to 8¢ can be made up of one 3¢ and one 5¢ stamps. Postage amounting to 9¢ can be made up of three 3¢ stamps. Postage amounting to 10¢ can be made up of two 5¢ stamps. For any amount over 10¢, an inductive argument shows that it can be made up using only 3¢ and 5¢ stamps, because the amount 3¢ less can be so made up.

Problem 83

Let $f(x) = x^3 + 3x - 1$. Then $f(0) = -1$ while $f(1) = 3$. Thus the equation $f(x) = 0$ has a solution between 0 and 1. Now the graph of $y = f(x)$ is not a straight line. Approximating the crucial portion of it between $x = 0$ and $x = 1$ by a straight line joining $(0,-1)$ to $(1,3)$, we see that this line $y = 4x - 1$ intersects the x-axis at the point $(1/4,0)$. Thus 1/4 is an approximate solution to the equation $x^3 + x - 1 = 0$.

Problem 84

Since 139 of 400 points fall within the figure, 139/400 would be a good approximation of the area of the figure.

Problem 85

Let l be the tangent to a circle with centre O, at the point P. Draw the diagram on a vertical plane so that l is horizontal and below O. Then P is the lowest point on the circle. Attach one end of a string of length OP to O and the other end to a weight. When released, the weight will settle in its lowest possible position. Since the weight is on the circle, it must be at P. Since the string is vertical and l is horizontal, the desired conclusion follows.

Problem 86

(Answer) The tapehead is back at cell 0 and there are no changes other than cell 0 being marked and then unmarked.

Problem 87

Let BE be the bisector of $\measuredangle ABC$ and CF be the bisector of $\measuredangle ACB$. Suppose $AB>AC$. Then $\measuredangle ABC<\measuredangle ACB$. Complete the parallelogram $BEGF$. Since $FG = BE = FC$, $\measuredangle FGC = \measuredangle FCG$. Since $\measuredangle FGE = \measuredangle FBE<\measuredangle FCE$, $\measuredangle CGE>\measuredangle GCE$ so that $CE>GE = BF$. In triangles BCF and CBE, we have $BC = CB$ and $BF = CE$. Since $\measuredangle EBC<\measuredangle FCB$, $EC<FB$, and we have a contradiction.

Problem 88

(Answer) $s_n = (3^{n+1} - 1)/2$.

Problem 89

Let C be the point on AB and D be the point on AB extended so that $AC/BC = AD/BD = 3$. We claim that the circle with CD as diameter is the desired locus. Draw BE parallel to CP and BF parallel to DP for an arbitrarily chosen point P on the circle, with E and F on AP. Since $\measuredangle CPD = 90°$, $\measuredangle EBF = 90°$. Now $AP/EP = AC/BC = AD/BD = AP/FP$. Hence $EP = FP = BP$ and $AP/BP = AC/BC = 3$. Conversely, for any point P such that $AP/BP = 3 = AC/BC$, PC bisects $\measuredangle APB$ and PD bisects the exterior $\measuredangle APB$. Hence $\measuredangle CPD = 90°$ and P lies on the circle.

Problem 90

If we unroll the cylindrical surface into a plane, the shortest distance between A and C is obviously the straight line. Since $BC = 8$ and $AB = 6$, $AC = \sqrt{8^2+6^2} = 10$. When rolled back onto the cylindrical surface, this shortest path is part of a spiral.

Problem 91

If $d = c^2$ for some integer c, then $(x + cy)(x - cy) = 1$. Hence both factors are 1 or -1. From $x + cy = x - cy$, we have $y = 0$. It follows easily that $x = \pm 1$.

Problem 92

Let T be the point diametrically opposite S. Then $\measuredangle STM' = \measuredangle SMT = 90°$. Hence $\measuredangle SM'N' = \measuredangle STM = \measuredangle SNM$. Similarly, $\measuredangle SN'T = \measuredangle STN$. It follows easily that $\measuredangle SN'M' = \measuredangle SMN$.

Problem 93

(Answer) The minimum value of 3 is attained at the point $(x,y) = (0,3)$.

Problem 94

We have
$A \cap (A \cup B) = (A \cup \phi) \cap (A \cup B) = A \cup (\phi \cap B)$
$= A \cup \phi = A$.

Problem 95

(Answer) Make the first two cuts as shown in the third frame. Make the third cut horizontally halfway between the top and the bottom of the cake.

Problem 96

(Answer) There are no missing dollars. The price structure has changed, even though it does not appear to be the case, as the clerk believes.

Problem 97

Let ABC be the equilateral triangle of side s. Let h_a, h_b and h_c be the distances of a point H to BC, CA and AB, respectively. Now the area of the triangle ABC is equal to the sum of the areas of the triangles HAB, HBC and HCA, which is $s(h_a + h_b + h_c)/2$. Since this area is constant and so is s, $h_a + h_b + h_c$ is also a constant. Hence any point on the island is as good as any other.

Problem 98

Adding the two equations yields $100x + 100y = 1000$ or $45x + 45y = 450$. Hence $10x = 70$ and $10y = 30$, so that $x = 7$ and $y = 3$.

Problem 99

(Answer) Draw a marble from the box labelled red/white.

Problem 100

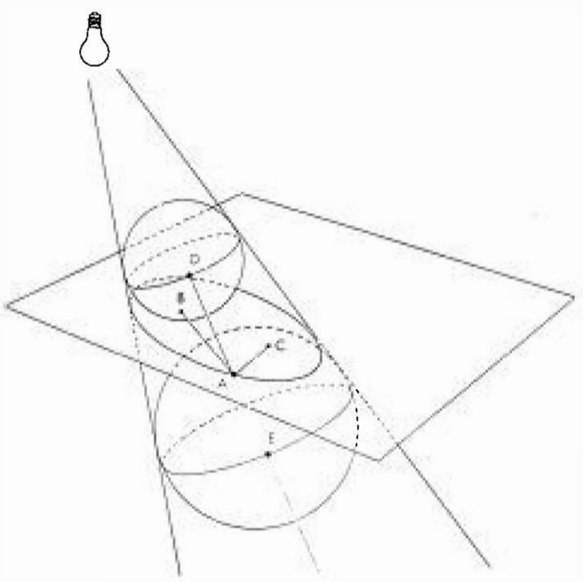
The computation on the left-hand column expresses a number as a sum of powers of 2. In the example featured in the statement of the problem, $35 = 17 \times 2 + 1 = 8 \times 2^2 + 2 + 1 = 4 \times 2^3 + 2 + 1 = 2 \times 2^4 + 2 + 2 = 2^5 + 2 + 1$. The computation on the right-hand column multiplies a number by powers of 2. In the featured example, $2 \times 56 = 112$, $2^2 \times 56 = 224$, $2^3 \times 56 = 448$, $2^4 \times 56 = 896$ and $2^5 \times 56 = 1792$. The overall computation works because multiplication is distributive over addition. In the featured example, we have $35 \times 56 = (2^5 + 2 + 1) \times 56 = 2^5 \times 56 + 2 \times 56 + 56 = 1792 + 112 + 56 = 1960$.

Problem 101

(Answer) $9567 + 1085 = 10652$.

Problem 102

Pretend that there is a larger sphere beneath the level surface inside the cone, tangent to both the level surface and the cone, as shown in the



Problem 102

illustration. Let B and C be the points of tangency with the level surface of the original sphere and the larger sphere, respectively. Let A be any point on the boundary of the shadow. Draw a line passing through A and the apex of the cone. This line will be tangent to the spheres at D and E, respectively. By symmetry, DE has constant length regardless of the position of A. Now $AB = AD$ since both are tangents from A to the original sphere. Similarly, $AC = AE$. Hence $AB + AC = AD + AE = DE$ is constant. It follows that the boundary of the shadow is an ellipse with B and C as foci.

Problem 103

Let ω be a root of the equation $x^3 - 1 = 0$ not equal to 1. Since $(\omega - 1)(\omega^2 + \omega + 1) = 0$ and $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$. In the binomial expansion for $(1 + x)^n$, substitute $x = 1$, $x = \omega$ and $x = \omega^2$ in turn to obtain $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, $\binom{n}{0} + \binom{n}{1}\omega + \binom{n}{2}\omega^2 + \cdots + \binom{n}{n}\omega^n = (1 + \omega)^n$ and $\binom{n}{0} + \binom{n}{1}\omega^2 + \binom{n}{2}\omega^4 + \cdots + \binom{n}{n}\omega^{2n} = (1 + \omega^2)^n$. Note that $1 = \omega^3 = \omega^6 = \cdots$, $\omega = \omega^4 = \omega^7 = \cdots$ and $\omega^2 = \omega^5 = \omega^8 = \cdots$. Adding the three identities and using $\omega^2 + \omega + 1 = 0$, we have $3[\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots] = 2^n + (-1)^n\omega^{2n} + (-1)^n\omega^n$. Hence $S = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots = 2^n + (-1)^n(\omega^n + \omega^{2n}))/3$. It follows that
if $n \equiv 0 \pmod 6$, $S = (2^n + 2)/3$;
if $n \equiv 1$ or 5, $S = (2^n + 1)/3$;
if $n \equiv 2$ or $4 \pmod 6$, $S = (2^n - 1)/3$;
and if $n \equiv 3 \pmod 6$, $S = (2^n - 2)/3$.

Problem 104

Let the bus routes be represented by straight lines, no two parallel and no three concurrent. Let the bus stops be at all the points of intersection of two routes and nowhere else. Suppose a route connecting two stops P and Q is closed. Now P lies on another route, as does Q, and these two routes have a common stop R. Hence it is possible to get from P to Q via R. On the other hand, a stop P is on exactly two routes. If both are closed, it is no longer possible to get from P to any other stop Q.

Problem 105

Let the sides of the smaller triangle be a, b and c with $a \leq b \leq c$. Then the sides of the larger triangle must be b, c and d with $a/b = b/c = c/d$. Let the common value be denoted by m/n in the lowest term. Then $a/d = m^3/n^3$. Hence $a = km^3$ and $d = kn^3$ for some integer k. Now

$k(n - m)(n^2 + nm + m^2) = 387 = 3^2 43$. Since $(n - m)^2 < n^2 + nm + m^2$, $n - m$ is either 1 or 3. If $n - m = 1$, $k(3m^2 + 3m + 1) = 387$. The only divisor of 387 greater than 1 that is congruent to 1(mod 3) is 43. However, we cannot have $3m^2 + 3m + 1 = 43$ as m would not be integral. Hence $n - m = 3$ and $k(m^2 + 3m + 3) = 43$. We must have $k = 1$ and $m^2 + 3m + 3 = 43$. Hence $m = 5$ or -8, and the negative root is rejected. It follows that $a = 125$, $b = 200$, $c = 320$ and $d = 512$.

Problem 106

Let such a polygon have n sides. Then the sum of its interior angles is given by $(n-2)180°$ and also by $120° + 125° + \cdots + (120° + (n - 1)5°) = n \cdot 120° + (n(n - 1)/2)5°$. It follows that $72(n - 2) = 48n + n(n - 1)$ or $n^2 - 25n + 144 = (n - 9)(n - 16) = 0$. Hence $n = 9$ or 16, and the polygon we seek has 16 sides. Note, however, that it is degenerate, because one of its angles is 180°.

Problem 107

Think of the three circles as the equators of three spheres. Then the three points in question are the apexes of the pairwise common tangent cones of these spheres. Hence, they all lie in a common external tangent plane of the three spheres. Since they also lie in the original plane and two planes intersect in a line, they are all on a line.

Problem 108

Let the triangle be ABC with $AB = AC$ and $BC = a$. Let E be the point on AC such that BE bisects $\measuredangle ABC$, with $BE = t$. Let $CE = x$. The construction is easy once x is determined. Extend BC to D so that $CD = x$. Then triangles BED and ECD are similar and we have $t^2 = x(a + x)$. On a circle with centre O and diameter a, take a point P and draw a tangent $PQ = t$. Join OQ, cutting the circle at R and S with R closer to Q. Then $PQ^2 = QR \cdot QS = QR(QR + a)$ so that $QR = x$.

Problem 109

We claim that no two consecutive terms are both odd. Otherwise, there must be a first such pair (x,y). It is not at the beginning of the sequence because the first two terms are 2 and 3. Hence this pair is generated. Now at least one of x and y is the last digit of a previous product but, since the product is odd, we must have an earlier odd pair, contradicting our assumption that (x,y) is the first such pair. This justifies our claim. Now if a 9 appears in the sequence, it must appear as the tens digit of a product, but this is impossible. If a 7 appears, it must be generated by $9 \times 8 = 72$, but this is impossible since there are no 9s. Finally, if a 5 appears, it must be generated by either $9 \times 6 = 54$ or $7 \times 8 = 56$, but neither is possible.

Problem 110

The two conscientious jurors in the three-person jury agree on the correct decision with a probability of p^2. They will disagree with a probability of $2p(1-p)$ and, after consulting the flippant juror, the probability of a correct decision is halved to $p(1-p)$. Hence the overall probability of a correct decision is $p^2 + p(1-p) = p$, the same as that of a one-person jury.

Problem 111

We use the substitution $x = t - 1$.
Then the equation becomes $t^4 - 9t^3 + 17t^2 - 18t + 13 = 0$. If $t \le 0$, the polynomial is clearly positive. For $0 < t < 1$, $t^4 - 9t^3 + 17t^2 - 18t + 4 = t^2(t - 1)(t - 9) + 4t^2 + (5t - 13)(t - 1) > 0$. Hence $t^4 - 9t^3 + 17t^2 - 18t + 13 = 0$ has no roots less than 1, and $x^4 - 5x^3 - 4x^2 - 7x + 4 = 0$ has no negative roots.

Problem 112

(Answer) A cylindrical spiral.

Problem 113

(Answer) The farmer can catch the hen and his wife can catch the rooster.

Problem 114

Since oil sells for twice as much as vinegar, twice as much vinegar as oil is sold. Thus the total number of gallons of oil and vinegar sold is a multiple of 3. When divided by 3, the numbers 15, 8, 17, 13, 19 and 31 leave remainders of 0, 2, 2, 1, 1 and 1, respectively. It follows that the barrel that is left contains 13, 19 or 31 gallons. If it is the 13-gallon barrel, then 30 gallons of oil is sold, but the total of 30 cannot be made up from the barrels. If it is the 31-gallon barrel, we have to make up a total of 24, which again is impossible. Hence the 19-gallon barrel is left. The customer buys $15 + 13 = 28$ gallons of oil and $17 + 8 + 31 = 56$ gallons of vinegar.

Problem 115

Each outside triangle has a pair of sides equal to a pair of sides of the central triangle and the two angles included by these pairs are supplementary. Hence all four triangles have equal area. To compute the area of the central triangle, consider a rectangle $ABCD$ with $AB = 4$ and $BC = 5$. Let E be a point on AB with $AE = 1$ and F be a point on BC with $BF = 3$. Then $DE = \sqrt{26}$, $EF = \sqrt{18}$ and $FD = \sqrt{20}$, so that DEF is congruent to the central triangle. The area of DEF is easily computed to be 9. Hence the total area of Farmer Wurzel's estate is $26 + 18 + 20 + 9 + 9 + 9 + 9 = 100$.

Problem 116

(Answer) 74369053 x 87956 = 6541204425668.

Problem 117

(Answer) Let the girls be A, B, C, D, E, F, G, H, I, J, K, L, M, N and O. A possible seven-day schedule is:
first day—(ABI)(CEM)(DHJ)(FGK)(LNO),
second day—(ACJ)(DFN)(EBK)(GHL)(MOI),
third day—(ADK)(EGO)(FCL)(HBM)(NIJ),
fourth day—(AEL)(FHI)(GDM)(BCN)(OJK),
fifth day—(AFM)(GBJ)(HEN)(CDO)(IKL),
sixth day—(AGN)(HCK)(BFO)(DEI)(JLM) and
seventh day—(AHO)(BDL)(CGI)(EFJ)(KMN).

Problem 118

Caius provided seven dishes and ate five, so that two were given to Titus. Titus got three dishes from Sempronius and should give him 18 of the 30 denarii.

Problem 119

The first player wins by taking 4 matches from the larger pile. After this move, the numbers of matches in the two piles are congruent (modulo 6). Whatever move the second player makes, this congruence cannot be maintained. On the other hand, the first player can always restore it on his subsequent move. Since at the end of the game, we have $0 \equiv 0 \pmod 6$, the first player must win.

Problem 120

(Answer) Label the coins 001, 010, 011, 012, 112, 120, 121, 122, 200, 201, 202, 220. First weigh 001, 010, 011 and 012 against 200, 201, 202 and 220. Then weigh 001, 200, 201 and 202 against 120, 121, 122 and 220. Finally, weigh 010, 120, 200 and 220 against 012, 112, 122 and 202.

Problem 121

Draw DE equal to the given perimeter. Draw a circular arc on DE subtending an angle of $90° + \alpha/2$, where α is the given angle. Draw a line parallel to DE and at a distance from DE equal to the given altitude, intersecting the circular arc at a point A. Let B and C be points on DE such that $BA = BD$ and $CA = CE$. Then triangle ABC has the correct perimeter and altitude from A. Note that $90° + \alpha/2 = \measuredangle DAE = \measuredangle BAD + \measuredangle BAC + \measuredangle CAE = (\measuredangle ABC)/2 + \measuredangle BAC + (\measuredangle ACB)/2 = 90° + (\measuredangle BAC)/2$. Hence $\measuredangle BAC = \alpha$ as desired.

Problem 122

(Answer) $(n + 1)/2n$.

Problem 123

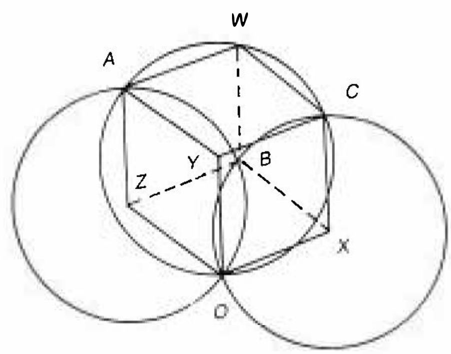
Of a total of 20 patients, 8 died. If there is really no difference between the two treatments, we may distribute the patients at random, provided that 9 received the old treatment and 11 the new one. The number of distributions in which none of the patients who died received the new treatment is $\binom{12}{11} = 12$. The number in which only 1 of those treated the new way died is $\binom{8}{1}\binom{12}{10} = 528$, and the number in which exactly two died is $\binom{8}{2}\binom{12}{9} = 6160$. Thus the desired probability is $(12 + 528 + 6160)/\binom{20}{11} = 335/8398$.

Problem 124

Ask each chicken to raise one foot and each rabbit to raise two feet. Then there are 70 feet touching the ground. Since there are only 50 heads, the 20 extra feet must be rabbit feet. Hence there are 20 rabbits and 30 chickens.

Problem 125

Let X, Y and Z be the circumcentres of triangles OBC, OCA and OAB, respectively. Join the lines as shown in the illustration.



Problem 125

Let W be the point other than Z such that $WA = WB = r$. Then $ZAWB$ is a rhombus. Hence WB is parallel to AZ, which is in turn parallel to YO and CX. Hence $WBXC$ is a parallelogram and $WC = BX = r$. Thus A, B and C lie on a circle of radius r and centre W.

Problem 126

(Answer) 10989.

Problem 127

(Answer) 25 x 3 = 75.

Problem 128

For such a number, n, $n^2 - n = n(n-1)$ must be divisible by 1000. Since n and $n-1$ are relatively prime, one of them is divisible by 125 and the other by 8. If n is divisible by 125, it can only be 125, 375, 625 or 875. Since 124, 374 and 874 are not divisible by 8, we must have $n = 625$. If $n-1$ is divisible by 125, then n is 126, 376, 626 or 876, but only 376 is divisible by 8. Conversely, if $n(n-1)$ is divisible by 1000, so will $n^k - n$ for $k > 2$ since $n(n-1)$ is a factor of $n^k - n$. Thus the only numbers we seek are 376 and 625.

Problem 129

(Answer) (e).

Problem 130

Let $n = 10x + y$ where $0 \leq y \leq 9$. Then $n^2 = 100x^2 + 20y + y^2$. The term $100x^2$ makes no contribution to the tens digit of n^2. The term $20xy$ contributes an even amount to the tens digit of n^2. Since 7 is odd, the odd amount must come from the term y^2.

It is routine to check that the only values of y for which the tens digit of y is odd are 4 and 6, with y^2 equal to 16 and 36, respectively. It follows that the unit digit of n^2, which is the same as the units digit of y^2, is 6.

Problem 131

Polynomials are continuous functions and compositions of polynomials are polynomials. Since $P(x) = Q(x)$ has no solutions, we may assume that $P(x) > Q(x)$ for all x. Suppose $P(P(x)) = Q(Q(x))$ for some x. Then for this x, $P(Q(x)) > Q(Q(x)) = P(P(x)) > Q(P(x)) = P(Q(x))$. This is a contradiction.

Problem 132

Represent the number n by n objects in a row. An expression of n as an ordered sum of k positive integers may be represented by the insertion of $k-1$ partition-markers, each between two adjacent objects. Since there are $n-1$ such spaces, there are 2^{n-1} ways of inserting from 0 to $n-1$ partition-markers. It follows that n can be expressed as an ordered sum of positive integers in 2^{n-1} ways.

Problem 133

(Answer) The first adviser wins by recommending the recruiting of an officer who takes over all but one of the soldiers.

Problem 134

(Answer) Denote by A a unit cube which forms part of a 1 by 1 by 3 block; denote by B a unit cube which forms part of the 1 by 2 by 2 block; by C for the 2 by 2 by 2 block; by D for any of the 1 by 2 by 4 blocks.

First layer

D	D	A	A	A
D	D	B	B	D
D	D	C	C	D
D	D	C	C	D
D	D	D	D	D

Second layer

D	D	D	D	D
D	A	B	B	D
D	A	C	C	D
D	A	C	C	D
D	D	D	D	D

Third layer

D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
A	D	D	D	D

Fourth layer

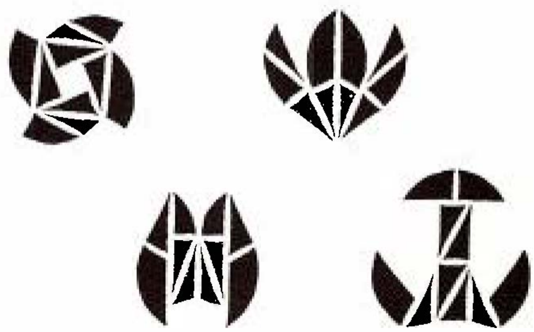
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
A	D	D	D	D

Fifth layer

D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
D	D	D	D	D
A	D	D	D	D

Problem 135

(Answer) See illustration.



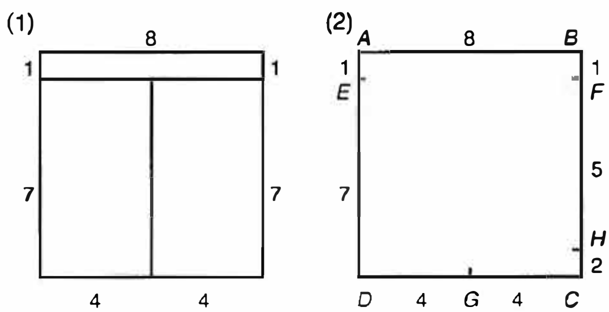
Problem 135

Problem 136

If n is divisible by 2 or 3, the tiling is clearly possible and we only need one kind of tiles. We now show that in all other cases the tiling is impossible. Color each cell of the n by n square black or white, black if the cell is on an odd-numbered row and white otherwise. Since n is not divisible by 2, the difference d between the numbers of visible black and white cells is equal to n at this stage. We now place the 2 by 2 and 3 by 3 squares to cover up the cells. A 2 by 2 square always covers up 2 cells of each color and has no effect on d. The placement of a 3 by 3 square either raises d by 3 or lowers it by 3. Since n is not divisible by 3, d can never be reduced to 0. This means that at least one cell is visible and a tiling is therefore impossible.

Problem 137

The first illustration shows that it is possible to divide the square into three rectangles each with diagonal $\sqrt{65}$. We now suppose that there is a division into three rectangles X, Y and Z each with diagonal less than $\sqrt{65}$. Of the four vertices of the square, two of them must belong to the same rectangle. Label the points as shown in the illustration, with A and B belonging to X. Since all of BE, AF and AG are at least $\sqrt{65}$, none of E, F and G can belong to X. G cannot belong to the same rectangle as E or F since both EG and FG are equal to $\sqrt{65}$. We may therefore put E and F in Y and G in Z. Since both BD and FD are greater than $\sqrt{65}$, D must belong to Z. Now all of AH, EH and DH are greater than $\sqrt{65}$. Hence H cannot belong to any of X, Y and Z, a contradiction.



Problem 137

Problem 138

(Answer) The claim is correct provided that the tripod's centre of gravity does not project outside the triangle determined by its three feet.

Problem 139

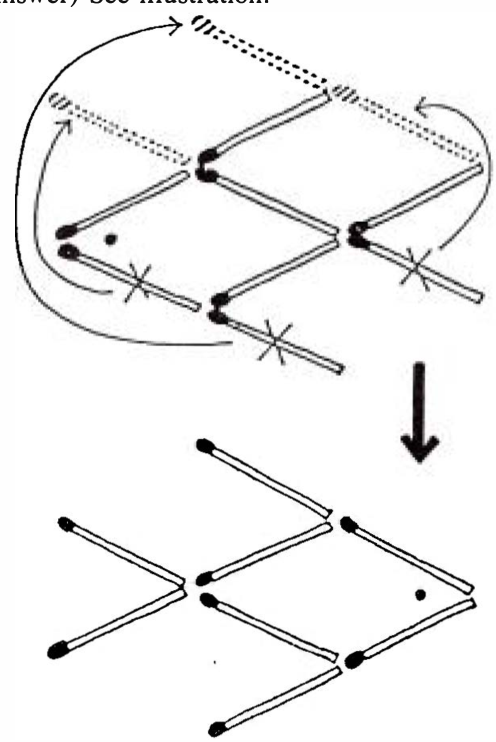
The division is impossible, because the sum of 25 odd numbers is odd, but 100 is even.

Problem 140

(Answer) The prisoner should get the keys in the following order: d, e (or e, d), c, a, b, f and g.

Problem 141

(Answer) See illustration.



Problem 141

Problem 142

Let x and y be the respective numbers of dollars and cents on the man when he entered the store. Then he had $100x + y$ cents, spent half of it and was left with $50y + x$. From $100x + y = 2(50y + x)$, we have $98x = 99y$. Since 98 and 99 are relatively prime, and x and y are positive integers under 100, we must have $x = 99$ and $y = 98$.

Problem 143

(Answer) 49 hours.

Problem 144

(Answer) There are no palindromic primes with an even number of digits apart from 11, because all palindromic numbers with an even number of digits are divisible by 11.

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