Rational Number Sense: Development and Assessment

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The development of number sense is one of the important objectives of mathematics today. Leutzinger and Bertheau define number sense as "sound judgment about the approximate quantity represented by a number" (1989). Children who possess number sense can form a mental image of a number and see relationships between numbers. A strong sense of numbers is needed to be able to perform mental computations, estimate answers and judge the reasonableness of results obtained using an algorithm or a calculator. Having a positive feeling about numbers helps children believe that mathematics is meaningful and makes them confident that they can understand procedures and solve problems.

Rational Number Considerations

Helping children make sense of rational numbers expressed as fractions or decimals is a particularly challenging task for teachers. Current research indicates that many students lack an adequate quantitative basis for thinking about fractions and decimals. They tend to apply rules without understanding why they work, and they have little confidence that the answers they produce will be sensible (Suydam 1984; Hiebert 1987). Rational number concepts and symbols are more difficult to teach and to learn than corresponding whole number ideas and numerals in several respects.

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First, the notation and its referents are more complex. The fraction symbol a/b is used to describe several different situations: a part of a whole, a part of a group, a measurement, a point on the number line and an indicated division. An appreciation of the size of the number represented by a fraction requires students to view the three parts of the symbol as a single entity based on the relationship between the numerator and the denominator (Behr, Post and Wachsmuth 1986). Decimal notation is a way of writing fractions with denominators that are a power of ten and a logical extension to the right of the place value system for whole numbers. Kieren (1984) has pointed out that the latter interpretation does not necessarily provide an easy transition for students because dividing up (going from ones to tenths to hundredths) is different from grouping (going from ones to tens to hundreds).

Second, while whole numbers have a unique standard representation, rational numbers have many names. A rational number can be expressed as a fraction or a decimal, and within each coding system an infinite number of equivalent names can be generated. A nontrivial concept is that the properties and the size of a number remain the same regardless of how it is named.

Third, ordering is a more complicated and difficult process in the rational number system than in the set of whole numbers. Creative thinking, based on visualization and sound understanding, is needed to construct strategies for comparing pairs of fractions and/or decimals in a variety of situations. Standard procedures rely on writing equivalent forms with like denominators or on place value considerations. In addition, rational numbers are dense: between any two rational numbers is another rational numbers.

Unlike whole numbers, there is no number "next to" a given number, but sequences of numbers "closer and closer" to a number can be found.

Number Sense with Fractions and Decimals

Rational number sense has two related components, conceptual understanding and quantitative awareness and involves several abilities. Students display sense with rational numbers when they

1. represent numbers using words, models, diagrams and symbols and make connections among various representations;
2. give other names for numbers within and between the two notational systems and justify the procedures used to generate the equivalent forms;
3. describe the relative magnitude of numbers by comparing them to common benchmarks, giving simple estimates, ordering a set of numbers and finding a number between two numbers.

A Teaching Experiment

As part of a project designed to study the thinking strategies used by children as they construct rational number concepts, a teaching experiment was conducted in the fall of 1988 with six Grade 6 students. Three boys and three girls were taught 21 lessons and interviewed individually four times over a two-month period. The purpose of the instruction was to help the students develop basic concepts of rational numbers and acquire a quantitative feel for numbers expressed in fraction or decimal form.

The lessons included concrete, pictorial and symbolic representations for the numbers, equivalence and order relations and estimation. Operations were not taught, but story problems were used to investigate the possible transfer effects of the instruction. Fraction notation was introduced first; decimal numbers were then taught by building on the concepts, models and language (tenths, hundredths and thousandths) of fractions and by extending place value ideas. The importance of defining the unit and the notion of a variable unit were stressed throughout the experiment.

New ideas were presented as problems, and the students were challenged to use materials or extend their previous knowledge to devise solutions. Alternative methods were encouraged, discussed and evaluated. Students were sometimes asked to explain their answers and thought processes orally as well as in writing. During the interviews, the subjects were instructed to use concrete materials or diagrams to represent numbers or model relationships and to state and justify rules or thinking strategies used to obtain answers or make decisions.

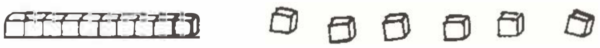
A Number Sense Interview

Tasks and questions that can be used to assess a variety of aspects of conceptual understanding and quantitative awareness of rational numbers follow. The material was taken from transcripts of lessons, interviews and from students' work. Spoken, written and concrete/pictorial responses are included.

Task 1: Represent 6/10

Interviewer: Show 6/10 using base-10 blocks, small cubes, long sticks, flats and cubes.

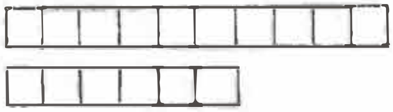
Student: If the long stick is one, then six small cubes would be 6/10.



Or if the flat is one, then I take six longs. If the cube is one, it would be six flats.

I: Can you show 6/10 using any of these strips? (Set of color-coded strips representing families of various fractions.)

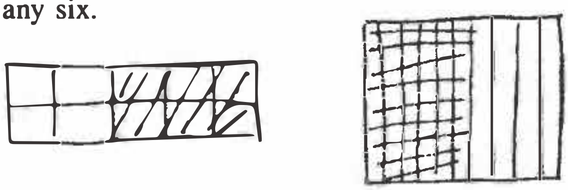
S: First I find the strip divided into 10 equal parts. Then I find a strip that has six parts the same size. It is 6/10 of the longer one.



This one is the same length, so it is 6/10 too, but it is also 3/5.

I: Make a sketch to show 6/10.

S: I divide a figure into 10 equal parts and shade any six.



I: If this is 6/10, sketch one.

S: You divide it into six equal parts and then add four more.

I: Show 6/10 on a number line.

S: You first mark the zero and the one. You divide that section into 10 equal parts, then you count over six from the left.

I: Write 6/10 as a number.

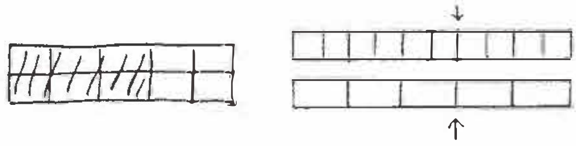
S: You can do it two ways.
6/10 and 0.6.

I: Why do these symbols mean 6/10?

S: For the fraction, the denominator indicates how many equal parts the whole is divided into, and the numerator indicates how many of the parts you take. For the decimal, the decimal points tells you that the zero is in the ones place. The place to the right must be tenths because it takes 10/10 to make one. So the six means 6/10.

I: Can you name 6/10 in another way?

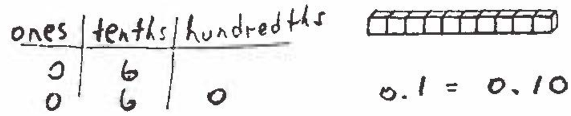
S: 6/10 = 12/20 = 18/30
To get equivalent fractions, you multiply the numerator and denominator by the same number. You can also make it 3/5 by dividing the 6 and the 10 by two. 3/5 is in simplest form. You can see that 3/5 = 6/10 from the diagram. The strips show it too.



You can also make equivalent decimals by adding zeros after the 6.

0.6 = 0.60 = 0.600

6/10 and 60/100 are the same because they just both have 6/10. You can see this with the blocks too. A 10-block has ten-hundredths.



I: Where would 18/30 go on the number line?

S: At the same point as 6/10. They are the same number, just different names.

Task 2: Model Decimals

I: Read this number, 1.3.

S: One and three-tenths.

I: Model it with these materials (long and small cubes only).

I: Can you read it or write it another way?

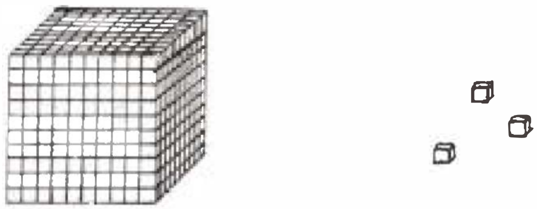
S: 13/10. As a fraction it is 1 3/10 and 13/10.

I: Read and model 1.03.

S: 1 and 3/100. You need the flat for the unit.

I: Read and model 1.003.

S: 1 and 3/1000. The cube is the unit.



I: Read and model 1.0003.

S: 1 and 3/10,000. To model you would need a unit that is ten cubes long. Or you could leave the cube as the unit and cut the small cube into 10 pieces to get ten-thousandths.

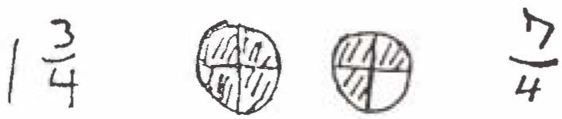
Task 3: Rename 1 3/4

I: Write your answers to the questions on this page.

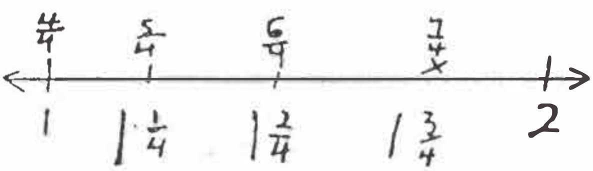
a. Write 1 3/4 as an improper fraction.

7/4

b. Draw a diagram to show that your answer is correct.



c. Mark a number line to show that your answer is correct.



Task 4: Order Numbers

I: Which is the greatest fraction?

1/7, 1/6, 1/8

S: 1/6. When the numerators are the same you go for the smallest denominator because there are less pieces so each piece is bigger.

I: 6/7, 8/9, 7/8

S: 8/9 is the biggest. It has the smallest piece left over.

I: 4/7, 3/8, 1/2

S: 4/7 is biggest because it's greater than a half, and 3/8 is smallest because it's less than half.

I: 2/3, 3/5

S: They are both over half and close together. They have different numerators and denominators. So you make equivalent fractions in fifteenths. 10/15 > 9/15.

I: 0.32, 0.302, 0.3

S: You look at the tenths place and if they're the same, you look at the hundredths place. 32/100 is the greatest and 3/10 is least. Or you can add a zero to 32/100, and add two zeros to 3/10. That one would be 3/100,000, and that one would be 320/1,000, and that would be 302/1,000.

I: Circle the least number, and write down how you decided.

S: 3.6, 3.58, 3.592.

I chose 3.58 for the least because it has 3 units, 5 tenths and 8 hundredths, and all the other ones have more tenths or hundredths.

I: 7/10, 0.68, 4/5.

S: You'd have to change 7/10 to a decimal and that's 68/100, so it would be 70/100, so that's the biggest one so far. 4/5 would be 80/100. 0.70, 0.68, 0.80.

Task 5: Between Two Numbers

I: Write a fraction between 3/5 and 4/5.

S: It could be 3 ½/5, but to name a fraction you make them 6/10 and 8/10, so 7/10 is between. You could also make them 30/50 and 40/50, and there are lots of numbers in between.

I: Write a decimal between 3.7 and 3.71.

S: 3.701. It's thousandths because that's 3 and 7/10 and that's 3 and 71/100. That's only one-hundredth more so you have to go to something smaller.

I: Write a number between 3/8 and 0.6.

S: 0.6 is the larger number because 3/5 is greater than 3/8. 3/7 is also between 3/5 and 3/8.

I: Write a number between 0 and 1/7.

S: 1/8.

I: Could you name another number even closer to zero?

S: 1/10, 1/100. You could go on forever.

Task 6: Estimate Sums

I: Estimate 1/3 + 0.327 + 4/12.

S: 1/3 + 1/3 + 1/3 = 1.

I: Estimate 0.243 + 15/31 + 1.019.

S: 1/4 + 1/2 + 1 = 1 3/4.

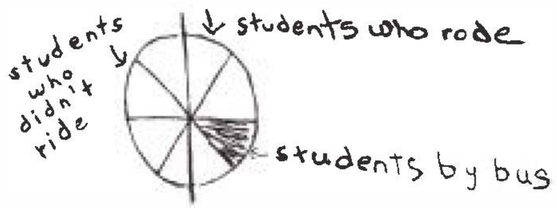
Task 7: Story Problems

I: Show how you arrived at the answers to the following questions.

Jack ate 1/2 of a cake and Mary ate 1/3 of the same cake. What fraction of the cake was eaten?

Answer: 5/6.

One-half of the students rode to school. One-quarter of the students who rode came by the bus. What fraction of all the students came by bus?



Answer: 1/8.

Conclusion

Students can build the mental imagery and understanding needed to make sense of rational numbers, but this awareness develops gradually and progress is often uneven. Obviously, all the students in this teaching experiment did not give all the above responses; answers often revealed lack of understanding or incorrect reasoning. Furthermore, concepts that were apparently well established one day would sometimes appear shaky when revisited at a later time or in a different context. Marked individual differences in the rate of learning, depth of understanding and strategy selection were also noted. Nevertheless, each of the six students made significant progress in acquiring knowledge about rational number size and relationships. Students also displayed increased confidence in their ability to learn and apply mathematics.

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