Canadian Mortgage Payments

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This paper shows how monthly mortgage payments are calculated, with special reference to Canadian mortgages. From this a ''loan progress formula'' is developed. I also explain the curious ''monthly interest factor'' that appears at the back of Canadian Mortgage Tables.

Let A_0 be the initial **mortgage amount** in dollars,
- p be the fixed **monthly payment** in dollars,
- n be the **number of months** of mortgage amortization and
- r be the **monthly interest rate**.

Monthly Payments

Our problem is to find p in terms of A_0, n and r.

Let A_k be the amount of principal owing immediately after the k^{th} (monthly) payment has been made, for $1 \leq k \leq n$. We see that

$$A_1 = A_0 + rA_0 - p = A_0(1+r) - p$$
$$A_2 = A_1 + rA_1 - p = A_1(1+r) - p = A_0(1+r)^2 - p - p(1+r)$$
$$A_3 = A_2 + rA_2 - p = A_2(1+r) - p = A_0(1+r)^3 - p - p(1+r) - p(1+r)^2$$

....

$$A_k = A_0(1+r)^k - p[1 + (1+r) + (1+r)^2 + \dots + (1+r)^{k-1}]$$

Then using the sum of a geometric progression

$$1 + a + a^2 + \dots + a^{k-1} = \frac{a^k - 1}{a - 1}\text{ , we get}$$

$$A_k = A_0(1+r)^k - p\,\frac{(1+r)^k - 1}{r}$$

After n months the mortgage is paid off, so that $A_n = 0$. That is

$$A_n = A_0(1+r)^n - p\,\frac{(1+r)^n - 1}{r} = 0$$

Hence the monthly payment is

$$p = A_0\,\frac{r(1+r)^n}{(1+r)^n - 1} \qquad \dots 1$$

Example 1

Find the monthly payment for a $100,000 mortgage @ 12% amortized over 10 years.

Here $A_0 = 100{,}000$, $n = 120$ months and $r = 1\% = 0.01$ per month.

Using equation 1 to calculate the monthly payment $p = \$100{,}000 \dfrac{(0.01)(1.01)^{120}}{(1.01)^{120} - 1} = \$1{,}434.71$

The Canadian Mortgage Tables give $1418.03 as the monthly payment for the above mortgage. This amount differs from that calculated above because the interest on Canadian mortgages is calculated "**semiannually,** not in advance." Thus, if the annual rate of a Canadian mortgage is R%, then the monthly rate is **not** $r = R/12\%$ (as in the U.S.), but r_c where $(1+r_c)^6 = 1 + R/2$.* That is

$$r_c = \sqrt[6]{1 + R/2} - 1 \qquad \ldots 2$$

Using $r_c = \sqrt[6]{1 + 0.06} - 1 = 0.009758794$ for r in equation **1**, for the above example ($A_0 = 100{,}000$, $R = 12\%$ and $n = 120$ months), we get

$p = \$100{,}000 \dfrac{(0.009758794)(1.009758794)^{120}}{(1.009758794)^{120} - 1} = \$1{,}418.03$, exactly as in the Tables.

Example 2

Calculate the monthly payment for a $75,000 Canadian mortgage over 25 years at 13.25%.

Using equation **1**, $p = A_0 \dfrac{r(1+r)^n}{(1+r)^n - 1}$ where $A_0 = 75{,}000$, $n = 300$, and $r = \sqrt[6]{1 + \dfrac{0.1325}{2}} - 1$,

that is, $r = 0.010748660$, $p = \$840.14$. (The Tables give $840.15.)

Loan Progress

After k monthly payments, the amount of principal owing is A_k. Recall

$$A_k = A_0 (1+r)^k - p\frac{(1+r)^k - 1}{r} \qquad \ldots 3$$

Example 3

Calculate the amount of principal owing on the mortgage in Example 2, after 10 years of payments. That is, $A_0 = \$75{,}000$, $p = \$840.14$, $r = 0.01074866$ and $k = 120$.

$A_{120} = \$75{,}000\,(1.01074866)^{120} - (840.14)\dfrac{(1.01074866)^{120} - 1}{0.01074866} = \$66{,}754.97$

(The Tables are only approximate and give $890 x 75 = $66,750.)

Example 4

Calculate the amount of principal owing on a $200,000 mortgage @ 14.75% over 40 years after 20 years of payments. (In the U.K. mortgages are often amortized over 35 and 40 years; 14.75% is a common rate.)

Here $A_0 = 200{,}000$, $n = 480$ and $r = \sqrt[6]{1 + \dfrac{0.1475}{2}} - 1 = 0.011930135$

Use equation **1** to find $p = \$2{,}394.10$, then equation **3** with $k = 240$, to get $A_{240} = \$189{,}021.57$, the amount of the $200,000 still owing after 20 years of payments.

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*r_c is such that the interest on the mortgage "when *compounded monthly* at a rate of r_c% per month" is **equivalent** to the interest on the mortgage "when **compounded semiannually** at a rate of R/2% per six months" (R% being the quoted annual rate for the mortgage).

Monthly Interest Factor

The monthly interest factors published at the back of Canadian Mortgage Tables are simply r_c for the appropriate quoted annual rate R. (See equation **2**.)

Example 5

For a mortgage with a 13.25% interest rate we saw in Example 2 that

$$r_c = \sqrt[6]{1 + \frac{0.1325}{2}} - 1 = 0.010748660.$$

This is the monthly interest factor for a 13.25% mortgage (see Tables). For a 14.75% mortgage we saw in Example 4 that

$$r_c = \sqrt[6]{1 + \frac{0.1475}{2}} - 1 = 0.011930135.$$

This is the monthly interest factor for a 14.75% mortgage (see Tables).

A **Hewlett Packard 27** calculator has the following buttons:

n	i	pmt	pv
months	$i = 100r_c$	p	A_0.

Note that the interest button i is not r_c but *100 times* r_c.

It is of interest that we could find
(a) A_0 in terms of p, n and r. Using equation 1, we see

$$A_0 = p\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$$

(b) n in terms of A_0, p and r. By rearranging equation 1 and taking logs (any base), we see

$$n = \frac{\log(p) - \log(p - rA_0)}{\log(1+r)}$$

To find r in terms of A_0, n and p is not so easy; we have to solve the polynomial equation

$$p = (1+r)^n(p - A_0 r)$$

to get r, which is difficult for $n > 2$; usually n is very large. An iterative process is used to approximate to the solution to the desired degree of accuracy.

Total Interest Paid

The total interest paid on a mortgage (over n months) is simply $np - A_0$.

Example 6

The interest paid on the $75,000 mortgage of Example 2 is
300 x $840.14 - $75,000 = $177,042.00.

The interest paid on the $200,000 mortgage of Example 4 is
480 x $2,394.10 - $200,000 = $949,168.00.

Monthly Interest Factors

Interest for one month at nominal annual rates shown,
based on interest compounded semiannually

12 %	.009 758 7942 ←	15⅛ %	.012 224 4297
12⅞ %	.010 452 2088	15¼ %	.012 322 4327
13 %	.010 551 0740	15⅜ %	.012 420 3883
13⅛ %	.010 649 8909	15½ %	.012 518 2966
13¼ %	.010 748 6596 ←	15⅝ %	.012 616 1575
13⅜ %	.010 847 3799	15¾ %	.012 713 9712
13½ %	.010 946 0522	15⅞ %	.012 811 7377
13⅝ %	.011 044 6762	16 %	.012 909 4570
13¾ %	.011 143 2522	16⅛ %	.013 007 1292
13⅞ %	.011 241 7802	16¼ %	.013 104 7543
14 %	.011 340 2602	16⅜ %	.013 202 3325
14⅛ %	.011 438 6923	16½ %	.013 299 8636
14¼ %	.011 537 0764	16⅝ %	.013 397 3478
14⅜ %	.011 635 4128	16¾ %	.013 494 7852
14½ %	.011 733 7014	16⅞ %	.013 592 1758
14⅝ %	.011 831 9423	17 %	.013 689 5196
14¾ %	.011 930 1355 ←	17⅛ %	.013 786 8166
14⅞ %	.012 028 2811	17¼ %	.013 884 0670
15 %	.012 126 3791	17⅜ %	.013 981 2708

Payment Table For Canadian Mortgages

Computed and Published by
Financial Publishing Company Massachusetts 1974

BASIC MONTHLY PAYMENT NECESSARY TO AMORTIZE A LOAN 12%

TERM AMOUNT	6 YEARS	7 YEARS	8 YEARS	9 YEARS	10 YEARS	11 YEARS	12 YEARS	13 YEARS	14 YEARS
25	.49	.44	.41	.38	.36	.34	.33	.32	.31
50	.97	.88	.81	.76	.71	.68	.65	.63	.61
75	1.46	1.32	1.21	1.13	1.07	1.02	.98	.94	.91
100	1.94	1.75	1.61	1.51	1.42	1.36	1.30	1.26	1.22
200	3.88	3.50	3.22	3.01	2.84	2.71	2.60	2.51	2.43
300	5.82	5.25	4.83	4.51	4.26	4.06	3.89	3.76	3.64
400	7.76	7.00	6.44	6.01	5.68	5.41	5.19	5.01	4.86
500	9.70	8.75	8.05	7.52	7.10	6.76	6.48	6.26	6.07
600	11.64	10.50	9.66	9.02	8.51	8.11	7.78	7.51	7.28
700	13.58	12.25	11.27	10.52	9.93	9.46	9.08	8.76	8.50
800	15.52	14.00	12.88	12.02	11.35	10.81	10.37	10.01	9.71
900	17.46	15.75	14.49	13.52	12.77	12.16	11.67	11.26	10.92
1000	19.40	17.50	16.10	15.03	14.19	13.51	12.96	12.51	12.14
2000	38.80	35.00	32.19	30.05	28.37	27.02	25.92	25.02	24.27
2500	48.50	43.75	40.24	37.56	35.46	33.77	32.40	31.28	30.34
3000	58.20	52.50	48.29	45.07	42.55	40.53	38.88	37.53	36.40
4000	77.60	70.00	64.38	60.09	56.73	54.03	51.84	50.04	48.53
5000	97.00	87.50	80.48	75.11	70.91	67.54	64.80	62.55	60.67
6000	116.40	104.99	96.57	90.13	85.09	81.05	77.76	75.05	72.80
7000	135.80	122.49	112.66	105.16	99.27	94.55	90.72	87.56	84.93
80000	1552.00	1399.87	1287.54	1201.72	1134.[illegible]	[illegible]	1036.77	1000.66	970.58
85000	1649.00	1487.36	1368.01	1276.83	1205.33	1148.11	1101.56	1063.20	1031.24
90000	1746.00	1574.85	1448.49	1351.94	1276.23	1215.64	1166.36	1125.75	1091.91
95000	1843.00	1662.35	1528.96	1427.05	1347.13	1283.18	1231.16	1188.29	1152.57
100000	1940.00	1749.84	1609.43	1502.15	1418.03	1350.71	1295.96	1250.83	1213.23

BASIC MONTHLY PAYMENT NECESSARY TO AMORTIZE A LOAN 13¼%

TERM AMOUNT	24 YEARS	25 YEARS	26 YEARS	27 YEARS	28 YEARS	29 YEARS	30 YEARS	35 YEARS	40 YEARS
25	.29	.29	.28	.28	.28	.28	.28	.28	.28
50	.57	.57	.56	.56	.56	.56	.55	.55	.55
75	.85	.85	.84	.84	.83	.83	.83	.82	.82
100	1.13	1.13	1.12	1.11	1.11	1.11	1.10	1.09	1.09
200	2.26	2.25	2.23	2.22	2.22	2.21	2.20	2.18	2.17
300	3.39	3.37	3.35	3.33	3.32	3.31	3.30	3.27	3.25
400	4.51	4.49	4.46	4.44	4.43	4.41	4.40	4.35	4.33
500	5.64	5.61	5.58	5.55	5.53	5.51	5.50	5.44	5.41
600	6.77	6.73	6.69	6.66	6.64	6.61	6.59	6.53	6.49
700	7.89	7.85	7.81	7.77	7.74	7.72	7.69	7.61	7.57
800	9.02	8.97	8.92	8.88	8.85	8.82	8.79	8.70	8.66
900	10.15	10.09	10.04	9.99	9.95	9.92	9.89	9.79	9.74
1000	11.27	11.21	11.15	11.10	11.06	11.02	10.99	10.88	10.82
2000	22.54	22.41	22.30	22.20	22.11	22.04	21.97	21.75	21.63
2500	28.17	28.01	27.87	27.75	27.64	27.54	27.46	27.18	27.04
3000	33.81	33.61	33.44	33.29	33.16	33.05	32.95	32.62	32.44
4000	45.07	44.81	44.59	44.39	44.22	44.07	43.94	43.49	43.26
5000	56.34	56.01	55.73	55.49	55.27	55.08	54.92	54.36	54.07
6000	67.61	67.22	66.88	66.58	66.32	66.10	65.90	65.23	64.88
7000	78.87	78.42	78.02	77.68	77.38	77.11	76.88	76.10	75.69
8000	90.14	89.62	89.17	88.77	88.43	88.13	87.87	86.97	86.51
9000	101.41	100.82	100.31	99.87	99.48	99.14	98.85	97.84	97.32
10000	112.67	112.02	111.46	110.97	110.54	110.16	109.83	108.71	108.13
11000	123.94	123.23	122.60	122.06	121.59	121.18	120.81	119.58	118.94
12000	135.21	134.43	133.75	133.16	132.64	132.19	131.80	130.45	129.76
55000	619.69	616.11	613.00	610.29	607.92	605.86	604.05	597.89	594.69
60000	676.02	672.12	668.72	665.77	663.19	660.93	658.96	652.24	648.76
65000	732.36	728.13	724.45	721.25	718.45	716.01	713.88	706.59	702.82
70000	788.69	784.14	780.18	776.73	773.72	771.09	768.79	760.95	756.88
75000	845.03	840.15	835.90	832.21	828.98	826.16	823.70	815.30	810.94
80000	901.36	896.16	891.63	887.69	884.25	881.24	878.62	869.65	865.01
85000	957.70	952.17	947.36	943.17	939.51	936.32	933.53	924.01	919.07
90000	1014.03	1008.18	1003.08	998.65	994.78	991.40	988.44	978.36	973.13
95000	1070.37	1064.19	1058.81	1054.13	1050.04	1046.47	1043.35	1032.71	1027.19
100000	1126.70	1120.20	1114.54	1109.61	1105.31	1101.55	1098.27	1087.06	1081.26

BASIC MONTHLY PAYMENT NECESSARY TO AMORTIZE A LOAN 14¾%

TERM AMOUNT	24 YEARS	25 YEARS	26 YEARS	27 YEARS	28 YEARS	29 YEARS	30 YEARS	35 YEARS	40 YEARS
25	.31	.31	.31	.31	.31	.31	.31	.31	.30
50	.62	.62	.62	.61	.61	.61	.61	.61	.60
75	.93	.93	.92	.92	.92	.91	.91	.91	.90
100	1.24	1.23	1.23	1.22	1.22	1.22	1.21	1.21	1.20
200	2.47	2.46	2.45	2.44	2.44	2.43	2.42	2.41	2.40
300	3.71	3.69	3.67	3.66	3.65	3.64	3.63	3.61	3.60
400	4.94	4.92	4.90	4.88	4.87	4.86	4.84	4.81	4.79
500	6.17	6.15	6.12	6.10	6.08	6.07	6.05	6.01	5.99
600	7.41	7.37	7.34	7.32	7.30	7.28	7.26	7.21	7.19
700	8.64	8.60	8.57	8.54	8.51	8.49	8.47	8.41	8.38
800	9.87	9.83	9.79	9.76	9.73	9.71	9.68	9.62	9.58
900	11.11	11.06	11.01	10.98	10.95	10.92	10.89	10.82	10.78
1000	12.34	12.29	12.24	12.20	12.16	12.13	12.10	12.02	11.98
2000	24.68	24.57	24.47	24.39	24.32	24.26	24.20	24.03	23.95
2500	30.84	30.71	30.59	30.48	30.40	30.32	30.25	30.04	29.93
3000	37.01	36.85	36.70	36.58	36.47	36.38	36.30	36.04	35.92
4000	49.35	49.13	48.94	48.77	48.63	48.51	48.40	48.06	47.89
5000	61.68	61.41	61.17	60.96	60.79	60.63	60.50	60.07	59.86
6000	74.02	73.69	73.40	73.15	72.94	72.76	72.60	72.08	71.83
7000	86.35	85.97	85.63	85.35	85.10	84.89	84.70	84.09	83.80
8000	98.69	98.25	97.87	97.54	97.25	97.01	96.80	96.11	95.77
9000	111.02	110.53	110.10	109.73	109.41	109.14	108.90	108.12	107.74
10000	123.36	122.81	122.33	121.92	121.57	121.26	121.00	120.13	119.71
11000	135.70	135.09	134.56	134.11	133.72	133.39	133.10	132.14	131.68
12000	148.03	147.37	146.80	146.30	145.88	145.51	145.20	144.16	143.65
80000	986.84	982.41	978.61	975.33	972.50	970.06	967.96	961.02	957.64
85000	1048.52	1043.81	1039.77	1036.28	1033.28	1030.69	1028.45	1021.08	1017.50
90000	1110.20	1105.22	1100.93	1097.24	1094.06	1091.32	1088.95	1081.14	1077.35
95000	1171.87	1166.62	1162.09	1158.20	1154.84	1151.95	1149.45	1141.20	1137.20
100000	1233.55	1228.02	1223.26	1219.16	1215.62	1212.58	1209.95	1201.27	1197.05

LOAN PROGRESS CHART

Showing the dollar balance remaining per $1000 of loan

INTEREST RATE	ORIG. TERM	ELAPSED TERM IN YEARS 2	5	8	10	12	15	18	20	22	25
10¾%	10	874	628	291							
	15	939	819	656	515	340					
	20	967	903	816	740	647	465	215			
	25	982	946	897	854	802	700	560	440	291	
	30	989	969	941	916	886	828	748	678	593	426
11%	10	875	631	293							
	15	940	822	660	519	344					
	20	968	906	820	745	652	470	218			
	25	982	948	900	858	807	706	566	445	295	
	30	990	970	943	920	890	833	754	685	600	432
13%	10	886	652	311							
	15	949	844	690	551	371					
	20	975	923	848	779	691	508	242			
	25	987	961	922	887	842	748	612	488	329	
	30	993	979	959	941	917	869	798	733	650	478
13¼%	10	888	655	313							
	15	950	846	694	554	374					
	20	976	925	851	783	695	513	245			
	25	988	962	924	890	846	753	618	493	333	
	30	994	980	961		920	873	803	739	656	484
14¾%	10	895	671	326							
	15	956	861	715	577	394					
	20	980	936	869	806	722	541	263			
	25	990	970	938	908	867	781	649	524	358	
	30	995	985	970	955	936	894	830	770	689	516