Another Proof of Heron's Formula for the Area of a Triangle

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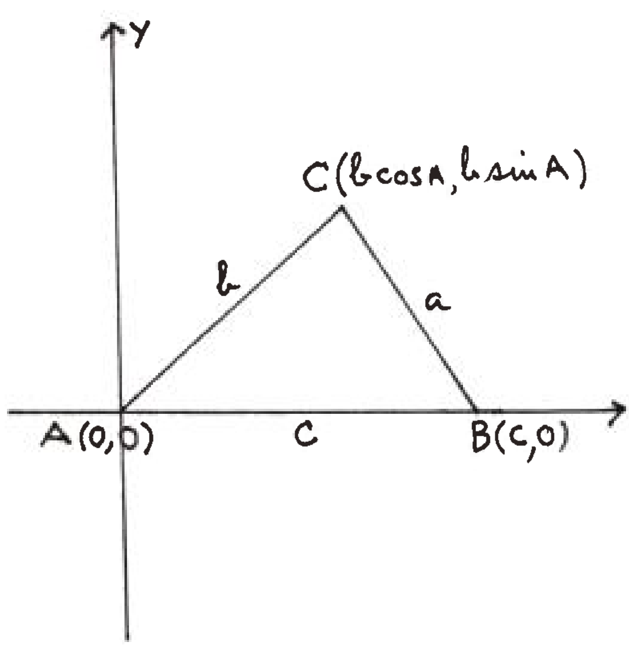
Heron of Alexandria (60 A.D.?) was a Greek mathematician. His mathematical works, included in *Metrica,* deal with mensuration of rectilinear figures and solids. His unique contribution to mathematics, however, is his famous relation for the area of a triangle in terms of its sides. Heron's formula states:

Given a triangle with sides equal in length to a, b and c, and s its semiperimeter, then the area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$

This article presents an analytical geometrical demonstration of the relation.



Let ABC be the triangle with sides AB = c, BC = a and CA = b, and coordinates of A = (0,0), B = (c,0) and C = (b cosA, b sinA). Then, by coordinate geometry, the area X of the triangle is given by:

$$X = \tfrac{1}{2}\begin{vmatrix} 0 & 0 \\ c & 0 \\ b\cos A & b\sin A \end{vmatrix}$$

$$= \tfrac{1}{2}\,[bc\sin A] \qquad \ldots 1$$

From triangle ABC, by cosine rule, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Therefore

$$\sin^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{(2bc)^2 - (b^2 - (b^2 + c^2 - a^2)^{\;2}}{4b^2c^2}$$

$$\sin^2 A = \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2}$$

$$= \frac{[(b + c)^2 - a^2]\,[(a^2 - (b - c)^2]}{4b^2c^2}$$

$$= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4b^2c^2}$$

$$= \frac{16\,s(s - a)(s - b)(s - c)}{4b^2c^2}$$

$$\sin A = \frac{2\sqrt{s(s - a)(s - b)(s - c)}}{bc}$$

From 1, area $X = \dfrac{\tfrac{1}{2}\,bc.2\sqrt{s(s - a)(s - b)(s - c)}}{bc}$

$$X = \sqrt{s(s - a)(s - b)(s - c)}$$

Thus Heron's formula for the area of a triangle is demonstrated.