A Circular Graphing Activity: Digital Roots of Powers of Natural Numbers

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Graphical representations are an increasingly important topic in contemporary mathematics curricular materials. *The Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) suggests that the middle school curriculum provide opportunities for students to describe, extend, analyze and create a wide variety of patterns and to describe and represent relationships with graphs. It is important that students have a variety of graphical experiences at all levels. We present a graphing activity for middle school students to do with the powers of natural numbers and their digital roots.

To find the digital root of a natural number, find the sum of its digits. If this sum is a single digit number, then this number is the digital root. If the sum is a number of two or more digits, repeat the process until a single digit number results. For example, the digital root of 34 is 3 + 4 or 7. The digital root of 896 is 8 + 9 + 6 = 23. As 23 is not a single digit number, repeat the process; that is 2 + 3 = 5. So 5 is the digital root of 896.

Activity 1

1. Consider the third powers (cubes) of the natural numbers and their corresponding digital roots as shown in Table 1.

   Note that the digital roots repeat indefinitely in a 1-8-9 pattern.
2. Graph these digital roots using a circle with nine equally spaced points marked on the circumference.

   The points are connected to make the graph easier to read; the 1-8-9 pattern is evident.

Table 1

Natural Number	Cubes of the Natural Numbers	Digital Roots of the Cubes
1	1	1
2	8	8
3	27	9
4	64	1
5	125	8
6	216	9
7	343	1
8	512	8
9	729	9
10	1000	1
11	1331	8
12	1728	9
13	2197	1
14	2744	8
15	3375	9
16	4096	1
17	4913	8
18	5832	9
19	6859	1
⋮	⋮	⋮

Activity 2

1. Consider the second powers (squares) and their digital roots as shown in Table 2.

   Observe that the digital roots repeat in a longer set of numbers. The pattern is 1-4-9-7-7-9-4-1-9.
2. Graph the digital roots as in Activity 1.

   The pattern of the digital roots can be seen from the graph. Note that this pattern does not form a simple closed curve, as in Graph 1.

Activity 3

Have your students construct the graphs of the digital roots of the following powers of the natural numbers 1, 4, 5, 6, 7, 8 and 9. The results should be as shown in Table 3 and Graphs 3 through 9.

Graph 1

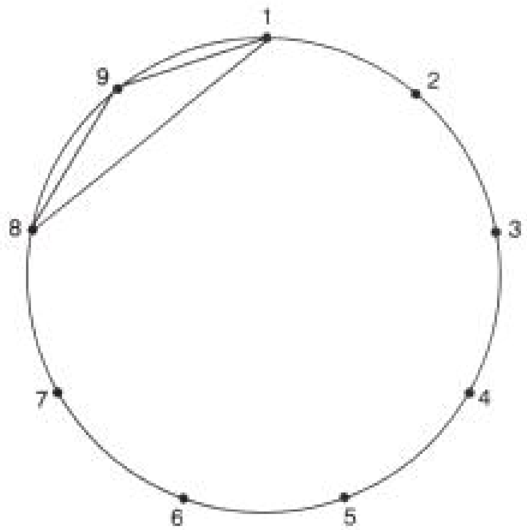
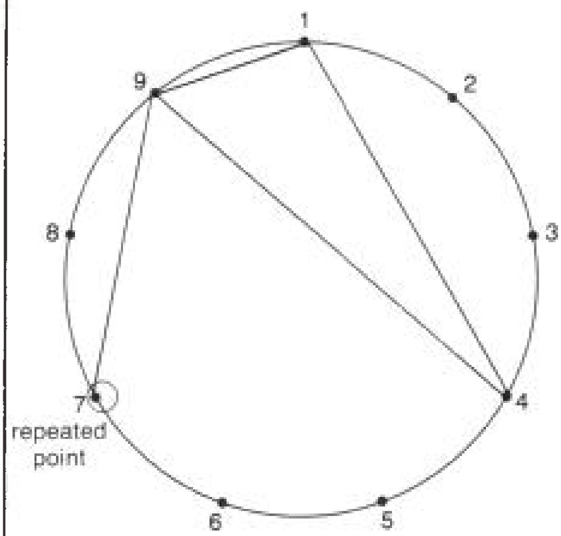


Table 2

Natural Number	Squares of the Natural Numbers	Digital Roots of the Squares
1	1	1
2	4	4
3	9	9
4	16	7
5	25	7
6	36	9
7	49	4
8	64	1
9	81	9
10	100	1
11	121	4
12	144	9
13	169	7
14	196	7
15	225	9
16	256	4
17	289	1
18	324	9
19	361	1
⋮	⋮	⋮

Graph 2



Graph 3
Digital Roots of First Powers

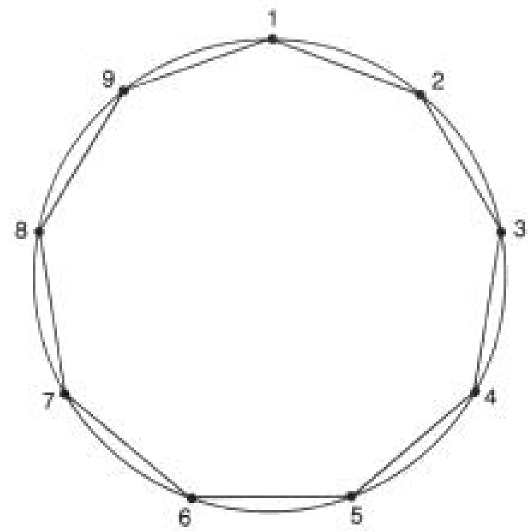


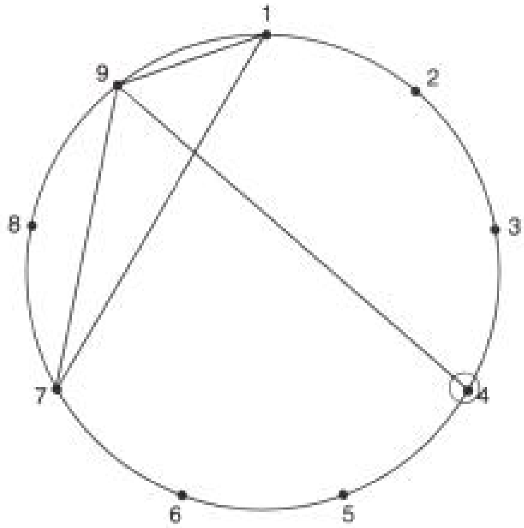
Table 3

Natural Number	First Powers	DR	Fourth Powers	DR	Fifth Powers	DR	Sixth Powers	DR	Seventh Powers	DR	Eighth Powers	DR	Ninth Powers	DR
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	16	7	32	5	64	1	128	2	256	4	512	8
3	3	3	81	9	243	9	729	9	2,187	9	6,561	9	19,683	9
4	4	4	256	4	1,024	7	4,096	1	16,384	4	65,536	7	262,144	1
5	5	5	625	4	3,125	2	15,625	1	78,125	5	390,625	7	1,953,125	8
6	6	6	1,296	9	7,776	9	46,656	9	279,936	9	1,679,616	9	10,077,696	9
7	7	7	2,401	7	16,807	4	117,649	1	823,543	7	5,764,801	4	40,353,607	1
8	8	8	4,096	1	32,768	8	262,144	1	2,097,152	8	16,777,216	1	134,217,728	8
9	9	9	6,561	9	59,049	9	531,441	9	4,782,969	9	43,046,721	9	387,420,489	9
10	10	1	10,000	1	100,000	1	1,000,000	1	10,000,000	1	100,000,000	1	1,000,000,000	1
11	11	2	14,641	7	161,051	5	1,771,561	1	19,487,171	2	214,358,881	4	*	8
12	12	3	20,736	9	248,832	9	2,985,984	9	35,831,808	9	429,981,696	9	*	9
13	13	4	28,561	4	371,293	7	4,826,809	1	62,748,517	4	815,730,721	7	*	1
14	14	5	38,416	4	537,824	2	7,529,536	1	105,413,504	5	*	7	*	8
15	15	6	50,625	9	759,375	9	11,390,625	9	170,859,375	9	*	9	*	9
16	16	7	65,536	7	1,048,576	4	16,777,216	1	268,435,456	7	*	4	*	1
17	17	8	83,521	1	1,419,857	8	24,137,569	1	410,338,673	8	*	1	*	8
18	18	9	104,976	9	1,889,568	9	34,012,224	9	612,220,032	9	*	9	*	9
19	19	1	130,321	1	2,476,099	1	47,045,099	1	893,871,739	1	*	1	*	1

* These numbers are not listed due to their length. Have your students calculate them. A computer might be useful for this task.

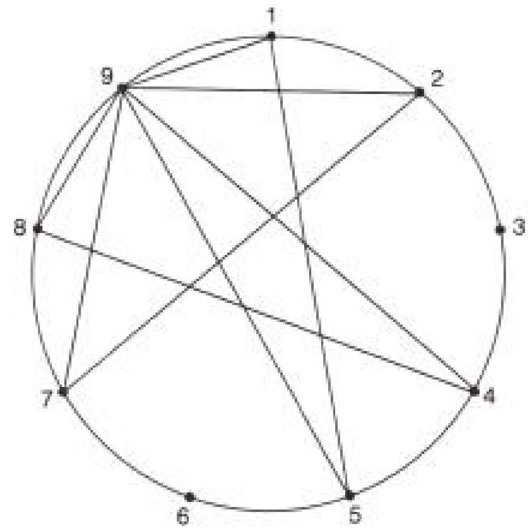
Graph 4

Digital Roots of Fourth Powers



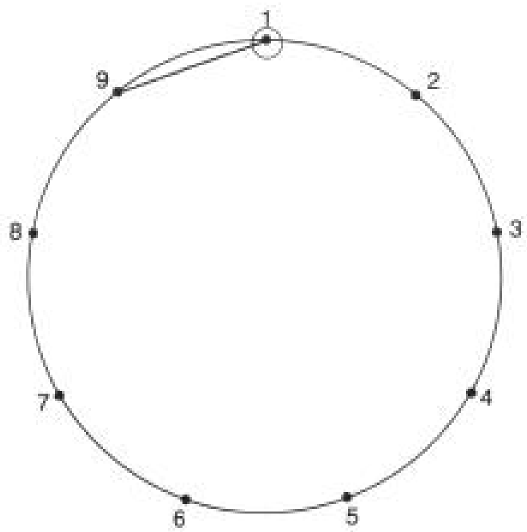
Graph 5

Digital Roots of Fifth Powers



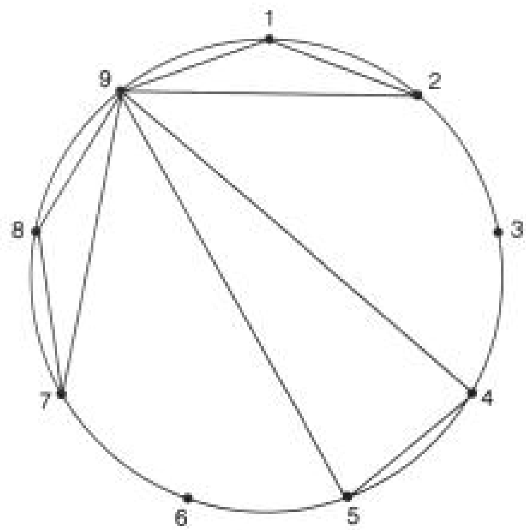
Graph 6

Digital Roots of Sixth Powers



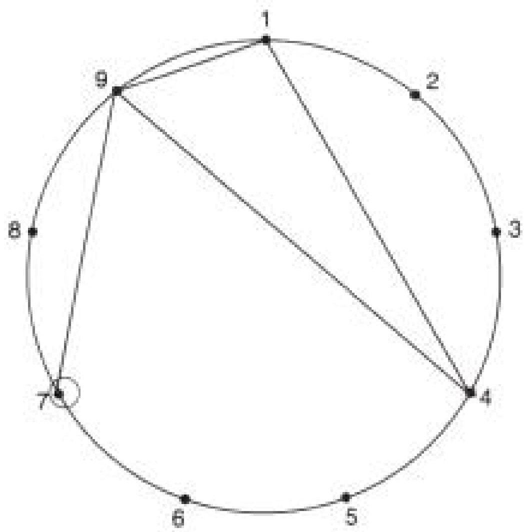
Graph 7

Digital Roots of Seventh Powers



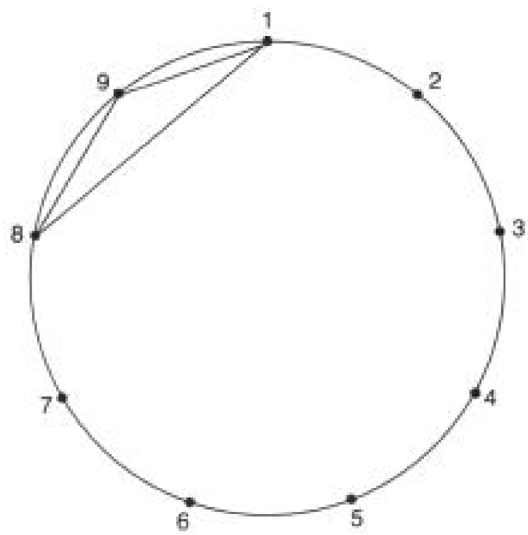
Graph 8

Digital Roots of Eighth Powers



Graph 9

Digital Roots of Ninth Powers



From the tables and graphs observe that the lengths and repetitive patterns of digital roots are

Powers	Length of
First	9 digits
Second	9 digits
Third	3 digits
Fourth	9 digits
Fifth	9 digits
Sixth	3 digits
Seventh	9 digits
Eighth	9 digits
Ninth	3 digits

The lengths are all 9 or a divisor of 9.

Challenges

Note that the digital root patterns for the second and eighth powers are identical, as they are for the third and ninth powers.

1. Have your students graph the tenth, eleventh, twelfth . . . powers. What further identical patterns emerge? (For $p > 1$, the digital roots of the pth powers and the $p + 6$th powers are identical.)
2. Compute the multiples of the natural numbers and graph their digital roots. What patterns emerge?

Reference

Curriculum and Evaluation Standards for School Mathematics. Commission on Standards for School Mathematics of the National Council of Teachers of Mathematics Staff, 1989.