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# PROBLEM Solving



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## EDITORIAL

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During the Calgary Conference of the National Council of Teachers of Mathematics, we were asked to become coeditors of *delta-K*, in lieu of Linda Brandau, who can no longer continue as editor because of illness. We would like to acknowledge her contribution, and we and the Mathematics Council of the ATA wish her a speedy recovery.

We believe *delta-K* should

- serve the mathematics teachers of Alberta,
- provide a vehicle for the exchange of ideas among mathematics teachers,
- communicate advances in mathematics instruction to teachers outside Alberta.

We anticipate publishing two more issues of this journal this school year. The section featuring teaching ideas, initiated by Dr. Brandau, will be retained. To achieve these ambitious objectives we need your cooperation in sending us your manuscripts. Don't be misled into thinking that membership in MCATA is limited to reading. Members' ideas and reactions are vital contributions.

In this issue

- Dale Burnett talks about the notion of randomness and demonstrates the power of guessing, testing, refining and discovery;
- Irvin Burbank suggests that a continuing focus on problem solving may have to be revisited in the '90s;
- Dennis Connolly presents an analysis of mortgage rates, a potential topic for senior grades;
- Gurcharan Singh Bhalla provides an analytic geometric proof of Heron of Alexandria's ancient formula for determining the area of a triangle given the length of its sides;
- David Duncan and Bonnie Litwiller, regular contributors to *delta-K*, continue their analysis of number patterns.

We contributed the last two articles as ideas for teaching. However, we would much rather publish *your* articles!

*A. Craig Loewen  
John B. Percevault*

# Computer Programming in the Schools: Emphasizing Structure—An Example Using “Inebriated Turtles”

*J. Dale Burnett*

*Dale Burnett is a professor of computer education at the University of Lethbridge, Lethbridge, Alberta.*

The following account should be interpreted at many levels. Life and “artificial life” are too complex to permit simpleminded generalizations and conclusions.

This article itself is a complex product. Obtained by incorporating one software package (Logo) into another (Word 4.0), and with structural flowcharts inserted from a third (More 3.0), it exemplifies some of the possibilities that are now within the reach of students and teachers with access to computers.

The vehicle I used for exploration was the Logo programming language. The activity I describe is thus a Logo programming exercise designed to solve the problem: How can you construct a set of Logo procedures that direct the turtle to perform a random walk within an enclosed space?

I look at random walks because this exercise provides an opportunity to develop an appreciation of randomness. What does it mean to talk about degrees of randomness? Are there different types of randomness? What do you notice when you look at a random path? Looking at randomness is also a way of exploring a new topic. How do you go about it? There are obviously many ways. What follows is a description of my first steps.

These procedures were originally written using Terrapin Logo for use on an Apple IIe. However, the examples of screen displays that follow were created using Logo (LCSI version 1.0) on a Macintosh computer. Some changes to the procedures may be necessary depending on the version of Logo that you use. This account builds on ideas from *Turtle Geometry* by Abelson and diSessa (pp. 56-57).

The concept of randomness may be approached from either end of the continuum of randomness. Thus one may either begin with

1. “pure” random behavior and slowly impose constraints, or
2. “pure” nonrandom behavior and slowly impose randomness.

## First Approach

Here I follow the first alternative (I leave the second for your consideration).

```
TO SIMPLE.RANDOM.WALK
  RT RANDOM 360
  FD RANDOM 20
  SIMPLE.RANDOM.WALK
END
```

Here are two samples of screen displays that were created by typing SIMPLE.RANDOM.WALK.



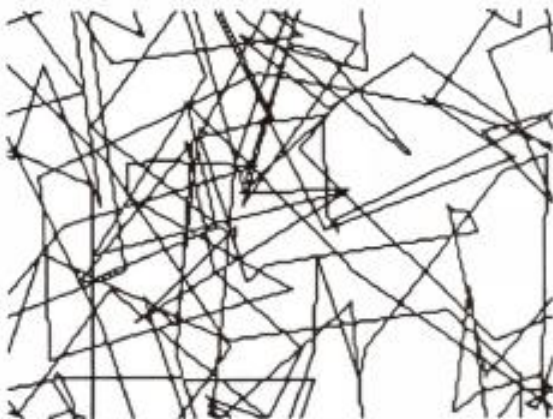
With Terrapin Logo, Ctl-G will stop the stager. Other versions have a similar program-interrupt feature. The “endless” feature results from the third line in the SIMPLE.RANDOM.WALK procedure, where the program calls itself (forever!). The term for a program calling itself as a subprogram is *recursion*.

A principle for exploration is *wherever there is a constant, substitute a variable*.

```
TO SRW :MAX.DEGREE.TURN :MAX.STEP.SIZE
  RT RANDOM :MAX.DEGREE.TURN
  FD RANDOM :MAX.STEP.SIZE
  SRW :MAX.DEGREE.TURN :MAX.STEP.SIZE
END
```

SRW stands for SIMPLE.RANDOM.WALK. Abbreviations are important when the name of a procedure must be typed repeatedly. :MAX.DEGREE.TURN and :MAX.STEP.SIZE are variable inputs. Meaningful labels for variables are also important, as they help one understand the logic of the procedures and aid in debugging. They should not be shortened. If you are not familiar with the idea of variables, refer to a Logo reference book for more information.

Try different values for :MAX.DEGREE.TURN and :MAX.STEP.SIZE. This permits you to control the degree of randomness. You may wish to save/print some pictures.



SRW 360 100



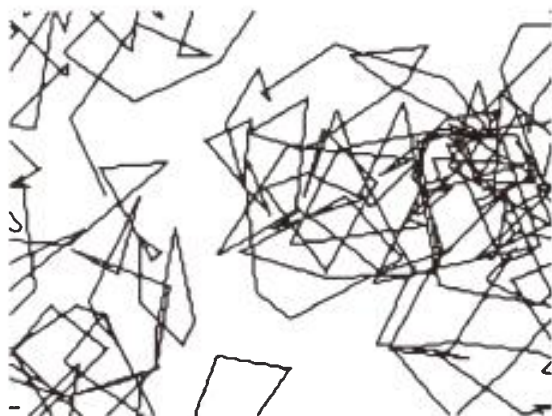
SRW 360 50



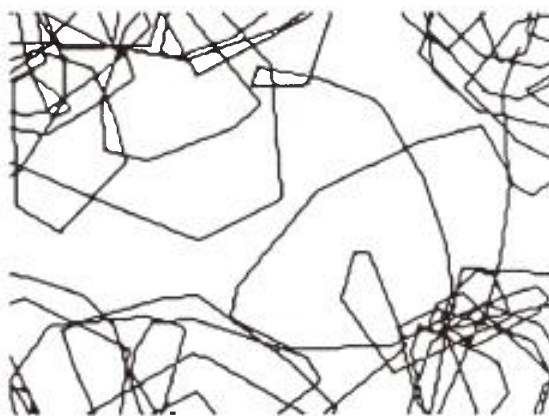
SRW 360 25



SRW 360 10



SRW 180 50



SRW 90 50



SRW 45 50



SRW 20 50

Notice how the turtle's path begins to approximate a curved line as the degree of randomness in the turn is restricted. If one were to restrict the randomness in the right turn to zero, the path would become a straight line. This example indicates a weakness in the randomness associated with the orientation of the turtle. It is better in the intuitive sense if the randomness were to fluctuate about zero (that is, permit right and left turns to occur with equal frequency) rather than about a restriction between 0 degrees and some other (positive) number. This is easily accommodated.

```

TO SRW :MAX.DEGREE.TURN :MAX.STEP.SIZE
  RT ( :MAX.DEGREE.TURN / 2) - RANDOM :MAX.DEGREE.TURN
  FD RANDOM :MAX.STEP.LENGTH
  SRW :MAX.DEGREE.TURN :MAX.STEP.SIZE
END

```

Do you appreciate what is happening in the first line of the procedure? Let's see what effect this change has on the paths of the turtle.



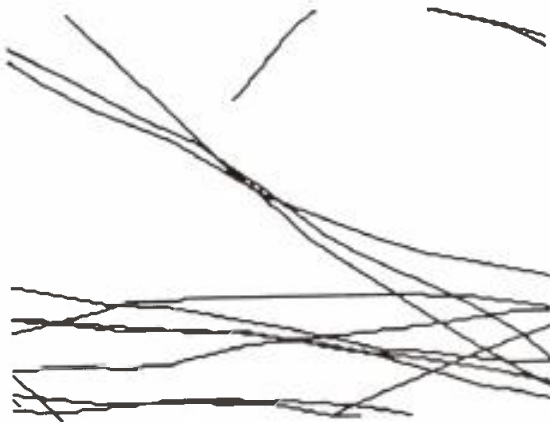
SRW 360 50



SRW 180 50

Do you see the difference? Do you understand the difference? Do you have a preference for one version of the SRW procedure? Why or why not? What would happen for SRW 20 50?

Here is what actually happened.



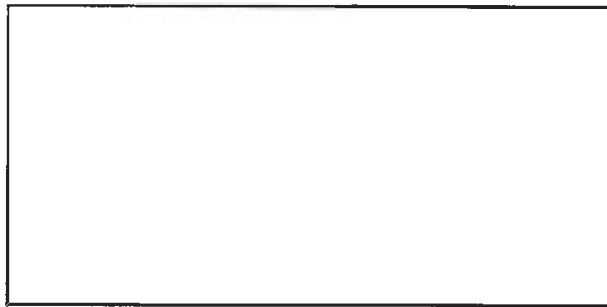
Are you surprised? Why? Why not?

Many people find the wrap feature, where the turtle disappears off one side of the screen and reappears on the other, to be annoying during such an exploration. This can be obviated by creating a room and placing the turtle inside it.

## Second Approach

Making a (rectangular) room is easy.

```
TO DRAW.ROOM :LEFT.SIDE :RIGHT.SIDE :TOP :BOTTOM
  CLEARSCREEN
  HT
  PU
  SETXY :LEFT.SIDE :BOTTOM
  PD
  SETXY :RIGHT.SIDE :BOTTOM
  SETXY :RIGHT.SIDE :TOP
  SETXY :LEFT.SIDE :TOP
  SETXY :LEFT.SIDE :BOTTOM
  PU
END
DRAW.ROOM -100 100 50 (-50)
```

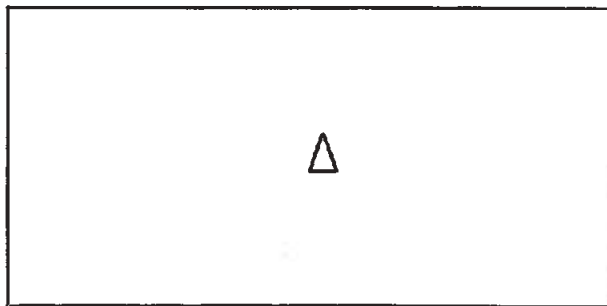


Note the use of variables! It is now easy to change the size of the room.

At first glance this is a much more cumbersome procedure than the familiar local geometry procedure of REPEAT 4 [FD 50 RT 90]. Looking ahead, we are going to need some way of testing whether or not the turtle is "hitting a wall." Logo does not have a pixel-detection primitive, so we will have to trick the computer into acting as if it had such a primitive. This can be accomplished by testing the  $x$  and  $y$  coordinates of the turtle's position against the *known* values of the boundaries of the room. Thus we want to make these coordinates explicit.

Placing the turtle inside the room is also easy.

```
TO PLACE.TURTLE :TURTLES.X.COORD :TURTLES.Y.COORD
  PU
  SETXY :TURTLES.X.COORD :TURTLES.Y.COORD
  PD
  ST
END
PLACE.TURTLE 0 0
```

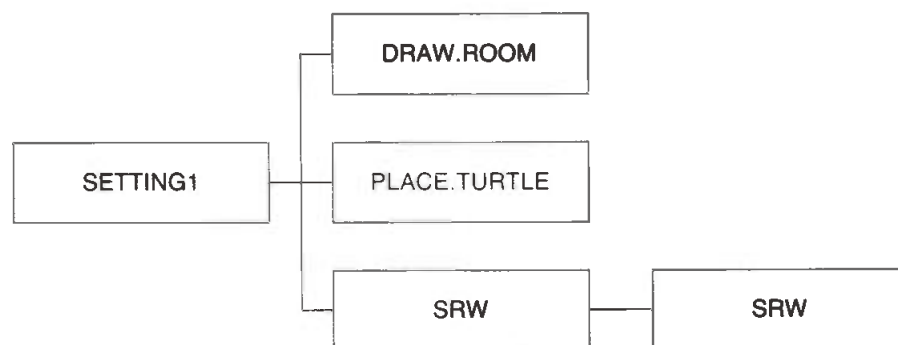




Now set up a master procedure that calls these three ideas.

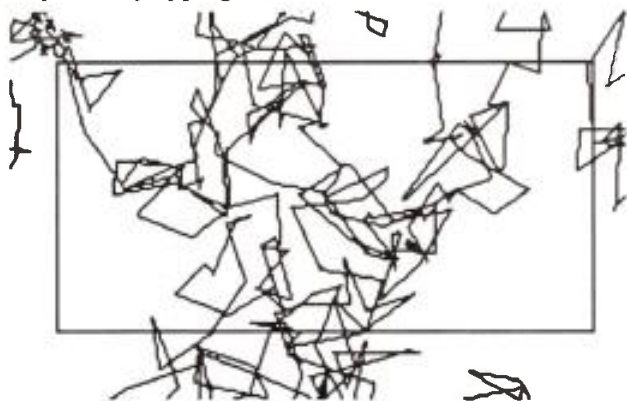
```
TO SETTING1
  DRAW.ROOM -100 100 50 (-50)
  PLACE.TURTLE 0 0
  SRW 360 30
END
```

It is often useful to visualize the structure of such a procedure. Here is an example of a chart that does this.



This diagram is read as follows: procedure SETTING1 calls procedure DRAW.ROOM, then procedure PLACE.TURTLE and then procedure SRW. Procedure SRW in turn calls itself (recursion). In Logo jargon, the task has been broken into “mind-size” chunks. Conceptually, the idea is to call one procedure, SETTING1, which then calls a sequence of other procedures successively to draw a room, place the turtle in it and then iterate a random walk.

Try this by typing SETTING1.



At least two problems immediately become apparent:

1. Ctl-G is a clumsy way to stop the turtle.
2. The walls have no effect!

Let's tackle the easy one first.

### Third Approach

One way to stop a turtle is to have it run out of gas (alcohol?!). Thus you must alter the main procedure to include a variable indicating the number of turtle steps you want the little fellow to take.

By editing SETTING1, and changing its name to SETTING2, I am able to modify the original procedure without having to retype the entire procedure. Altering the name (by changing the suffix to 2) leaves the original procedure intact in case I wish to go back to it later. It also gives me a form of development trail as new ideas occur to me.

```

TO SETTING2 :TOTAL.NO.OF.STEPS.TO.BE.TAKEN
  INITIALIZE
  DRAW.ROOM
  PLACE.TURTLE 0 0
  SRW2 :TOTAL.NO.OF.STEPS.TO.BE.TAKEN
END

```

The new procedure incorporates two ideas. One is maintaining a counter that will keep track of the number of steps that the turtle takes. The second is to create a separate procedure that will establish all the necessary initial values for running the procedures.

```

TO INITIALIZE
  MAKE "NO.OF.STEPS.TAKEN 0
  MAKE "LEFT.SIDE (-100)
  MAKE "RIGHT.SIDE 100
  MAKE "TOP 50
  MAKE "BOTTOM (-50)
  MAKE "MAX.DEGREE.TURN 360
  MAKE "MAX.STEP.SIZE 30
END

```

The input variable TOTAL.NO.OF.STEPS.TO.BE.TAKEN specifies the value of the number of steps that you want the turtle to take and the variable NO.OF.STEPS.TAKEN acts as the counter that will be incremented by one each time the turtle takes a step. LEFT.SIDE, RIGHT.SIDE, TOP, and BOTTOM refer to the sides of the room. MAX.DEGREE.TURN and MAX.STEP.SIZE refer to the parameters controlling the degree of randomness the turtle will exhibit.

DRAW.ROOM must also be modified to change the way the room's dimensions are specified.

```

TO DRAW.ROOM2
  CLEARSCREEN
  HT
  PU
  SETXY :LEFT.SIDE :BOTTOM
  PD
  SETXY :RIGHT.SIDE :BOTTOM
  SETXY :RIGHT.SIDE :TOP
  SETXY :LEFT.SIDE :TOP
  SETXY :LEFT.SIDE :BOTTOM
  PU
END

```

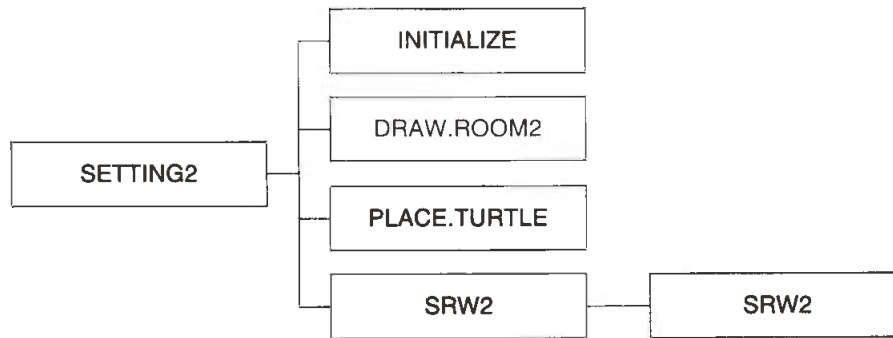
Finally, SRW must be modified to keep track of the number of steps that the turtle takes.

```

TO SRW2 :NO.OF.STEPS.TO.BE.TAKEN
  RT (:MAX.DEGREE.TURN / 2) - RANDOM :MAX.DEGREE.TURN
  FD RANDOM :MAX.STEP.SIZE
  MAKE "NO.OF.STEPS.TAKEN :NO.OF.STEPS.TAKEN + 1
  IF :NO.OF.STEPS.TAKEN = :NO.OF.STEPS.TO.BE.TAKEN STOP
  SRW2 :NO.OF.STEPS.TO.BE.TAKEN
END

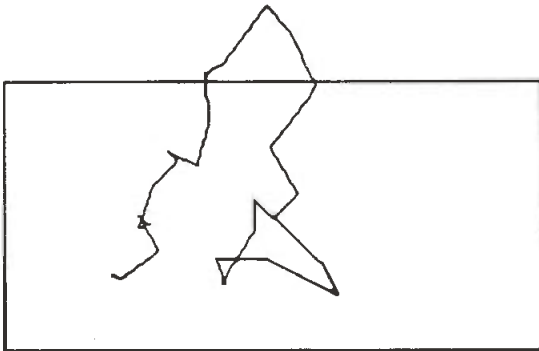
```

The corresponding procedural flowchart is:



Although there has been a number of changes to the Logo procedures, the logical flow of the task remains essentially the same.

Here is an example of running SETTING2 30.



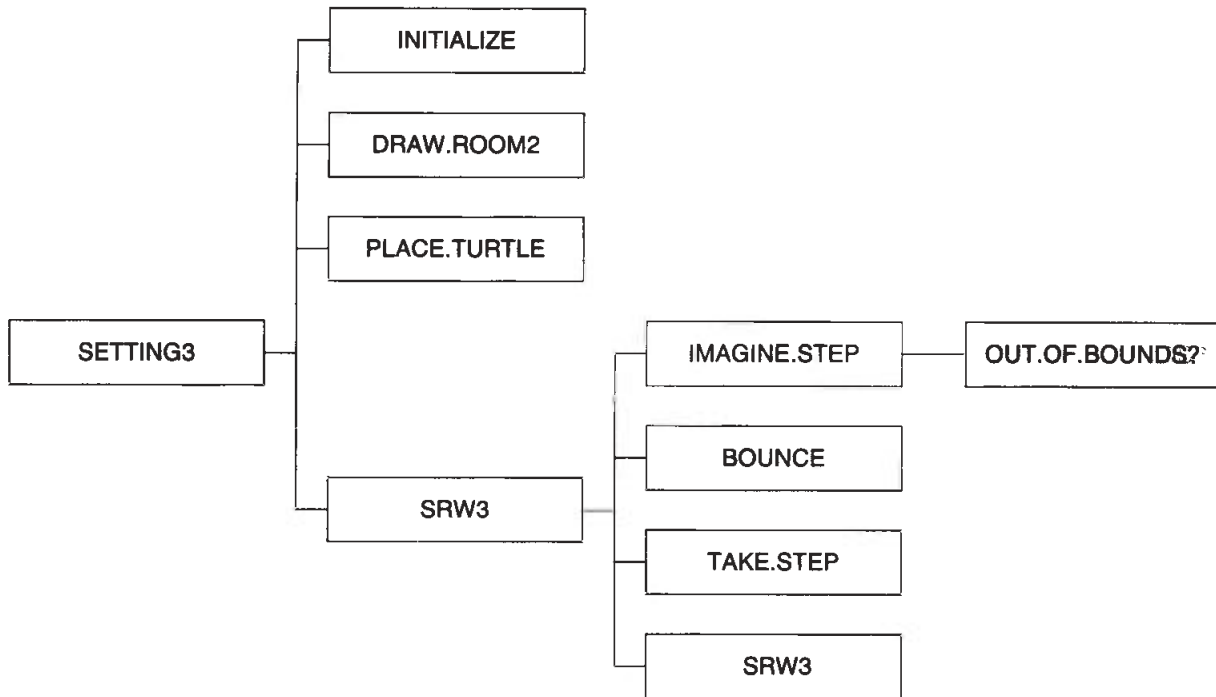
The “running out of gas” feature is working, but the walls still need more substance.

## Fourth Approach

Let's now return to the problem of penetrable walls.

One way to approach the problem is to take an imaginary step first and then test if the potential position is outside the wall. If it is, you want to do something about it. Otherwise you simply take the step. Taking an imaginary step is a way of tricking the computer. The idea is to hide the turtle, put the pen up (so the turtle does not leave a trail), take a step and determine whether or not you are still inside the room. If you are, then the step will be okay, so you back up, put the pen down, show the turtle and retake the step. If you are outside the room, then do not take the step. Instead, one possibility is to back up, rotate the turtle by 180 degrees, put the pen down, show the turtle and try again.

Let's look at the structural chart first.



The actual procedures are

```
TO IMAGINE.STEP
  MAKE "PRESENT.HEADING HEADING
  MAKE "PRESENT.X XCOR
  MAKE "PRESENT.Y YCOR
  PU
  HT
  FD :STEP.SIZE
  MAKE "TOO.FAR OUT.OF.BOUNDS?
  BK :STEP.SIZE
  PD
  SETXY :PRESENT.X :PRESENT.Y
  SETHEADING :PRESENT.HEADING
END

TO OUT.OF.BOUNDS?
  IF XCOR > :RIGHT.SIDE OUTPUT "TRUE
  IF XCOR < :LEFT.SIDE OUTPUT "TRUE
  IF YCOR > :TOP OUTPUT "TRUE
  IF YCOR < :BOTTOM OUTPUT "TRUE
  OUTPUT "FALSE
END
```

```

TO SRW3 :NO.OF.STEPS.TO.BE.TAKEN
  RT (:MAX.DEGREE.TURN / 2) - RANDOM :MAX.DEGREE.TURN
  MAKE "STEP.SIZE RANDOM :MAX.STEP.SIZE
  IMAGINE.STEP
  IF :TOO.FAR THEN BOUNCE ELSE TAKE.STEP
  IF :NO.OF.STEPS.TAKEN = :NO.OF.STEPS.TO.BE.TAKEN STOP
  SRW3 :NO.OF.STEPS.TO.BE.TAKEN
END

```

```

TO BOUNCE
  RT 180
END

```

```

TO TAKE.STEP
  FD :STEP.SIZE
  MAKE "NO.OF.STEPS.TAKEN NO.OF.STEPS.TAKEN + 1
END

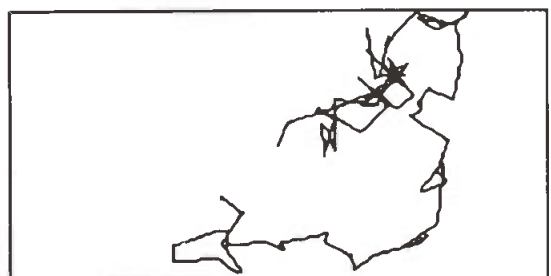
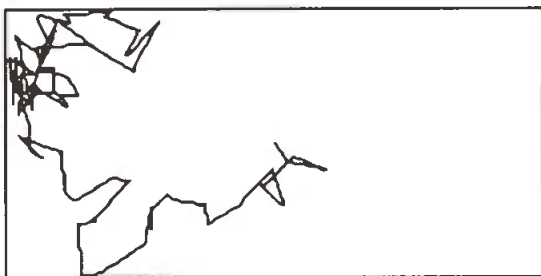
```

```

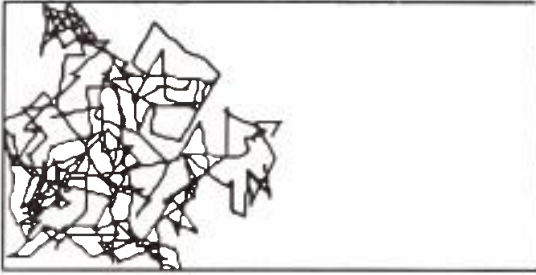
TO SETTING3 :NO.OF.STEPS.TO.BE.TAKEN
  INITIALIZE
  DRAW.ROOM2
  PLACE.TURTLE 0 0
  SRW3 :NO.OF.STEPS.TO.BE.TAKEN
END

```

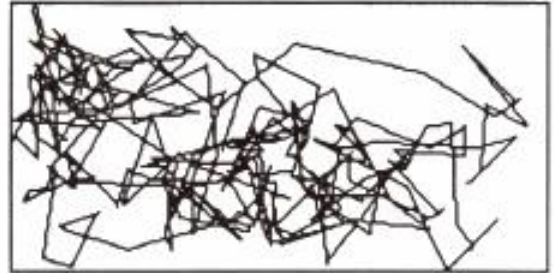
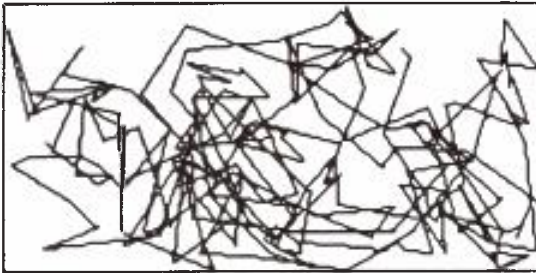
Now we can return to the original problem of studying random motion. The intensity of the randomness is determined by the two input variables :MAX.DEGREE.TURN and :MAX.STEP.SIZE to the procedure SRW3. Here are a few sample runs of 100 turtle steps, with :MAX.STEP.SIZE 20 and :MAX.DEGREE.TURN 360.



Increasing the number of steps to 300 yields



Increasing the :MAX.STEP.SIZE to 40, gives (for 300 steps)

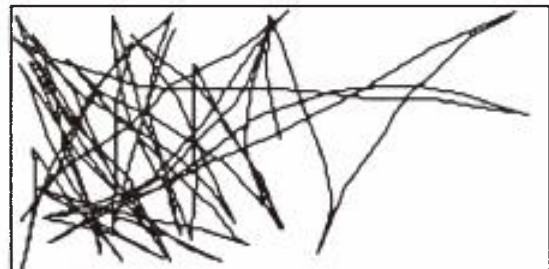


Let's try this with 10,000 steps!!



What conclusions do you draw from this?

Let's return to journeys of 300 steps, leave the :MAX.STEP.SIZE at 30 and vary the :MAX.DEGREE.TURN.



:MAX.DEGREE.TURN 90

:MAX.DEGREE.TURN 30

## Additional Approaches

Another area for experimentation occurs when the turtle hits the wall. It is possible to write procedures that have different "bounce" properties. How many can you think of?

As a Logo exercise, write some procedures that have the effect of sobering the turtle up! Can different turtles have different tolerances to alcohol (or to random procedures)?

## Meta-Approaches

There are many ways to investigate the turtle's behavior. One approach is simply to run the procedure repeatedly, print out the final path and then examine it. This seems like a good approach until you have 10 or 20 printouts on your lap!

Another approach is to ignore the paths and just note the destination of the turtle after a preset number of steps. You then write a meta-procedure that reruns the main procedure a large number of times, say 500 or 1,000, and examine the resulting array of dots for a pattern.

What began as an apparently simple task has exploded into a number of exciting and substantive investigations. There are at least three distinct levels of investigation:

1. To construct a set of procedures that work properly
2. To use these procedures to investigate another topic (randomness)
3. To construct meta-procedures to run the procedures many times

There is room for improvement and innovation at all levels. You may see a better way of constructing the original set of procedures. You may have a better set of questions to ask about how to examine the nature of randomness. You may have a better way of looking at meta-issues. You may even have a set of considerations for looking at this topic in a fourth (or more!) dimension.

In my case, the sequel was to examine the behavior of "hungry" turtles. One does not live by wine alone!

## Postscript

As I indicated at the outset, this account may be interpreted at many levels. From a pedagogical perspective, the same software (More 3.0) that produced the structural flowcharts may be used to produce a series of screen displays suitable for presenting the Logo procedures. Such displays can facilitate class discussion of the various features the procedures incorporate. These displays may also be printed on a laser printer and used as masters for making overheads. Alternatively, with the proper acetate, overheads can be produced directly on the laser printer. Clearly such an approach can be generalized and extended to other programming tasks and languages.

## Reference

Abelson, H., and A. diSessa. *Turtle Geometry*. Cambridge, Mass.: MIT Press, 1981.

# Problem Solving: Some Points To Consider

*Irvin K. Burbank*

*Irvin Burbank is a professor of elementary education at the University of Victoria, B.C., and senior author of Houghton Mifflin Mathematics (1-8).*

Generally, educators agree that a major purpose (if not the goal) of school mathematics is to teach and encourage students to solve problems. Achieving success in problem solving continues to be a major challenge for teachers and students at all grade levels. To meet this challenge, one first needs to find satisfying answers to the following questions:

- What is problem solving?
- What is a problem?

George Poyla, a math educator who has devoted a lifetime to the study of problem solving, has provided its simplest description. He has defined it as finding a way out of difficulty, where no way is known offhand; or as finding a way around an obstacle; or as a way of attaining a desired end that is not attainable by obvious means. Because answers do not always come easily or quickly, problem solving can be frustrating for students, who then become impatient and try to avoid the activity.

In answering the first question, Poyla hints at the answer to the question: What is a problem? A problem must be defined in terms of the student. For a situation to constitute a problem, it must meet the following criteria:

1. The student must have a desire to solve the problem. If a student has no interest in finding the prime factors of 60, the situation is not a problem for that student. In addition to wanting to solve the problem, the student needs to feel that the problem is within his/her capability and be willing to attack the problem.

2. The situation must be new or unfamiliar, with no immediate, obvious or apparent solution. If a student has encountered this kind of situation before and resolved it, it may not present a problem now.
3. The situation must involve an initial state and a goal state with obstacles in between that present a challenge to be overcome.



Objective: To logically bridge the gap from point A to point B

The following diagrams illustrate what often occurs in the process of solving problems.

Diagram 1 shows the student starting at point A (given state) and endeavoring to get to point B (goal state). This is a new situation the student has not encountered before. He/she gathers his/her resources and ventures out in an attempt to reach point B. The path followed may lead in a direction away from point B (path I).

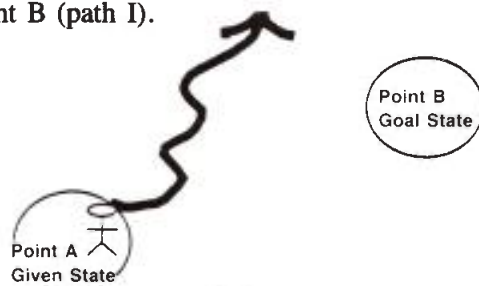


Diagram 1



Realizing he/she is not progressing, the student backtracks to point A, regroupes, gains more information and starts off on another path in the attempt to reach point B. However, this time the selected paths may prove to be dead ends, as illustrated by paths II and III in Diagram 2.

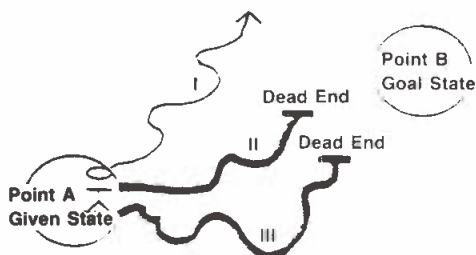


Diagram 2

Once again the student backtracks, regroupes, redefines and gets more information before starting off on another path. It may take a long time, but he/she eventually reaches the goal, as illustrated by the winding path IV, in Diagram 3.

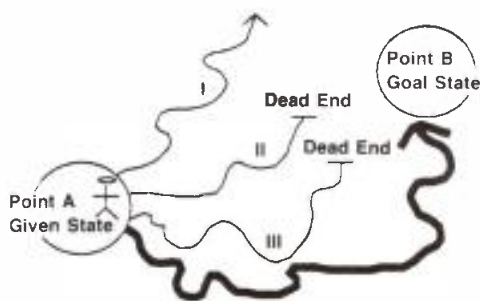


Diagram 3

Once a student arrives at point B, it is often possible to look back and see bridges and shortcuts. For example, he/she may see how paths III and IV can be bridged, and likewise where to build a bridge between II and III, as illustrated in Diagram 4.

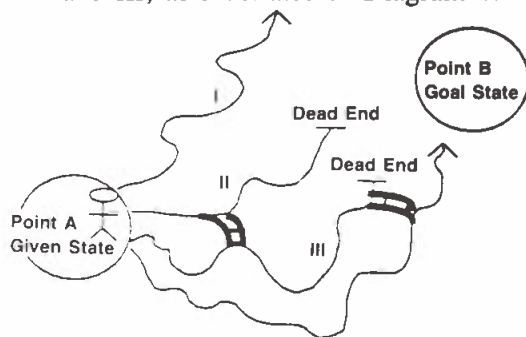


Diagram 4

The next time the student is faced with this type of problem, he/she will start on path II, cross over to path III, cross the bridge to path IV, and then proceed to point B, instead of taking the original long path IV, as shown in Diagram 5.

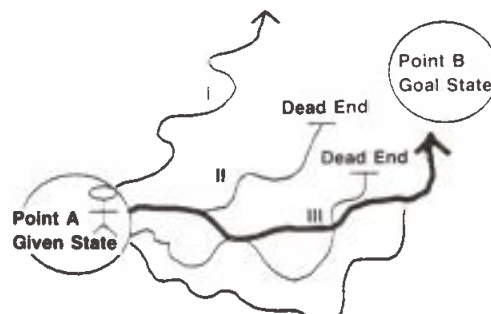


Diagram 5

The student uses the new, shorter path over and over again when solving similar problems. Interestingly, the task is no longer a problem: it has become an exercise.

So much for attempts to answer the questions: What is problem solving? and What is a problem? However, other factors also are involved in the development of problem solving skills. In the rest of this article I will discuss some of them.

### Factor 1

Often teachers do not have a clear understanding of the nature of problem solving, and thus may give the students a false impression of what it entails. For example, a student is asked to go to the chalkboard to demonstrate his/her solution to a problem. The student fills up two boards with diagrams, steps and calculations before arriving at the solution. The teacher steps forward, impatiently takes the chalk and quickly solves the problem in four short steps. The student replies, "But sir, you asked to see *my* solution to the problem, you did not ask me to show your solution to the problem. If I could've solved it in four steps, I would've solved it in four steps, but this is the first time I have solved this problem."

In this example, the student was correct. He/she was engaged in problem solving: the teacher was doing a four-step exercise. If the truth be known, it is highly likely the teacher did not solve the problem in four steps the first time he/she solved it. Thus we can conclude that true problem solving takes time and that good problem solvers must put up with frustration as they strive to move from the given state (point A) to the goal state (point B). If the teacher conveys to the student that problem solving is a quick,

direct progression from point A to point B, he/she misleads the student about the nature of problem solving. For example, the teacher has the following routine:

1. Reads the question
2. Writes the formula or equation
3. Gets the answer

The student is led to believe this is the way problems are solved. He/she tries this routine:

1. Reads the question
2. No formula pops into his/her head
3. Reads the question
4. No equation comes to mind
5. Reads the question
6. . . .

The student concludes, "I can't solve problems!" and is correct, because no one solves problems this way: these are the steps of an exercise, not strategies for solving a problem.

### Factor 2

Intimidation is a major factor hindering the development of problem solving skills in the classroom. Unfortunately, it is often quite unintentional. For example, the teacher who solves a problem in 4 steps and 10 seconds flat often does not realize how this intimidates a child who has just struggled through many steps to get the solution.

Suppose a teacher prefaces the problem solving with a statement such as, "Here is an easy problem for you, Walter." This is a devastating remark to make to Walter because it implies that he could not possibly solve an ordinary problem. If Walter can solve the "easy" problem then he gains no satisfaction from it. If he cannot solve it, then that confirms for him that he is a dunce. It would have been better to have said, "Here is a tough (or challenging) problem for you." If Walter can then solve it, he will feel a lot of satisfaction; if he cannot, he can conclude, without loss of confidence, that he could not solve it because it was too tough, but that he is still learning, and one day he may be able to solve it.

### Factor 3

Students often witness the teacher and other students doing mathematical exercises, but they seldom see the teacher in the process of solving problems. They do not witness what the teacher does when the attempt does not seem to be leading to a solution, or when the attempt reaches a dead end. A child cannot learn to kick, catch or throw a ball by engaging in worksheet activity, only by observing others practising these skills. Hence it is not surprising that students have difficulty solving problems if they seldom see others doing it and are only exposed to problem solving through worksheets.

### Factor 4

If students' exposure to problem solving comes only in the form of a worksheet, then they are not involved in the critical problem solving activities of asking questions and vocalizing the terms related to the problem. If students ask: Why? How come? How much? How many? then they are at the first and crucial stage in problem solving. The importance of asking questions and vocalizing is evidenced by the fact that often by the time a student has said the last word, the answer pops into his/her mind. The reason may be that this was the first time the student had verbalized the math terms or phrases.

### Factor 5

If the teacher is not keen on problem solving, he/she will probably find it hard to motivate the students to develop the skills and strategies necessary for solving problems. Unfortunately, many teachers share their students' fear and dislike of problem solving.

### Factor 6

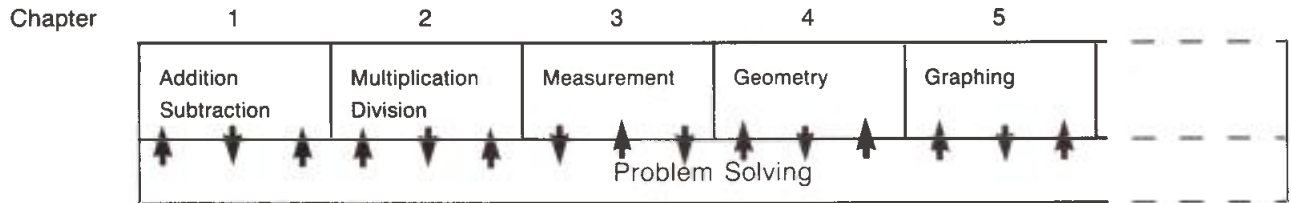
This last factor has to do with how problem solving fits into the elementary math curriculum. Some of the old math programs placed problem solving in the last chapter of the textbook. For example:

Chapter	1	2	3	4	5	6
	Addition Subtraction	Multiplication Division	Measurement	Geometry	Graphing	Problem Solving

Placing problem solving in the last chapter often meant the class never studied it. Other programs placed problem solving at the end of each chapter, a slight improvement over the first format:

Chapter	1	2	3	4	5					
	Addition Subtraction	Problem Solving	Multiplication Division	Problem Solving	Measurement	Problem Solving	Geometry	Problem Solving	Graphing	Problem Solving

However, it is generally agreed that problem solving should be interwoven throughout the math program, as follows:



In this type of program the introduction to a lesson involves problem solving; teaching the lesson involves problem solving; and practice, reinforcement and enrichment activities all use various elements of problem solving.

After studying problem solving and some of the factors to be considered in promoting it, teachers will, I hope, conclude that problem solving has a dual personality: it is the goal of mathematics education and the means of achieving the goal. It is the product and the process. Moreover, carrying out problem solving is the best method for teaching problem solving.

# Canadian Mortgage Payments

*Dennis Connolly*

*Dennis Connolly is associate professor of mathematical science at the University of Lethbridge, Lethbridge, Alberta. This paper was presented at the NCTM Conference in Calgary, in October 1990.*

This paper shows how monthly mortgage payments are calculated, with special reference to Canadian mortgages. From this a "loan progress formula" is developed. I also explain the curious "monthly interest factor" that appears at the back of Canadian Mortgage Tables.

Let  $A_0$  be the initial mortgage amount in dollars,  
 $p$  be the fixed monthly payment in dollars,  
 $n$  be the number of months of mortgage amortization and  
 $r$  be the monthly interest rate.

## Monthly Payments

Our problem is to find  $p$  in terms of  $A_0$ ,  $n$  and  $r$ .

Let  $A_k$  be the amount of principal owing immediately after the  $k^{\text{th}}$  (monthly) payment has been made, for  $1 \leq k \leq n$ . We see that

$$\begin{aligned} A_1 &= A_0 + rA_0 - p = A_0(1+r) - p \\ A_2 &= A_1 + rA_1 - p = A_1(1+r) - p = A_0(1+r)^2 - p - p(1+r) \\ A_3 &= A_2 + rA_2 - p = A_2(1+r) - p = A_0(1+r)^3 - p - p(1+r) - p(1+r)^2 \\ &\dots \dots \dots \end{aligned}$$

$$A_k = A_0(1+r)^k - p[1 + (1+r) + (1+r)^2 + \dots + (1+r)^{k-1}]$$

Then using the sum of a geometric progression

$$1 + a + a^2 + \dots + a^{k-1} = \frac{a^k - 1}{a - 1}, \text{ we get}$$

$$A_k = A_0(1+r)^k - p \frac{(1+r)^k - 1}{r}$$

After  $n$  months the mortgage is paid off, so that  $A_n = 0$ . That is

$$A_n = A_0(1+r)^n - p \frac{(1+r)^n - 1}{r} = 0$$

Hence the monthly payment is

$$p = A_0 \frac{r(1+r)^n}{(1+r)^n - 1}$$

### Example 1

Find the monthly payment for a \$100,000 mortgage @ 12% amortized over 10 years.  
Here  $A_0 = 100,000$ ,  $n = 120$  months and  $r = 1\% = 0.01$  per month.

Using equation 1 to calculate the monthly payment  $p = \$100,000 \frac{(0.01)(1.01)^{120}}{(1.01)^{120} - 1} = \$1,434.71$

The Canadian Mortgage Tables give \$1418.03 as the monthly payment for the above mortgage. This amount differs from that calculated above because the interest on Canadian mortgages is calculated "semiannually, not in advance." Thus, if the annual rate of a Canadian mortgage is  $R\%$ , then the monthly rate is not  $r = R/12\%$  (as in the U.S.), but  $r_c$  where  $(1+r_c)^6 = 1 + R/2$ .<sup>\*</sup> That is

$$r_c = \sqrt[6]{1 + R/2} - 1 \quad \dots 2$$

Using  $r_c = \sqrt[6]{1 + 0.06} - 1 = 0.009758794$  for  $r$  in equation 1, for the above example ( $A_0 = 100,000$ ,  $R = 12\%$  and  $n = 120$  months), we get

$$p = \$100,000 \frac{(0.009758794)(1.009758794)^{120}}{(1.009758794)^{120} - 1} = \$1,418.03, \text{ exactly as in the Tables.}$$

### Example 2

Calculate the monthly payment for a \$75,000 Canadian mortgage over 25 years at 13.25%.

Using equation 1,  $p = A_0 \frac{r(1+r)^n}{(1+r)^n - 1}$  where  $A_0 = 75,000$ ,  $n = 300$ , and  $r = \sqrt[6]{1 + \frac{0.1325}{2}} - 1$ ,  
that is,  $r = 0.010748660$ ,  $p = \$840.14$ . (The Tables give \$840.15.)

### Loan Progress

After  $k$  monthly payments, the amount of principal owing is  $A_k$ . Recall

$$A_k = A_0 (1+r)^k - p \frac{(1+r)^k - 1}{r} \quad \dots 3$$

### Example 3

Calculate the amount of principal owing on the mortgage in Example 2, after 10 years of payments. That is,  $A_0 = \$75,000$ ,  $p = \$840.14$ ,  $r = 0.01074866$  and  $k = 120$ .

$$A_{120} = \$75,000 (1.01074866)^{120} - (840.14) \frac{(1.01074866)^{120} - 1}{0.01074866} = \$66,754.97$$

(The Tables are only approximate and give  $\$890 \times 75 = \$66,750$ .)

### Example 4

Calculate the amount of principal owing on a \$200,000 mortgage @ 14.75% over 40 years after 20 years of payments. (In the U.K. mortgages are often amortized over 35 and 40 years; 14.75% is a common rate.)

Here  $A_0 = 200,000$ ,  $n = 480$  and  $r = \sqrt[6]{1 + \frac{0.1475}{2}} - 1 = 0.011930135$

Use equation 1 to find  $p = \$2,394.10$ , then equation 3 with  $k = 240$ , to get  $A_{240} = \$189,021.57$ , the amount of the \$200,000 still owing after 20 years of payments.

<sup>\*</sup> $r_c$  is such that the interest on the mortgage "when compounded monthly at a rate of  $r_c\%$  per month" is equivalent to the interest on the mortgage "when compounded semiannually at a rate of  $R/2\%$  per six months" ( $R\%$  being the quoted annual rate for the mortgage).

## Monthly Interest Factor

The monthly interest factors published at the back of Canadian Mortgage Tables are simply  $r_c$  for the appropriate quoted annual rate  $R$ . (See equation 2.)

### Example 5

For a mortgage with a 13.25% interest rate we saw in Example 2 that

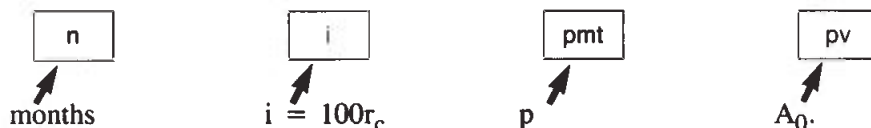
$$r_c = \sqrt[6]{1 + \frac{0.1325}{2}} - 1 = 0.010748660.$$

This is the monthly interest factor for a 13.25% mortgage (see Tables). For a 14.75% mortgage we saw in Example 4 that

$$r_c = \sqrt[6]{1 + \frac{0.1475}{2}} - 1 = 0.011930135.$$

This is the monthly interest factor for a 14.75% mortgage (see Tables).

A Hewlett Packard 27 calculator has the following buttons:



Note that the interest button  $i$  is not  $r_c$  but *100 times*  $r_c$ .

It is of interest that we could find

(a)  $A_0$  in terms of  $p$ ,  $n$  and  $r$ . Using equation 1, we see

$$A_0 = p \left[ \frac{(1+r)^n - 1}{r(1+r)^n} \right]$$

(b)  $n$  in terms of  $A_0$ ,  $p$  and  $r$ . By rearranging equation 1 and taking logs (any base), we see

$$n = \frac{\log(p) - \log(p - rA_0)}{\log(1+r)}$$

To find  $r$  in terms of  $A_0$ ,  $n$  and  $p$  is not so easy; we have to solve the polynomial equation

$$p = (1+r)^n(p - A_0r)$$

to get  $r$ , which is difficult for  $n > 2$ ; usually  $n$  is very large. An iterative process is used to approximate to the solution to the desired degree of accuracy.

## Total Interest Paid

The total interest paid on a mortgage (over  $n$  months) is simply  $np - A_0$ .

### Example 6

The interest paid on the \$75,000 mortgage of Example 2 is  
 $300 \times \$840.14 - \$75,000 = \$177,042.00$ .

The interest paid on the \$200,000 mortgage of Example 4 is  
 $480 \times \$2,394.10 - \$200,000 = \$949,168.00$ .

## Monthly Interest Factors

Interest for one month at nominal annual rates shown,  
based on interest compounded semiannually

12 %	<u>.009 758 7942</u> ←	15½ %	.012 224 4297
12¾ %	.010 452 2088	15¼ %	.012 322 4327
13 %	.010 551 0740	15⅜ %	.012 420 3883
13⅛ %	.010 649 8909	15½ %	.012 518 2966
13¼ %	<u>.010 748 6596</u> ←	15⅝ %	.012 616 1575
13⅜ %	.010 847 3799	15¾ %	.012 713 9712
13½ %	.010 946 0522	15⅞ %	.012 811 7377
13⅝ %	.011 044 6762	16 %	.012 909 4570
13¾ %	.011 143 2522	16⅛ %	.013 007 1292
13⅞ %	.011 241 7802	16¼ %	.013 104 7543
14 %	.011 340 2602	16⅜ %	.013 202 3325
14⅛ %	.011 438 6923	16½ %	.013 299 8636
14¼ %	.011 537 0764	16⅝ %	.013 397 3478
14⅜ %	.011 635 4128	16¾ %	.013 494 7852
14½ %	.011 733 7014	16⅞ %	.013 592 1758
14⅝ %	.011 831 9423	17 %	.013 689 5196
14¾ %	<u>.011 930 1355</u> ←	17⅛ %	.013 786 8166
14⅞ %	.012 028 2811	17¼ %	.013 884 0670
15 %	.012 126 3791	17⅝ %	.013 981 2708





# Another Proof of Heron's Formula for the Area of a Triangle

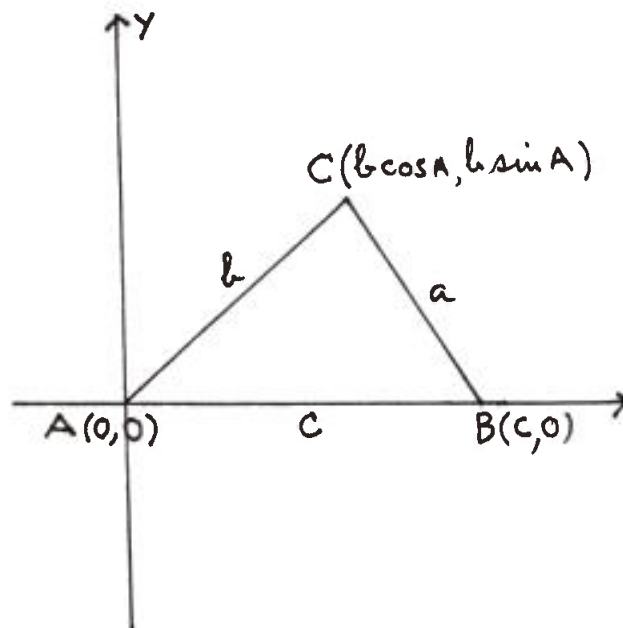
*Gurcharan Singh Bhalla*

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Heron of Alexandria (60 A.D.?) was a Greek mathematician. His mathematical works, included in *Metrica*, deal with mensuration of rectilinear figures and solids. His unique contribution to mathematics, however, is his famous relation for the area of a triangle in terms of its sides. Heron's formula states:

Given a triangle with sides equal in length to  $a$ ,  $b$  and  $c$ , and  $s$  its semiperimeter, then the area of a triangle is  $\sqrt{s(s-a)(s-b)(s-c)}$

This article presents an analytical geometrical demonstration of the relation.



Let ABC be the triangle with sides  $AB = c$ ,  $BC = a$  and  $CA = b$ , and coordinates of  $A = (0,0)$ ,  $B = (c,0)$  and  $C = (b \cos A, b \sin A)$ . Then, by coordinate geometry, the area  $X$  of the triangle is given by:

$$X = \frac{1}{2} \begin{vmatrix} 0 & 0 \\ c & 0 \\ b \cos A & b \sin A \end{vmatrix}$$

$$= \frac{1}{2} [bc \sin A]$$

... 1

From triangle ABC, by cosine rule, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Therefore

$$\sin^2 A = 1 - \left( \frac{b^2 + c^2 - a^2}{2bc} \right)^2 = \frac{(2bc)^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$\begin{aligned} \sin^2 A &= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2} \\ &= \frac{[(b + c)^2 - a^2][a^2 - (b - c)^2]}{4b^2c^2} \\ &= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4b^2c^2} \\ &= \frac{16 s(s - a)(s - b)(s - c)}{4b^2c^2} \end{aligned}$$

$$\sin A = \frac{2\sqrt{s(s - a)(s - b)(s - c)}}{bc}$$

$$\text{From 1, area } X = \frac{1}{2} bc \cdot \frac{2\sqrt{s(s - a)(s - b)(s - c)}}{bc}$$

$$X = \sqrt{s(s - a)(s - b)(s - c)}$$

Thus Heron's formula for the area of a triangle is demonstrated.

# A Circular Graphing Activity: Digital Roots of Powers of Natural Numbers

David R. Duncan and Bonnie H. Litwiller

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Graphical representations are an increasingly important topic in contemporary mathematics curricular materials. *The Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) suggests that the middle school curriculum provide opportunities for students to describe, extend, analyze and create a wide variety of patterns and to describe and represent relationships with graphs. It is important that students have a variety of graphical experiences at all levels. We present a graphing activity for middle school students to do with the powers of natural numbers and their digital roots.

To find the digital root of a natural number, find the sum of its digits. If this sum is a single digit number, then this number is the digital root. If the sum is a number of two or more digits, repeat the process until a single digit number results. For example, the digital root of 34 is  $3 + 4 = 7$ . The digital root of 896 is  $8 + 9 + 6 = 23$ . As 23 is not a single digit number, repeat the process; that is  $2 + 3 = 5$ . So 5 is the digital root of 896.

## Activity 1

1. Consider the third powers (cubes) of the natural numbers and their corresponding digital roots as shown in Table 1.

Note that the digital roots repeat indefinitely in a 1-8-9 pattern.

2. Graph these digital roots using a circle with nine equally spaced points marked on the circumference.

The points are connected to make the graph easier to read; the 1-8-9 pattern is evident.

Natural Number	Cubes of the Natural Numbers	Digital Roots of the Cubes
1	1	1
2	8	8
3	27	9
4	64	1
5	125	8
6	216	9
7	343	1
8	512	8
9	729	9
10	1000	1
11	1331	8
12	1728	9
13	2197	1
14	2744	8
15	3375	9
16	4096	1
17	4913	8
18	5832	9
19	6859	1
...	...	...

## Activity 2

1. Consider the second powers (squares) and their digital roots as shown in Table 2.

Observe that the digital roots repeat in a longer set of numbers. The pattern is 1-4-9-7-7-9-4-1-9.

2. Graph the digital roots as in Activity 1.

The pattern of the digital roots can be seen from the graph. Note that this pattern does not form a simple closed curve, as in Graph 1.

### Activity 3

Have your students construct the graphs of the digital roots of the following powers of the natural numbers 1, 4, 5, 6, 7, 8 and 9. The results should be as shown in Table 3 and Graphs 3 through 9.

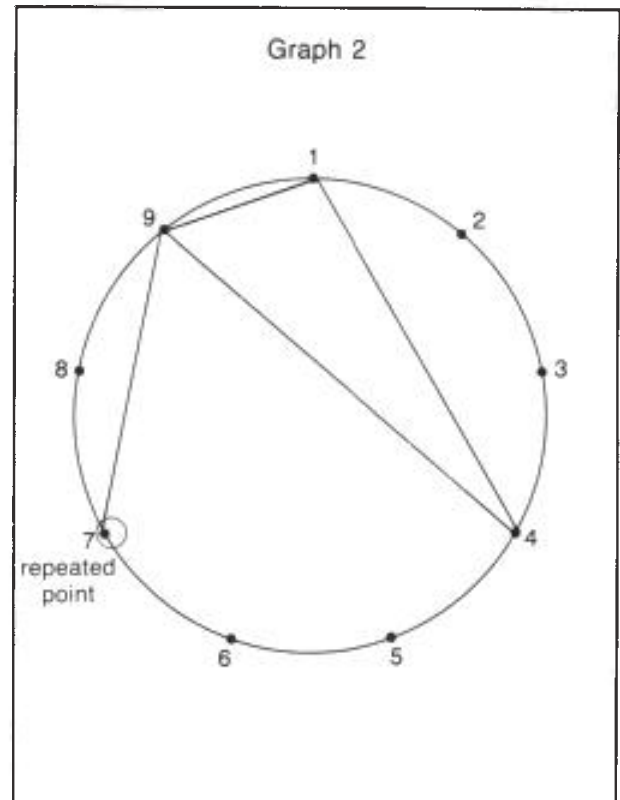
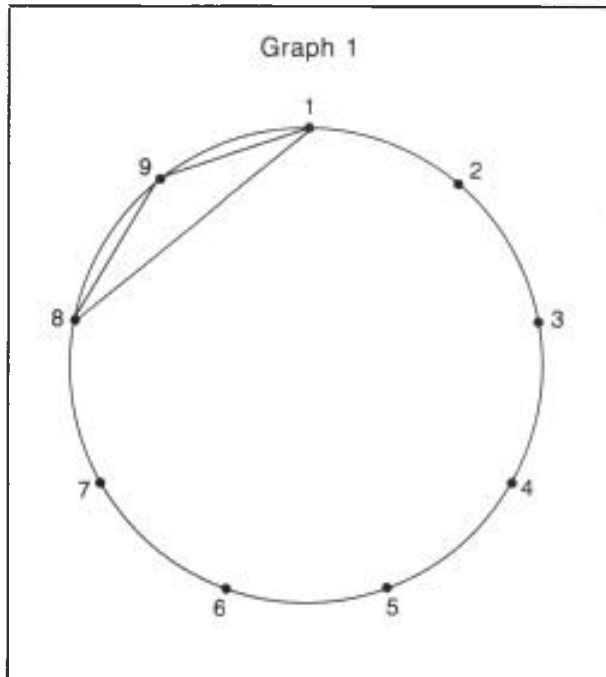


Table 2

Natural Number	Squares of the Natural Numbers	Digital Roots of the Squares
1	1	1
2	4	4
3	9	9
4	16	7
5	25	7
6	36	9
7	49	4
8	64	1
9	81	9
10	100	1
11	121	4
12	144	9
13	169	7
14	196	7
15	225	9
16	256	4
17	289	1
18	324	9
19	361	1
⋮	⋮	⋮
⋮	⋮	⋮

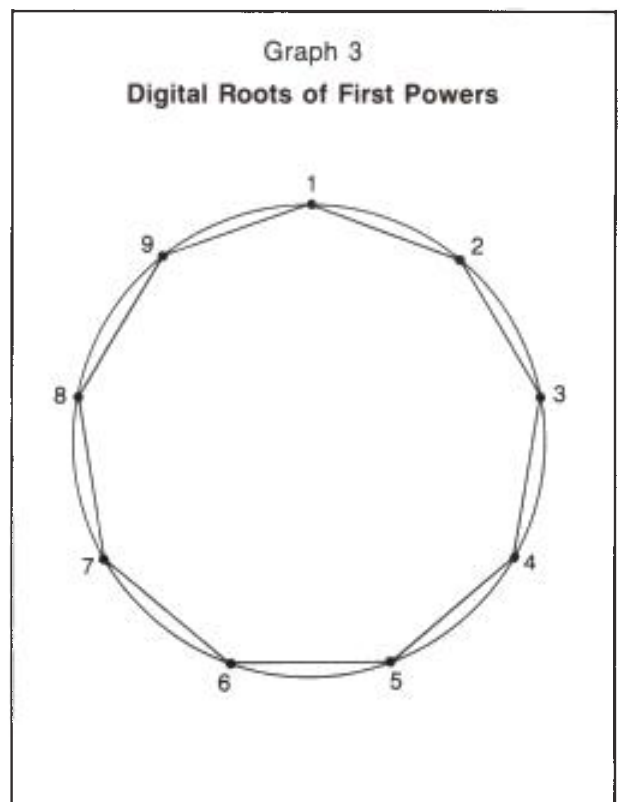
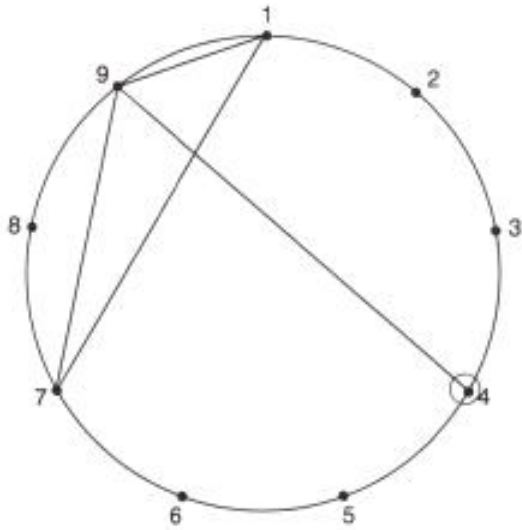


Table 3

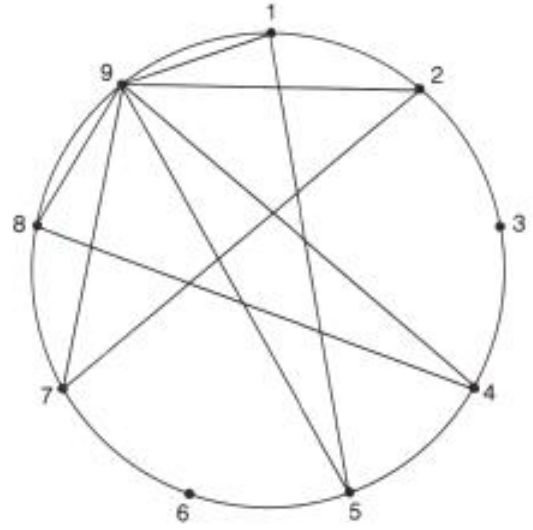
Natural Number	First Powers	DR	Fourth Powers	DR	Fifth Powers	DR	Sixth Powers	DR	Seventh Powers	DR	Eighth Powers	DR	Ninth Powers	DR
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	16	7	32	5	64	1	128	2	256	4	512	8
3	3	3	81	9	243	9	729	9	2,187	9	6,561	9	19,683	9
4	4	4	256	4	1,024	7	4,096	1	16,384	4	65,536	7	262,144	1
5	5	5	625	4	3,125	2	15,625	1	78,125	5	390,625	7	1,953,125	8
6	6	6	1,296	9	7,776	9	46,656	9	279,936	9	1,679,616	9	10,077,696	9
7	7	7	2,401	7	16,807	4	117,649	1	823,543	7	5,764,801	4	40,353,607	1
8	8	8	4,096	1	32,768	8	262,144	1	2,097,152	8	16,777,216	1	134,217,728	8
9	9	9	6,561	9	59,049	9	531,441	9	4,782,969	9	43,046,721	9	387,420,489	9
10	10	1	10,000	1	100,000	1	1,000,000	1	10,000,000	1	100,000,000	1	1,000,000,000	1
11	11	2	14,641	7	161,051	5	1,771,561	1	19,487,171	2	214,358,881	4	*	8
12	12	3	20,736	9	248,832	9	2,985,984	9	35,831,808	9	429,981,696	9	*	9
13	13	4	28,561	4	371,293	7	4,826,809	1	62,748,517	4	815,730,721	7	*	1
14	14	5	38,416	4	537,824	2	7,529,536	1	105,413,504	5	*	7	*	8
15	15	6	50,625	9	759,375	9	11,390,625	9	170,859,375	9	*	9	*	9
16	16	7	65,536	7	1,048,576	4	16,777,216	1	268,435,456	7	*	4	*	1
17	17	8	83,521	1	1,419,857	8	24,137,569	1	410,338,673	8	*	1	*	8
18	18	9	104,976	9	1,889,568	9	34,012,224	9	612,220,032	9	*	9	*	9
19	19	1	130,321	1	2,476,099	1	47,045,099	1	893,871,739	1	*	1	*	1

\* These numbers are not listed due to their length. Have your students calculate them. A computer might be useful for this task.

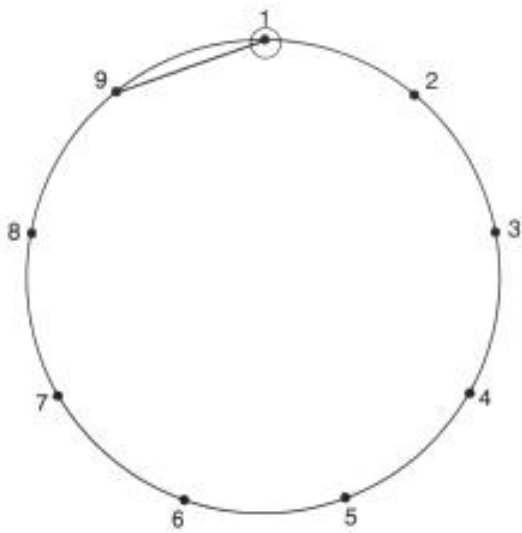
Graph 4  
Digital Roots of Fourth Powers



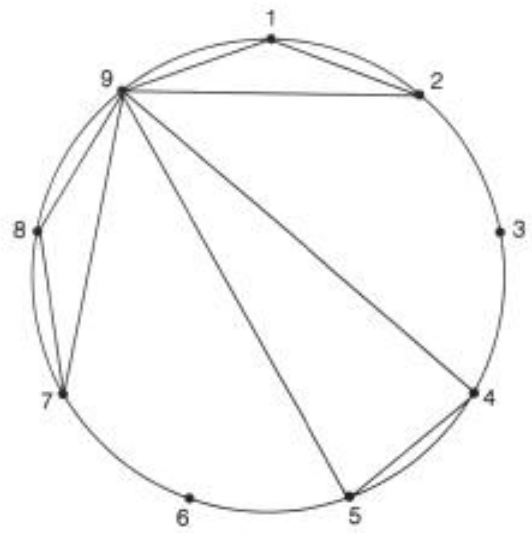
Graph 5  
Digital Roots of Fifth Powers

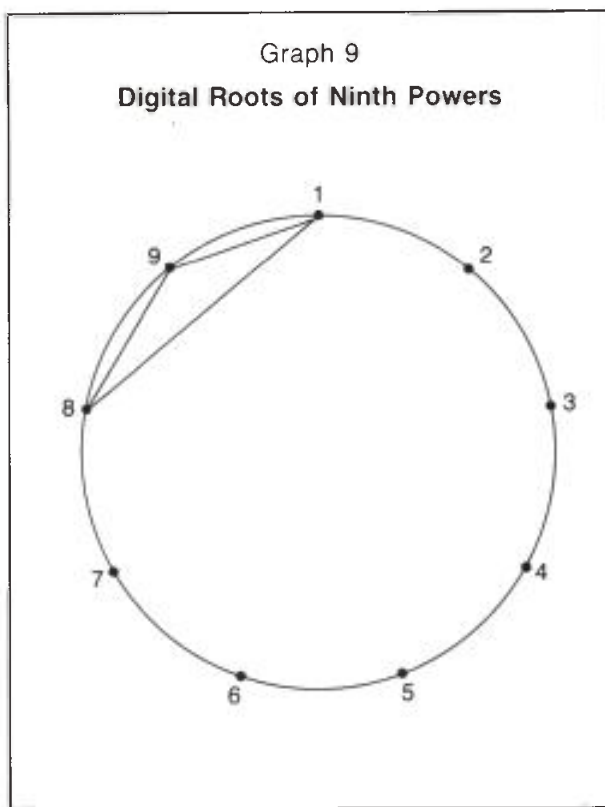
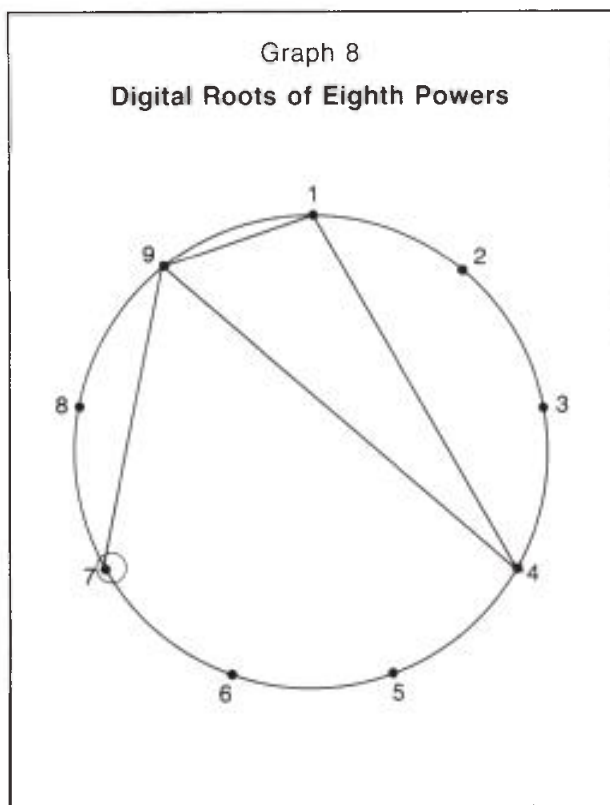


Graph 6  
Digital Roots of Sixth Powers



Graph 7  
Digital Roots of Seventh Powers





From the tables and graphs observe that the lengths and repetitive patterns of digital roots are

Powers	Length of
First	9 digits
Second	9 digits
Third	3 digits
Fourth	9 digits
Fifth	9 digits
Sixth	3 digits
Seventh	9 digits
Eighth	9 digits
Ninth	3 digits

The lengths are all 9 or a divisor of 9.

### Challenges

Note that the digital root patterns for the second and eighth powers are identical, as they are for the third and ninth powers.

1. Have your students graph the tenth, eleventh, twelfth . . . powers. What further identical patterns emerge? (For  $p > 1$ , the digital roots of the  $p$ th powers and the  $p + 6$ th powers are identical.)
2. Compute the multiples of the natural numbers and graph their digital roots. What patterns emerge?

### Reference

*Curriculum and Evaluation Standards for School Mathematics.*  
Commission on Standards for School Mathematics of the  
National Council of Teachers of Mathematics Staff, 1989.

# Include Fractions in the Elementary School Curriculum

*John B. Percevault*

*John Percevault is a retired professor of mathematics education from the University of Lethbridge, Lethbridge, Alberta.*

During the sixties, Alberta's elementary mathematics curriculum was critically reviewed. As a result of the decision to adopt the Metric System (S.I.) in Canada, work with common fractions was eliminated. The focus was on decimal numeration and the extension to S.I. prefixes, while instruction in common fractions was limited to halves, thirds, fourths, fifths and tenths.

The current Alberta curriculum encourages integration and continuity of instruction. Integration should not only be across subjects but also within a subject. Continuity rightly includes the continued use of the same materials or models throughout the grades. It should also include students' developing ideas and the ability to express those ideas clearly and concisely. Manipulatives provide just such an opportunity for integration and continuity within the mathematics curriculum. The best opportunity for language development occurs in a small-group situation, where the size of the group may be less important than the opportunity for discussion.

The ideas that follow are not new. Many of the diagrams can be found in elementary or Grade 7 mathematics texts. However, the idea of students developing manipulatives rather than using prepared materials may be new to some readers. Students enjoy using materials they or their colleagues have developed. I have not indicated the grade levels for which these exercises are appropriate. I have tried them out on Grade 4 students, who were excited by and enjoyed developing and using a fraction kit consisting of strips of paper. Could some of these

experiences be tried at a lower level? The question, I hope, is rhetorical.

## Developing the Kit and Related Activities

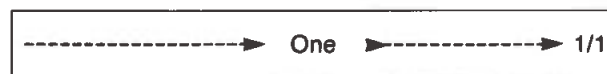
### I. Whole

Provide each student with a uniform set of rectangular paper strips.

Have students explore some of the questions that follow:

1. Are the strips the same size? Why?
2. Are the measures of opposite sides the same?
3. How do the measures of the angles compare?
4. Do the strips have the same area?

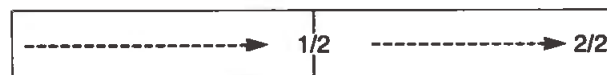
These are properties of congruency. After the exploration label the unit strip.



*Illustration 1*

### II. Halves

Ask each student to take a new paper strip. Estimate where the halfway point of the strip would be. Check. Challenge students to find the halfway point another way (measure, fold, refine estimate). Accept all correct responses.



*Illustration 2*



Discuss properties of congruent figures developed with the unit strip. Ask:

1. How many halves in one?
2. If a strip (unit) is divided into two equal (congruent) parts, what is the size of each part?

Label as in Illustration 2. Is there another term for one?

Note that the order of developing fractions on the paper strip that I follow here is completely arbitrary. Teachers could develop thirds next.

### III. Fourths

Before providing students with a new strip ask them if they can divide a strip into four equal (congruent) parts. Have each of them justify or illustrate the method used. Accept all correct responses.

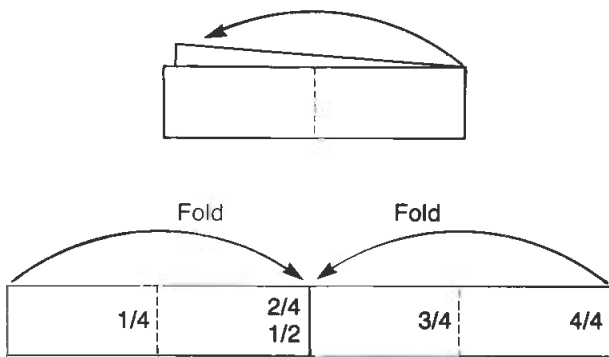


Illustration 3

Possible questions to explore include:

1. How many equal parts are there?
2. What is the name of each part? Quarters?
3. Compare: How many halves in one?  
How many quarters in one?
4. How much of the whole one is  
 $1/2 + 1/4$ ?  
 $1/4 + 1/2$ ? Prove.
5. How many ways of naming one?  
1,  $2/2$ ,  $4/4$ , . . .
6. Extend to eighths and sixteenths. Some students may do this on their own.

### IV. Thirds and Sixths

After thirds, sixths and possibly twelfths may be developed.

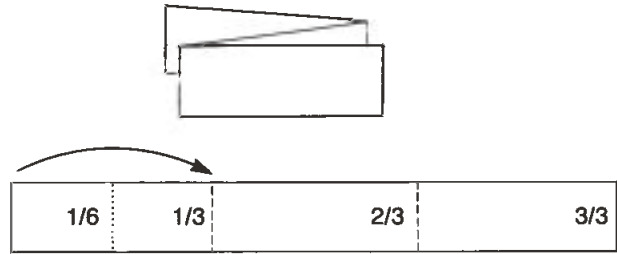


Illustration 4

Continue developing equivalent fractions. Challenge the students to determine whether they have formed a generalization. Verify with fraction strips or diagrams.

### V. Denominators $\geq$ Five

As the number of folds increases, the measures become less precise. A handy, accurate measuring device can be developed using parallel lines. Figure 1 shows a line segment divided into fractions. The fractions equivalent to one-half are also marked. Other equivalents are evident. Incidentally, why this works might be a good investigation for a high school geometry class.

### VI. Decimal Fractions

Comparing the halves and fifths to the tenths strip generates the decimal equivalents. Additions and subtractions can be made, such as:

$$\begin{aligned} 1/2 + 1/5 &= 0.5 + 0.2 \\ 1/2 - 1/5 &= 0.5 - 0.2 \\ 1/10 + 0.5 &= 1/10 + 1/2 \end{aligned}$$

### VII. Fraction Chart

Students should be encouraged to develop their own charts. (See Figures 2 and 2A).

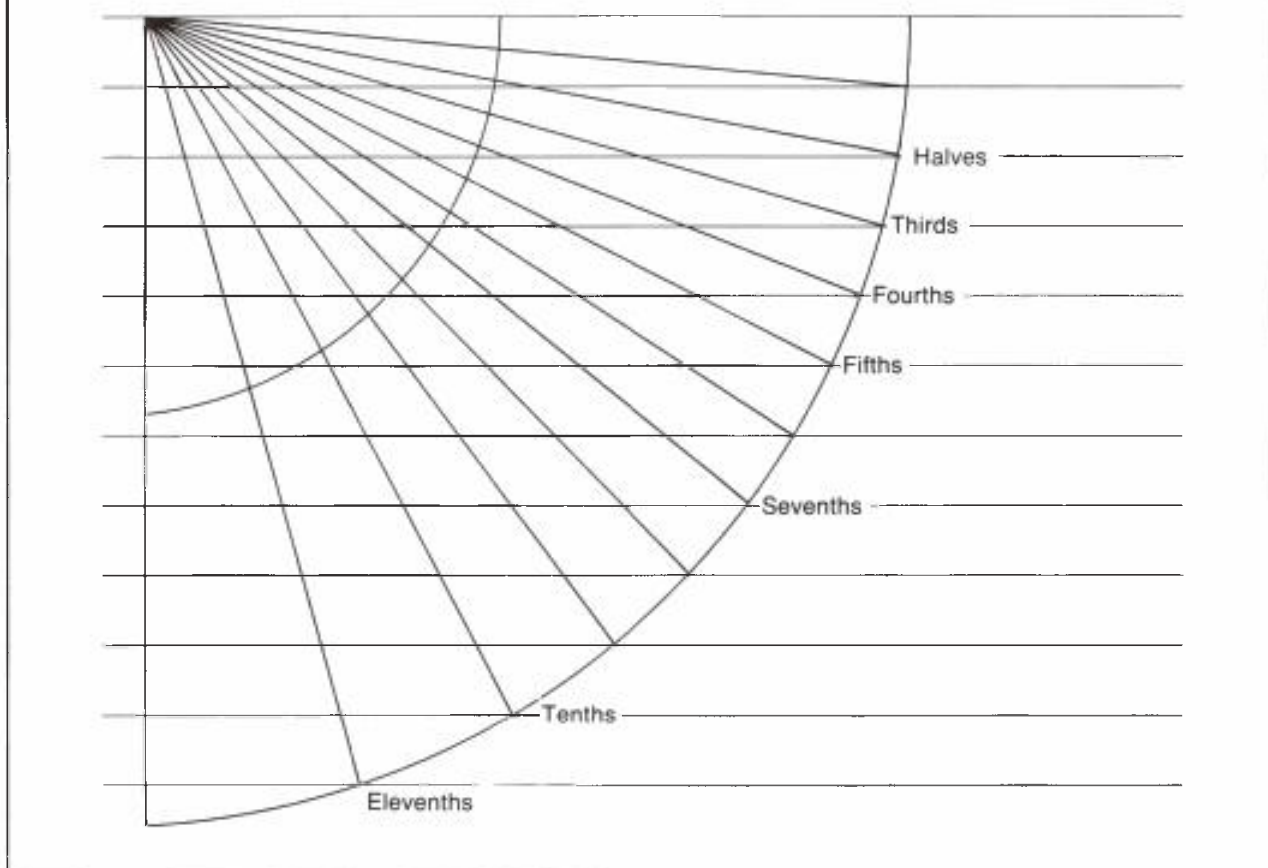
List equivalent fractions. Challenge the students.

$$\begin{aligned} 1/2 &= 2/4 \\ 1/2 &= ?/3 \\ 1/2 &= 3/6 \\ 1/2 &= 4/8 \\ &\vdots \\ 1/2 &= ?/16 \end{aligned}$$

The chart also shows the following:

$$\begin{aligned} 1/2 - 1/3 &= 2/3 - 1/2 \\ 1/3 - 1/4 &= 3/4 - 2/3 \\ 1/4 - 1/5 &= 4/5 - 3/4 \\ &\vdots \\ &\vdots \\ \frac{1}{n} - \frac{1}{n+1} &= \frac{n}{n+1} - \frac{n-1}{n} \end{aligned}$$

Figure 1  
Fraction Generator



Identify the line segments. Estimate. Use strips to verify. Allow students to determine the fractional equivalents (common denominator) for each minuend and subtrahend.

Sums and minuends greater than one can be illustrated by placing two strips end to end. It can be shown that:

$$\begin{aligned} 1/2 + 7/10 &= 5/10 + 7/10 \\ &= 12/10 \end{aligned}$$

or

$$0.5 + 0.7 = 1.2$$

Multiplication can be incorporated:

3 x 1/2 means placing 3 units of 1/2 end to end

4 x 2/3 means . . .

Division can also be shown. Consider the following questions:

1. How many quarters are in one-half?  
 $1/2 \div 1/4 = ?$
2. What is the size of each part when one-half is divided into two equal parts?  
 $1/2 \div 2 = ?$

By now some readers are probably saying: But, this isn't part of the elementary curriculum! I'll concede that! Perhaps, however, the teaching of fractions in elementary grades should be reintroduced if integration, continuity, estimation, use of manipulatives, thinking and language development are part of the curriculum. Evaluation procedures must allow the use of manipulatives. Developing and applying rules of operation should be left for the junior high school. Rather, inductive, problem solving strategies should be used to set the stage for the deductive approach to teaching mathematics in the elementary grades.

Figure 2  
 Fraction Chart

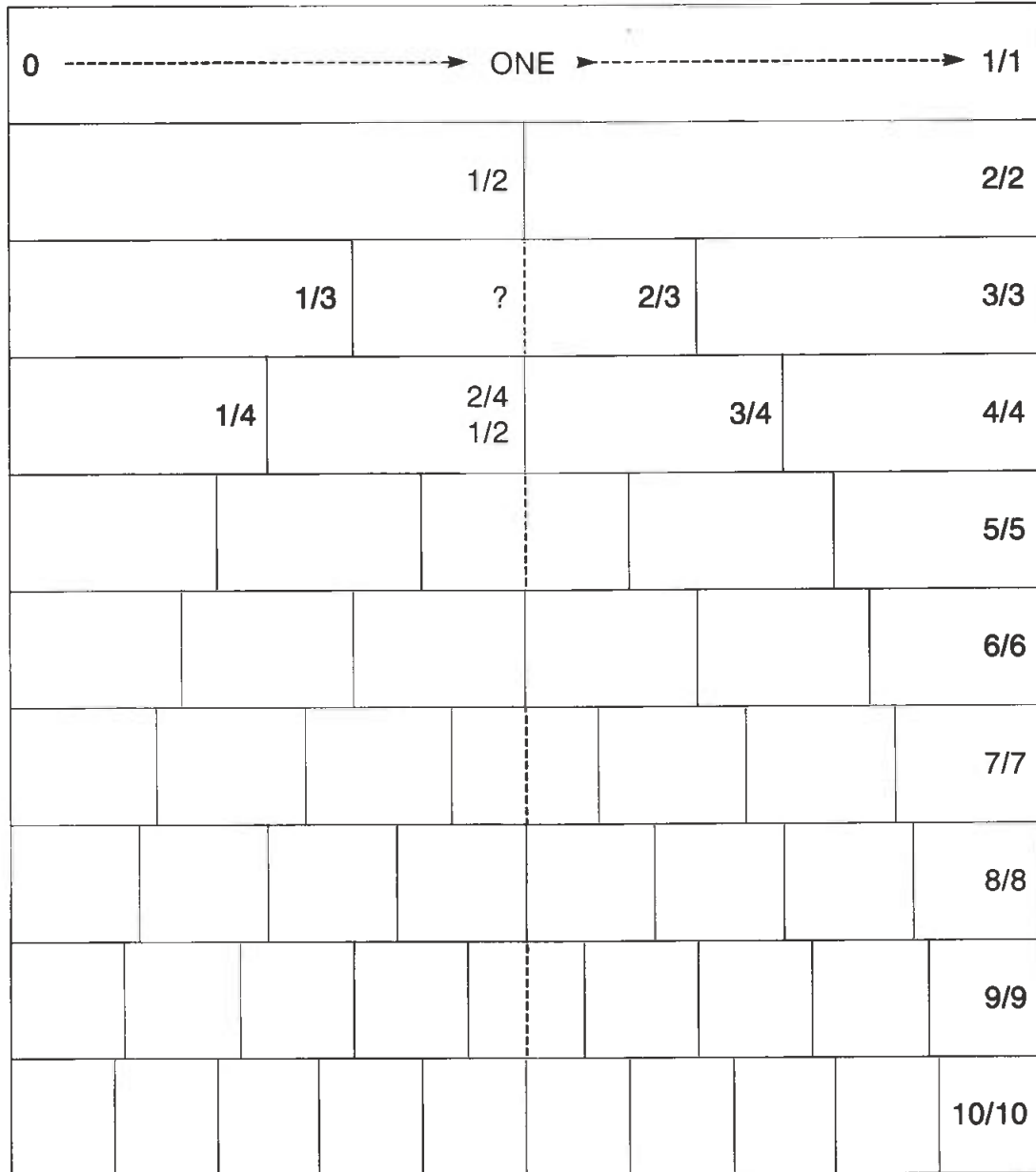
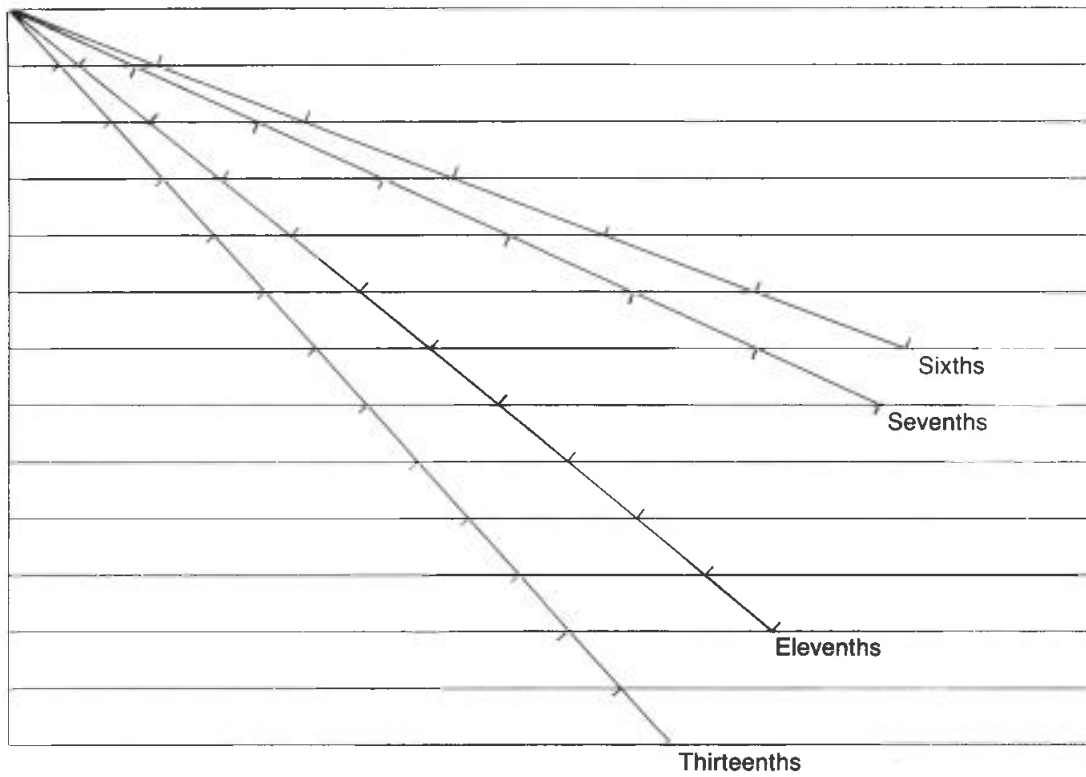


Figure 2A



An adaptation of Figure 1, this provides number lines for the fractions given.  
The lines are 14 cm long as are the strips in Figure 2.

# M & M & Ms: An Alternative Context for Teaching Mean, Median and Mode

*A. Craig Loewen*

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Too much mathematics instruction occurs by means of traditional modes of direct instruction. That is, as teachers we regularly expect our students to sit quietly and absorb our explanations of mathematical procedures and algorithms, and then apply and practise their new learnings by solving word problems. Several items are missing from this scenario: student discussion and talk, involvement and reflection, and (most important) motivation and enjoyment. Cognitive psychology tells us that language and learning are integrally linked. If this is so, then we all need an opportunity to talk about our ideas and to share our generalizations. We also need an opportunity to evaluate our learning and reconsider what we have believed to be true: reflection is an essential element in the revision, sorting and linking of new concepts. We also know that where students are not interested in instruction and not involved in the activities prepared for them, they will not learn. We cannot force students to talk about mathematics, to involve themselves in our lessons or to reflect, and we cannot force them to like our classes. We can, however, provide the most enjoyable and stimulating context that will encourage them to talk, reflect and participate, and thus to learn.

Games provide one possible context in which students may encounter mathematics in a more dynamic manner. Games provide drill and practice around specific objectives. They are fun and exciting. They give students a chance to talk, share and generalize about new ideas while using key mathematical

terms. Games seem to be widely adopted in the lower elementary classroom but disappear rapidly from instruction in the upper grade levels. Why are they not more commonly used? One reason may be that teachers believe students no longer enjoy games once they have left elementary school: students feel too grown up. Yet, we adults enjoy games —why wouldn't our students? A second reason for this phenomenon may be that games are more difficult to devise for the upper grade levels. This article presents an example of a suitable game.

M & M & Ms is used to help teach the concepts of mean, median and mode. It provides students with an opportunity to practise calculating each measure of central tendency with simple, small sets of values. The game also provides students with an opportunity to generalize about how large and small numbers in a set affect the resulting central tendency of the entire set. Finally, the game provides a context for the process of comparing mean, median and mode.

To implement the game effectively, the teacher will need to discuss the algorithms for the computation of each measure. The concept of mean will undoubtedly be familiar to the students, so little time need be spent there. After reviewing the concept of mean, the class should be divided into groups of three or four, given a deck of cards and allowed to play one round of the game in their groups (game instructions and card decks are included in the following pages). The teacher should provide a description and demonstration of a median and repeat the game allowing the students (in their groups) to play for the highest median. Again, the teacher should be sure to discuss the strategy involved by providing and discussing

some examples. Repeat the process a second time when dealing with the concept of mode.

At first the students will probably believe that the only strategy necessary is always to replace a low card with a high card; however, this is true only with mean. Let's take an example. Assume you have the following cards in your hand: 2, 9, 10. The next card is face up on the discard pile: 5. Your current mean is 7, and your current median is 9. If you are playing for the highest mean, replacing the 2 card in your hand with the 5 card will improve your mean; that is, simply to trade the lowest card in your hand for a higher card is a productive move. If you are playing for the highest median, then trading the 2 card for the 5 card will not improve your score. Therefore, a different strategy may be necessary for the median game than for the mean game.

It should be obvious from the above that discussion is an important dimension of this activity. During the discussion the teacher should try to maximize the number of generalizations derived by the students rather than provide generalizations to the class. Students may need to play the game more than once to make the relevant observations. Some possible questions to help focus the discussion are:

- Is it always true that you should exchange a low card for a high card? When would you want to do so?
- How would you make a decision between drawing a card or taking a card from the discard pile

when you are playing for the highest mean? highest median? highest mode?

- What is the effect of a high card (50 card) on the mean of a set of numbers? the median of the set? the mode of the set?
- What is the effect of a low card (0 card) on the mean of a set of numbers? the median of the set? the mode of the set?
- If all three of your cards are different, what is your mode? How many like cards are necessary to create a mode?
- How does the mode of the set of numbers change if you have three like cards rather than two like cards?
- Which is easiest to get: a high mean, median or mode? Why?
- If you had a 7 and two 10s in your hand and you drew the 50 card, which would change the most, the mean, the median, or the mode?

Games represent one method by which teachers can give students a chance to talk, compare and generalize from their experiences. As we continue to look for new contexts in which students can encounter and verbalize mathematical concepts we increase their opportunity to learn effectively. The measures of central tendency can be taught through a direct do-as-I-say approach; the important question is whether they can best be learned that way.

## M & M & Ms

### Objectives

- (a) To calculate the mean or median or mode of a set of numbers
- (b) To compare the relative impact of a small or large number on the mean, median and mode of a set of numbers

### Rules

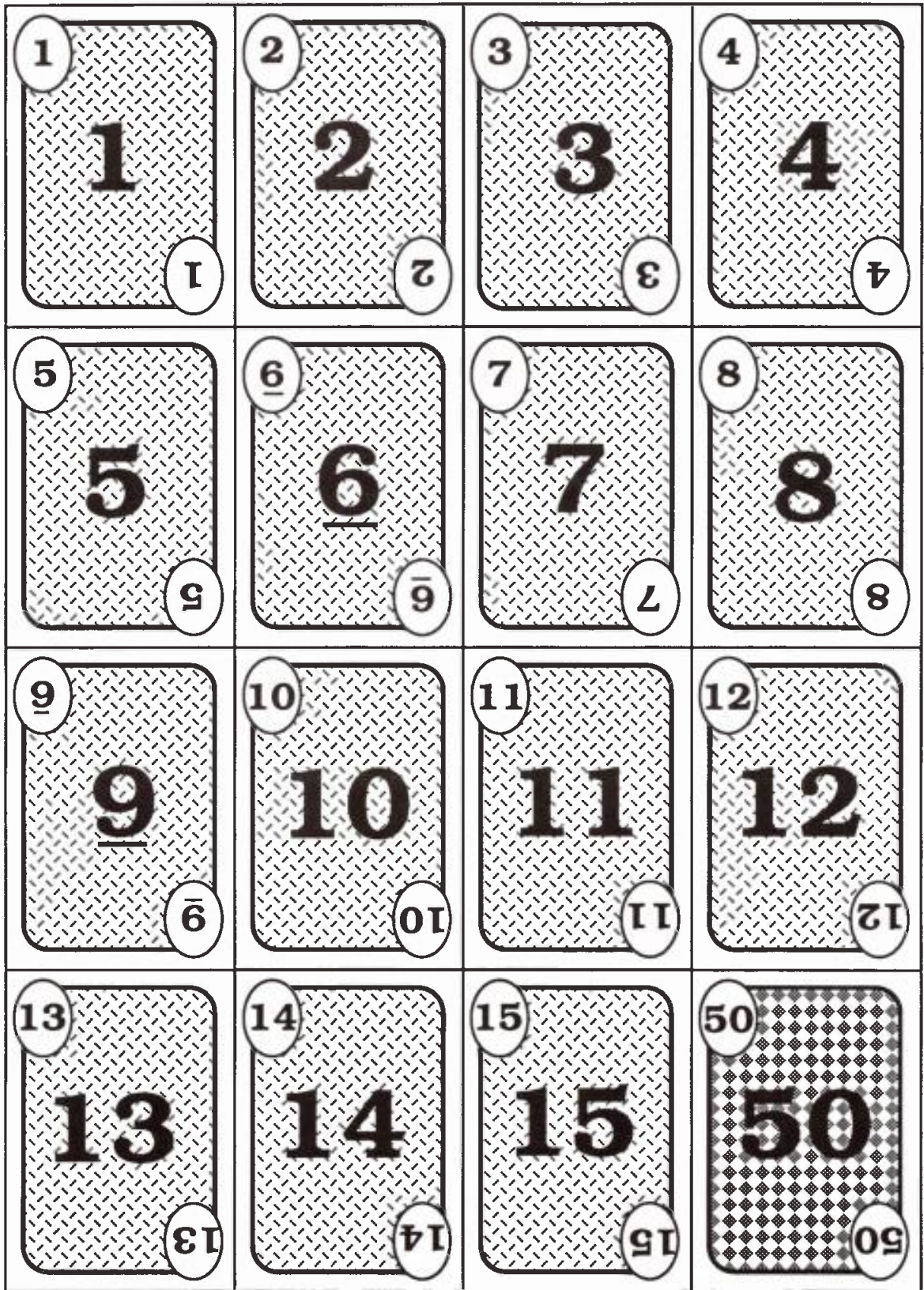
- (a) Players decide whether they are playing for the highest mean, median or mode.
- (b) Three cards are dealt face down to each player. The remaining cards are then placed face down in a stack to create the draw pile. The top card is turned over and placed beside the draw pile to start the discard pile.
- (c) The person to the left of the dealer begins. This player has the choice of either taking the top card from the draw pile, the top card from the discard pile or knocking.
- (d) If the player chooses to take the top card from the draw pile, he/she adds it to his/her hand, and then must discard (face up in the discard pile) one card from his/her hand. Play then proceeds to the left.
- (e) If the player chooses to take the top card from the discard pile, he/she exchanges it with one card from his/her hand. Play proceeds to the left.

- (f) When a player decides that the cards in his/her hand probably have a higher mean (or median or mode) than that of any of the other players, he/she knocks on the table and passes play to the left. Each of the remaining players gets one more turn.
- (g) After each player has had one final turn (except the person who knocked), all players lay down their hands and compare to see who has the highest mean, median or mode. The player with the highest score wins.
- (h) If all the cards in the draw pile have been used up before anyone knocks, the cards in the discard pile are shuffled and turned over to form a new draw pile.

### Game Variations

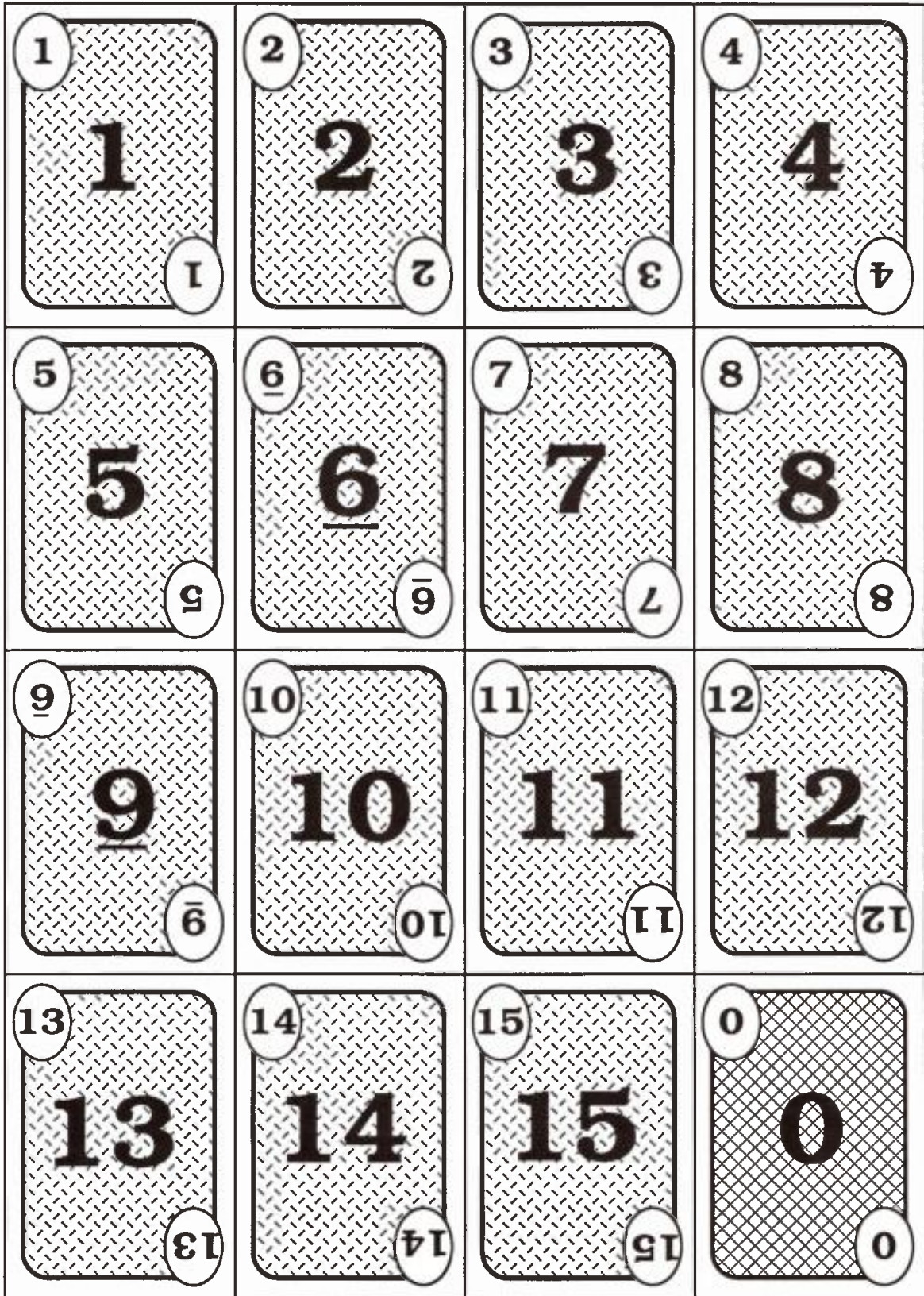
- (a) Players are dealt more than three cards, such as four or five. Not too many cards should be dealt, however, as it makes it too difficult to calculate the mean mentally.
- (b) Each player is initially dealt four cards, three of which he/she keeps in his/her hand and one of which remains face down on the table in front of the player. When the mean, median or mode is calculated at the end of the game, this card is added into his/her hand.

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