Area of *Taxicab* Geometry Circles

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Mathematics teachers are always interested in finding activities that lead their students to apply familiar concepts in novel settings. Geometry provides a rich source of such activities.

The concept of the area of a circle is a familiar geometric idea. Most students can recall the familiar formula, $A = \pi r^2$, where A is the area of the circle, r is its radius, and π is the constant = 3.1416.

What would happen if these same concepts were considered in a *taxicab* geometry setting, that is, a setting in which distances are only measured along preestablished horizontal and vertical routes? We shall consider two examples of taxicab geometry, using square dot paper and isometric dot paper.

Square Dot Paper

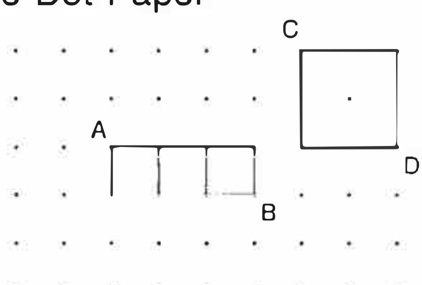

Figure 1

Figure 1 depicts square dot paper with all possible route lines of minimal length drawn from A to B. In this illustration the distance between A and B is 4, and there are four separate routes of that minimal length. The distance between C and D is also 4, but there are now six different routes of that length. Also note that there are many additional routes from A to B whose lengths are greater than 4 units; however, a shortest route will define the distance between A and B or between any two distinct points.

Recall that a circle is the set of points equidistant from a fixed point, called its centre. Using this definition, what would a circle look like in a square dot domain? Figure 2 shows Circle 1 of radius 3 with centre *0*. Each of the 12 points that lie on the circle has a *taxicab* distance of 3 units from *0*. We have connected the 12 points in Circle 2 for counting units of area.

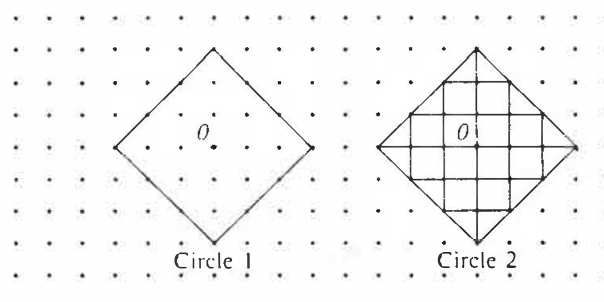

Figure 2

Let a one-unit square be the unit of area. Counting the squares and half-squares (in Circle 2), the area is found to be 18 square units.

Figure 3 displays circles of varying radii, and Table 1 reports the radius and area for each circle.

Table 1

Circle	Radius	Area in Squares
A	1	2
B	2	8
C	3	18
D	4	32
E	5	50
F	6	72

What general formula could be used to compute these areas? Can your students conjecture, from the data in Table 1, that $A = 2r^2$? This taxicab formula has the same form as the familiar $A = \pi r^2$ in Euclidean geometry, with π replaced by the constant 2.

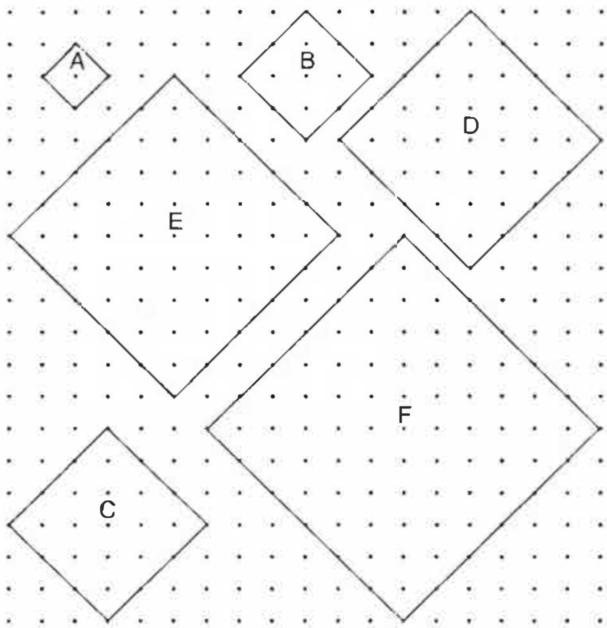

Figure 3

Isometric Dot Paper

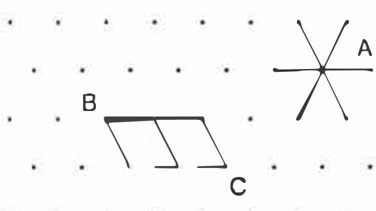

Figure 4

Figure 4 displays the basics of line segments on isometric graph paper. There are six possible one-unit line segments from point A, as shown. Figure 4 also depicts all possible routes drawn from B to C. The distance between B and C is three because the shortest distance from B to C uses three units. Note that there are three separate routes of that minimal length. We also note that there are many additional routes whose lengths are greater than three units. Again, the shortest route will define the distance between A and B or between any two points.

In this setting, what is the analogue to the Euclidean circle? Figure 5 shows Circle 1 of radius 2 with centre *0*. Each of the 12 points that lie on the circle has a taxicab distance of two units from *0*.

Let an equilateral triangle of side one-unit be the unit of area. Counting the triangles in Circle 2 of Figure 5 yields an area of 24 triangular units.

Figure 6 displays circles of varying radii, and Table 2 reports the radius and area for each circle.

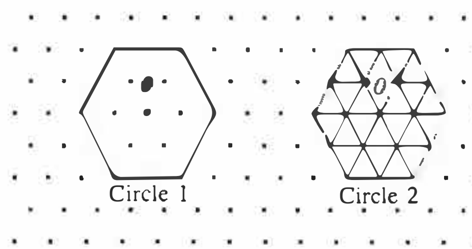

Figure 5

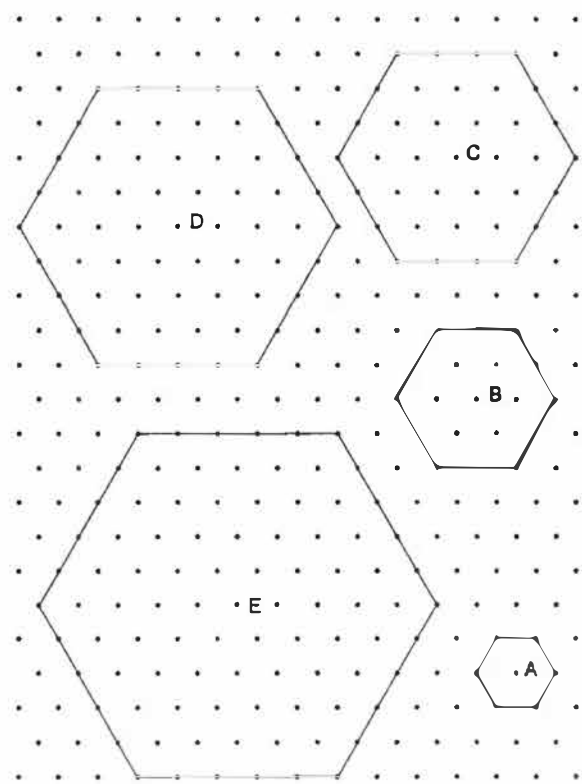

Figure 6

Table 2

Circle	Radius	Area in Triangular Squares
A	1	6
B	2	24
C	3	54
D	4	96
E	5	150

Using the data from Table 2, can your students conjecture that $A = 6r^2$? Again, this taxicab formula is similar to the familiar $A = \pi r^2$, except that π is replaced by the constant 6 in this new setting.

Challenges for the reader and his/her students:

1) Check the formulas that we conjectured using circles of other radii.
2) Conjecture formulas for the circumference of the taxicab circles of Figures 3 and 6.
3) Are there three-space, four-space or *n*-space analogues to the point lattices discussed in this article?

References

Kramer, E. F. "Taxicab Geometry." *Mathematics Teacher* 66 (1973): 695-706.

Sowell, K. O. "Taxicab Geometry, A New Slant." *Mathematics Magazine* 62 (1989): 238-48.