Counting the Complement: The Probability of Consecutive Numbers in a Lotto Drawing

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Teachers are always looking for ways of incorporating ideas of probability into their mathematics classes. It is especially beneficial if a given probability situation can motivate further questions for study. We shall present one such situation.

Many states and provinces have introduced Lotto games in an attempt to raise revenue. Each of these Lotto games begins with an initial set of n numbers. To play the game, a player selects a subset of size r from the initial set. At specified intervals, the Lotto officials select and publicize a winning combination of r numbers. Players win various monetary prizes depending on the number of winning numbers that they match on tickets they have purchased.

An obvious problem for your class is to compute the probability of winning various prizes. This problem has been dealt with in numerous publications. The problem that we now wish to consider is one that arises from observing the winning sets of numbers over a span of time.

Frequently, the winning set contains at least two consecutive numbers. What is the probability that this will happen? We shall solve this problem using the specific Lotto game played in our location, in which six winning numbers are selected randomly from the set 1, 2, 3, . . . , 39. What is the probability that at least two of these winning numbers are consecutive? For example, the winning set might be 3, 6, 12, 13, 25, 31.

To solve this problem, we must proceed indirectly. First, calculate the number of ways the six winning numbers can be selected, whether or not they are consecutive. This can be done in

$C(39, 6) = \frac{39!}{6!\,33!} = 3{,}262{,}623$ ways.

Next, determine how many of these 3,262,623 combinations contain *no* consecutive numbers; this event is the complement of the event whose probability we are seeking. Then subtract the number of combinations in which no consecutive numbers appear from 3,262,623 to identify the number of combinations containing at least two consecutive numbers.

Proceed as follows to count the number of combinations that contain *no* consecutive numbers:

A. Identify the largest number to be selected in the winning set; call this number L.

B. For each case resulting from step A, determine the number of ways in which the remaining five numbers can be selected so that each is less than L and none of the six numbers is consecutive. To do so, visualize each of these five numbers as having a one integer "buffer" to its right on the natural number line. This buffer cannot be used for any of the other winning numbers. For example, if 13 were one of the winning numbers, then 14 cannot also be a winning number. Note that only a right-hand buffer is needed for each of the five smaller winning numbers; if 13 is a winning number then 12 cannot also be a winning number, because its right-hand buffer would then be 13.

The method of step B must be performed separately for each of the values of L as identified in step A.

Case 1

L (the largest winning number) is 39. This leaves 38 numbers apparently available for selection. However, each of the five smaller winning numbers has a right-hand buffer and consequently can be thought of as "using up" two spaces on the natural number line.

Thus, the 38 apparent available numbers now become only 33. The number of ways that five numbers can be selected from 33 is then $C(33, 5) = 237{,}336$. Specifically, this identifies the number of winning combinations having no consecutive numbers and for which the largest number is 39.

Case 2

L is 38. By the same reasoning of Case 1, the number of winning combinations having no consecutive numbers, and for which the largest number is 38, is $C(32, 5) = 201{,}376$.

Case 3

L is 37. Again using the reasoning of Case 1, the number of winning combinations having no consecutive numbers and for which the largest number is 37 is $C(31, 5) = 169{,}911$.

Following this technique for all possible values of L yields Table 1.

Table 1

L (Largest Winning Number)	Number of Winning Combinations with No Consecutive Numbers and Having Largest Value L	
39	$C(33, 5) =$	237,336
38	$C(32, 5) =$	201,376
37	$C(31, 5) =$	169,911
36	$C(30, 5) =$	142,506
35	$C(29, 5) =$	118,755
34	$C(28, 5) =$	98,280
33	$C(27, 5) =$	80,730
32	$C(26, 5) =$	65,780
31	$C(25, 5) =$	53,130
30	$C(24, 5) =$	42,504
29	$C(23, 5) =$	33,649
28	$C(22, 5) =$	26,334
27	$C(21, 5) =$	20,349
26	$C(20, 5) =$	15,504
25	$C(19, 5) =$	11,628
24	$C(18, 5) =$	8,568
23	$C(17, 5) =$	6,188
22	$C(16, 5) =$	4,368
21	$C(15, 5) =$	3 003
20	$C(14, 5) =$	2,002
19	$C(13, 5) =$	1,287
18	$C(12, 5) =$	792
17	$C(11, 5) =$	462
16	$C(10, 5) =$	252
15	$C(9, 5) =$	126
14	$C(8, 5) =$	56
13	$C(7, 5) =$	21
12	$C(6, 5) =$	6
11	$C(5, 5) =$	1
		1,344,904

We shall verify four of the entries of Table 1 by listing the combinations satisfying the "no consecutives" condition.

- L is 11. The only possible combination is 1, 3, 5, 7, 9, 11.
- L is 12. The six possible combinations are
  1, 3, 5, 7, 9, 12
  1, 3, 5, 7, 10, 12
  1, 3, 5, 8, 10, 12
  1, 3, 6, 8, 10, 12
  1, 4, 6, 8, 10, 12
  2, 4, 6, 8, 10, 12
- L is 13. The 21 possible combinations are
  1, 3, 5, 7, 9, 13
  1, 3, 5, 7, 10, 13
  1, 3, 5, 7, 11, 13
  1, 3, 5, 8, 10, 13
  1, 3, 5, 8, 11, 13
  1, 3, 5, 9, 11, 13
  1, 3, 6, 8, 10, 13
  1, 3, 6, 8, 11, 13
  1, 3, 6, 9, 11, 13
  1, 3, 7, 9, 11, 13
  1, 4, 6, 8, 10, 13
  1, 4, 6, 8, 11, 13
  1, 4, 6, 9, 11, 13
  1, 4, 7, 9, 11, 13
  1, 5, 7, 9, 11, 13
  2, 4, 6, 8, 10, 13
  2, 4, 6, 8, 11, 13
  2, 4, 6, 9, 11, 13
  2, 4, 7, 9, 11, 13
  2, 5, 7, 9, 11, 13
  3, 5, 7, 9, 11, 13
- L is 14. The 56 possible combinations are
  1, 3, 5, 7, 9, 14
  1, 3, 5, 7, 10, 14
  1, 3, 5, 7, 11, 14
  1, 3, 5, 7, 12, 14
  1, 3, 5, 8, 10, 14
  1, 3, 5, 8, 11, 14
  1, 3, 5, 8, 12, 14
  1, 3, 5, 9, 11, 14
  1, 3, 5, 9, 12, 14
  1, 3, 5, 10, 12, 14
  1, 3, 6, 8, 10, 14
  1, 3, 6, 8, 11, 14
  1, 3, 6, 8, 12, 14
  1, 3, 6, 9, 11, 14
  1, 3, 6, 9, 12, 14

1, 3, 6, 10, 12, 14
1, 3, 7, 9, 11, 14
1, 3, 7, 9, 12, 14
1, 3, 7, 10, 12, 14
1, 3, 8, 10, 12, 14
1, 4, 6, 8, 10, 14
1, 4, 6, 8, 11, 14
1, 4, 6, 8, 12, 14
1, 4, 6, 9, 11, 14
1, 4, 6, 9, 12, 14
1, 4, 6, 10, 12, 14
1, 4, 7, 9, 11, 14
1, 4, 7, 9, 12, 14
1, 4, 7, 10, 12, 14
1, 4, 8, 10, 12, 14
1, 5, 7, 9, 11, 14
1, 5, 7, 9, 12, 14
1, 5, 7, 10, 12, 14
1, 5, 8, 10, 12, 14
1, 6, 8, 10, 12, 14
2, 4, 6, 8, 10, 14
2, 4, 6, 8, 11, 14
2, 4, 6, 8, 12, 14
2, 4, 6, 9, 11, 14
2, 4, 6, 9, 12, 14
2, 4, 6, 10, 12, 14
2, 4, 7, 9, 11, 14
2, 4, 7, 9, 12, 14
2, 4, 7, 10, 12, 14
2, 4, 8, 10, 12, 14
2, 5, 7, 9, 11, 14
2, 5, 7, 9, 12, 14
2, 5, 7, 10, 12, 14
2, 5, 8, 10, 12, 14
2, 6, 8, 10, 12, 14
3, 5, 7, 9, 11, 14
3, 5, 7, 9, 12, 14
3, 5, 7, 10, 12, 14
3, 5, 8, 10, 12, 14
3, 6, 8, 10, 12, 14
4, 6, 8, 10, 12, 14

Recall that there are 3,262,623 possible winning combinations altogether, and that 1,344,904 of these combinations have no consecutive numbers. Therefore, 3,262,623 – 1,344,904 = 1,917,719 combinations have at least two consecutive numbers. The probability that at least two consecutive numbers will appear is then

$$\frac{1{,}917{,}719}{3{,}262{,}623} = 0.588 \approx 59 \text{ percent of the time.}$$

Cases in which the winning numbers contain at least two consecutive numbers should, therefore, be quite common.

To test this, we kept track of the local winning Lotto numbers during October, November and December 1991 and January 1992. For the 35 drawings that took place, 21 had at least two consecutive numbers. Thus in $\frac{21}{35}$ or $\frac{3}{5}$ or 60 percent of the time, "consecutives" occurred in the winning set of numbers. This is very close to the predicted 58.8 percent.

Challenges for readers and their students:

1. Redo the consecutive number problems for a Lotto game where you live or one which you play.
2. Verify your prediction which results from challenge problem 1 by checking actual Lotto results over a span of time.
3. For a given value of n (total number of numbers available), how large must r (number of numbers selected) be so that the probability of at least two consecutive numbers first exceeds $\frac{1}{2}$?
4. For a given value of r, how large must n be so that the probability of at least two consecutive numbers first exceeds $\frac{1}{2}$?
5. Use the computer to simulate problems of this type.