delta-k

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The probability of consecutive numbers in a Lotto drawing.

- Making Sense of Mathematics
- Recreational Mathematics
- Teaching Ideas

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EDITORIAL

Congratulations to the executive on another excellent mathematics conference in Medicine Hat on November 6 and 7, 1992. The *Math Fare* was a truly unique and most useful conference for any teachers seeking new and creative teaching ideas. Bravo to Diane Congdon and her associates on such an interesting and well-organized conference! We look forward to the 1993 Calgary meeting with equal enthusiasm.

The articles in this issue have been divided into three sections: Making Sense of Mathematics, Recreational Mathematics and Teaching Ideas. The first section includes three articles that deal with the attempts made by some students and teachers to make sense of their mathematical experiences. The article by Liedtke addresses the question of number sense and provides some interesting examples of discussion between teachers and students. In the second article, Hahn describes some of her first positive experiences involving the learning of mathematics, achieved through exploration in a Logo problem solving context. The third article, written by Fischer, provides some helpful advice to teachers for developing and maintaining a positive problem solving environment.

The second section Recreational Mathematics includes two interesting applications of mathematics. The first article, by Duncan and Litwiller, discusses the probabilities of particular number combinations in lotteries. In the second article, contributed by Lauber, the age-old technique of casting out nines is reexamined within the context of polynomial arithmetic. Both applications could encourage interesting discussions in the senior high classroom.

Teaching Ideas, the final section, includes two classroom-ready activities. The first activity, contributed by Neufeld, challenges students to apply their addition and subtraction skills to create a graphic. The second activity is a game that represents an application of several mathematical concepts (graphing, the four basic operations and simple probability) to the stock market context. The game is both challenging and fun, suitable for the junior high mathematics student.

As always, we are interested in receiving articles from all of our readers, but we are especially interested in hearing from Alberta educators. If you were one of the presenters in Medicine Hat, jot down your contribution and forward it to the Mathematics Council. Your ideas count, and someone is waiting to use them in his or her classroom!

Happy reading!

A. Craig Loewen

Erratum

John Percevault was the primary editor of the last issue of *delta-K* (Volume 31, Number 1, December 1992).

Making Sense Out of Number Sense

Werner Liedtke

In the focus issue of the *Arithmetic Teacher*, Howden (1989, 6) reports that the Standard from the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) dealing with number sense "raised the most questions from teachers, parents and administrators. These four questions were most frequently asked: What is number sense? Why is number sense important? How is number sense taught? How is number sense measured?"

The editorial panel of the *Arithmetic Teacher* states that "although mathematics educators agree that the development of number sense is important, no single definition is universally accepted" (Thompson and Rathmell 1989, 2). Hope (1989) agrees when he advises that "number sense is considered a desirable trait to foster, although its meaning as other notions of thinking . . . can be defined only broadly." The same author goes on to explain that "number sense cannot be defined precisely, but situations where it is evidently lacking can easily be recognized" (p. 12).

In a midterm, when asked to "define number sense in your own words" and to "illustrate the definition with an example," one mathematics education student's opinion in a way opposed Hope's. She declared, "I find it very difficult to define number sense, but I would surely recognize its presence when I encountered it." She then described an interview response from an elementary school student to illustrate her declaration.

While taking the above-mentioned mathematics course, student teachers have an opportunity to watch interviews of students from different grades. As part of the follow-up, responses that indicate the absence and presence of number sense are discussed. The student teachers identified several elementary school students who provided responses indicative of the presence of number sense. The excerpts from three different interviews involving students from three different grade levels follow. As you read the excerpts, try to identify indicators of the presence of number sense.

The Grade 3 girl knew that two types of subtraction tasks exist. She knew of more than one strategy to find answers for basic facts. Throughout the interview, she appeared very confident, was more than willing to talk about her knowledge, and expressed herself in a charming and unique way. When asked to find the answer for 56–23, she recorded it vertically, and the following exchange took place (T—teacher; S—student):

T: Why did you write it like that?
S: That's the way I write it.

T: Why?
S: It's an easier way to write it.

T: What makes it easier?
S: Nothing really—just an easier way.

T: So where do you start?
S: With the six and the three.

T: Why do you start on that side?
S: Because it gets sort of complicated if you start there (the tens), because sometimes the three could be here (at the top) and the six could be here and you would have to add it. But you start with this one, then you don't know if you have to add or not (records answer 33).

T: Let's do another one (62 – 27 = is presented). What do you put here (at the end), a box or a line?
S: Nothing, I just put the answer there.

T: Show me how you would find the answer.
S: Do it the same way I did before?

T: Yes.
S: (writes $\begin{array}{r}62\\-27\end{array}$.)

This paper is based on a presentation made at the Mathematics Council Conference in Edmonton in 1991.

T: I would like you to talk to me while you are doing it.

S: OK. Two take away seven you can't do it so you cross the six and put a five above it, and you put a ten there so it gets to twelve. Then you add seven to twelve and that's four, no five (records five), and then you take away the five to the two and that's three (records three in tens place).

T: How did you know this was five (the five in the ones place is pointed to)? How did you figure that out?

S: I just go from seven to twelve.

T: Show me how you do that.

S: I go from seven (seven said with emphasis and holds up one hand and begins to count showing one finger for each of) eight, nine, ten, eleven, twelve (five fingers are held up).

T: So you count up?

S: Yeah. I counted up instead of down. But if it's like 12 take away 1, just . . . or, if it's like 12 take away . . . just pretend I don't know the answer . . . if it's 12 take away 2 . . . and just pretend I don't know (giggles) . . . 12 take away 2 and I take away . . . and I go from 12 down to take "awaying" 2 . . . like 12, 11, 10 . . . but since it's not that I am not going to do it, since it's like a higher number than 3 or 4 I won't do it.

The Grade 5 boy was a learning disabled student. He did not find it easy to talk about his knowledge. Long pauses existed between questions and answers, and between parts of the same answer. During the major part of the interview on multiplication, he kept confusing the terms *multiplied by* and *divided by*. Both verbal interpretations were used for the multiplication symbol.

When the interview dealt with the basic multiplication facts, the student classified the items as easy and difficult. On completion of this task, answers for the facts identified as easy were solicited. All responses were correct, including the answers for 8 x 9 and 6 x 9. The answers for the facts identified as difficult were not known. However, the student was willing to "make an estimate." The following conversation was part of this setting:

T: If you forget an answer or don't know an answer, what would you do (7 x 8 was selected)?

S: I'd work on it.

T: How? How would you work on it?

S: I would go 9 x 7 equals 63 – 8.

T: That's clever. Did you teach that to yourself or did someone show that to you?

S: I teached it to myself.

T: (The flashcard showing 8 x 8 was selected.) What do you think is close to the answer for this?

S: 60.

T: How would you work on this one?

S: Ah, . . . I'd turn that to nine and I'd divide it by seventy-two . . . I mean the answer is seventy-two and I minus it by eight.

T: And you taught this to yourself?

S: Yes.

T: That's fascinating. Let's pretend you forgot the answer for 7 x 6 (selected from the difficult group). You tell me how you would figure it out.

S: How to figure it out?

T: Yes.

S: I'd make that a five (the six) and keep that a seven and I'd . . . and it would be . . . thirty-five, and I'd add a seven onto it.

During the interview dealing with decimal fractions, the boy from Grade 7 appeared confident and was very willing to talk. While the understanding of decimals, and then addition and subtraction were dealt with, part of his mathematical behavior seemed to be rule bound. Reasons were given for most of the students' rules. Other rules simply existed. For example, when the addition of "ragged" decimals (0.86 + 0.4 + 2.0 + 6.125) was discussed, he insisted that "you always write the biggest numbers first. You just do!" As the discussion turned to multiplication, the following dialogue took place:

T: Have you ever multiplied decimals before?

S: Yes.

T: Without finding the answer, try to tell me something about the answer for 0.7 x 0.8.

S: Oh . . . it's just like a normal, just 7 x 8 is 56, but when you multiply decimals . . . you've got to . . . like . . . when you have got the answer . . . you've got to take how many places it is in the whole question . . . two places to the left . . . so 56 is the answer . . . and then you go (begins to write) . . . you go 0.7 and 0.8 . . . you just forget about the decimals . . . and you get

56, but there is 2 places—so you got to go 1 . . . 2 (counts from right) put the decimal there and your answer is 0.56.

T: Is that a rule you were taught?

S: Yes.

T: How would that rule work for 3.05 x 0.9?

S: OK . . . yeah . . . it doesn't matter where the decimal place is. You don't have to line them up. You can just go like (writes 3.05 x 0.9 and multiplies to find the answer, counts decimals places and records 2.745).

The request to place a decimal point into the answer for 15.5 x 8.24 = 12772, resulted in the immediate response, "Right here (12.772) . . . because 8.24 has two decimal places in the number and 15.5 has one place . . . though . . . 8 x 15 = 120 . . . so it would go there—127.72. Oh . . . yeah!"

It was fascinating to note that division of decimals was dealt with by changing every decimal numeral to a fraction numeral. The rationale given was "That's how it is done!"

Why do the student teachers suggest that some of the students' responses indicate the presence of number sense? Do you agree? If so, which responses do you think indicate the presence of number sense? Do any of the responses indicate a lack of number sense?

Number sense is a new expression. Most teachers-in-training would likely not have encountered it as part of their training or by paging through mathematics methods texts (published prior to 1990). That is not to say that these books do not deal with the notion of number sense.

According to the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, 39–40),

> number sense is an intuition about numbers that is drawn from all the various meanings of number. It has five components:
>
> 1. Developing number meanings
> 2. Exploring number relationships with manipulatives
> 3. Understanding the relative magnitude of numbers
> 4. Developing intuitions about the relative effect of operating on numbers
> 5. Developing referents for measures of common objects and situations in their environment

Developing number sense is not just an important component of Standard 6 for K to 4: Number Sense and Numeration (ibid., 38–40), but also is part of Standard 12 for K to 4, which states that students should develop number sense for fractions and decimals (p. 57). For Grades 5 to 8 (Standard 5: Number and Number Relationships), it is proposed that "the mathematics curriculum should include the continued development of number and number relationships so that students can develop number sense for whole numbers, fractions, decimals, integers, and rational numbers" (p. 87).

As suggested, the expression *number sense* may be new, but important ideas related to the notion have been dealt with by different authors under different headings, for example, *concept of number*; *understanding numbers*; Howden (1989) talks about Wirtz teaching *friendliness with numbers*; *making numbers come alive* (May 1978; Liedtke 1983); many primary teachers talk about attempting to develop *a feeling for numbers* or they comment that some students lack such a *feeling*.

Whatever number sense or the intuition about numbers and relationships is defined as, authors who have written about it and teachers who attempt to teach it would agree with Thornton and Tucker (1989) who state that it develops over time. The authors propose that for some students number sense develops and matures naturally but "for others it will happen only if the teacher plans ahead to be ready to capture the opportunity of the moment, in the mathematics lesson and beyond" (ibid., 21). This planning by the teacher "should begin in the first grade with appropriate tasks that call on number sense and give children a less mechanical view of mathematics" (Markovits, Hershkowitz and Bruckheimer 1989). As a matter of fact, it would be advantageous if this view of mathematics and a focus on number sense would be part of a child's preschool experience. (This viewpoint will be discussed when a few specific examples and strategies are discussed.)

What about number sense as part of a theoretical framework? Van de Walle (1990a, 12) reminds us that "all theories are just that—theories." The author points out that "if, however, a theory of learning is found to be useful in effectively helping us to be better teachers, then a theory is worthy of consideration." According to Van de Walle, a cognitive theory can heighten teachers' awareness to integrate new ideas with existing knowledge and provides the basis for developmental teaching. The objective of teaching mathematics developmentally is relational understanding. The goal of relational understanding is described in terms of a three-part objective: well-integrated conceptual knowledge (concept, relationships), well-developed procedural knowledge (symbolism, rules, procedures) and clearly developed connections between concepts and procedures.

Any attempt to define Van de Walle's terms *well integrated*, *well developed* and *clearly developed* would likely make it difficult, if not impossible, to do so without referring to number sense. Perhaps relational understanding and number sense are to a degree synonymous, or at least number sense is an important component or subset of relational understanding. The observations that make student teachers suggest that some evidence exists for the presence of number sense for the three students who were featured in the interview excerpts usually fall into a category that includes statements labeled well integrated, well developed and/or clearly developed.

As part of a chapter on "concepts of number," Van de Walle (1990b) presents a list of reasons for an added emphasis on number sense. This list includes the point that "number sense contributes directly to problem-solving abilities and flexible thinking in numerical situations" (p. 63). According to Van de Walle, "without a major commitment by a curriculum to experiences that develop number sense, many children will never understand number sense in any other way than by counting" (p. 64).

What are some general as well as specific teaching-learning strategies and settings that are conducive to the development of number sense? Van de Walle (1990a, 18–19) identifies hallmarks of teaching developmentally. The characteristics of this approach that could be considered especially important for the development of number sense include

> understanding that existing ideas give meaning to new ones; encouraging children to talk about concepts and relationships; using manipulative models as a major tool to create linkage between conceptual and procedural knowledge; capitalizing on children's oral language in the promotion of relational understanding; being careful to see that conceptual knowledge is developed prior to the introduction of symbolism; and avoiding an overemphasis on mindless drill.

The authors who wrote articles for the *Arithmetic Teacher* focus issue on number sense mention many ideas that are related to teaching strategies and classroom settings. These include "doing mathematics" in an environment that fosters curiosity and exploration (Howden 1989, 11); exposing children to "messier" aspects of everyday problem solving and placing more emphasis on thinking about various procedures that can be used to solve a problem and on interpreting the answers that these procedures produce (Hope 1989, 16); promoting discussion through appropriate questioning techniques and planning for the development of number sense for all parts or throughout the lesson (Thornton and Tucker 1989, 19 and 21); having students discuss the application of number-sense concepts to a word problem to contribute to the understanding of the "hows" and "whys" of numbers (Dougherty and Crites 1989, 25); creating a climate that encourages pupils to ask "why" (Whitin 1989, 29); selecting activities that give students the opportunity to verbalize relationships that demonstrate the acquisition of good number sense (Glatzer and Glatzer 1989, 38); revising measurement-based curricular applications to take actual measurement practices into account, encouraging students to work on tasks in groups of three or four, relating the operations of arithmetic to real-world models, making children aware that not every mathematics problem has a single correct answer, enlisting students to help uncover "old-fashioned" or unrealistic material in their textbook, and challenging students to make generalizations (Kastner 1989, 46); provoking each pupil into constructing his or her own knowledge of numbers and the relations among them (Ross 1989, 50); selecting appropriate activities conducive to the development of number sense (Markovits, Hershkowitz and Bruckheimer 1989, 55).

The characteristics of teaching developmentally suggested by Van de Walle and the list of hints from the different authors clearly point to the importance of language or the creation of a teaching environment where students are given an opportunity to talk as they exchange ideas. This role of language is supported in Standard 2: Mathematics as Communication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). It is advocated that students in Grades K to 4 should be able to "relate their everyday language to mathematical language and symbols" and to "realize discussing, reading, writing and listening to mathematics are a vital part of learning and using mathematics"

(p. 26). Students in Grades 5 to 8 should be able to "discuss mathematical ideas, and make conjectures and convincing arguments" (p. 78). The goal that students can "reflect and clarify their thinking about mathematical ideas and situations" is included for Grades K to 8 (pp. 26 and 78).

Many or most of the teaching strategies that have been advocated and the suggested focus on language can best be accommodated in an environment where students cooperate with each other. As Willoughby (1990, 58) suggests, "Mathematics is not a solitary activity. It should be done and learned with others. . . . This was true ten years ago, it is true today, and it will be true in ten years—whether or not cooperative learning happens to be in vogue."

This emphasis on talk or on students talking does by no means imply increased passivism by a teacher. To get students to clarify or modify their thinking as they are involved in mathematical situations, teachers need to listen carefully as they attempt to accommodate responses at various ability levels. An interactive environment is required where leads change back and forth from student to student as well as from teacher to student. In terms of a two-dimensional teaching model consisting of four quadrants (Calkins 1986), where low and high student input is plotted along the vertical axis and low and high teacher input along the horizontal axis, it is likely that the goals from the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) related to language and thinking can best be met in a setting advocated for the quadrant of high-student—high-teacher input.

A synthesis of the different suggestions for teaching ideas that have been made by the various authors points to the importance of appropriate teaching and to the teacher's role as the key to the acquisition and/or development of number sense. Telling and showing may be part of a mathematics lesson, but as Ross (1989, 50) reminds us, "When a teacher shows students something, students do not have to think; they simply follow directions."

Romberg (1990, 472) states that "the notion that mathematics is a set of rules and formalisms invented by experts, which everyone else is to memorize and use to obtain unique, correct answers, must be changed." Teaching about and for problem solving are parts of mathematics lessons; however, teaching via problem solving (Schroeder and Lester 1989) and via thinking is essential when the building of number sense is the goal.

Howden (1989, 7–8) states that "children discover new relationships and properties with numbers when they use concrete materials." Manipulative materials and models by themselves are likely not to contribute to the development of number sense. To design the "mind on" setting that is required, a knowledgeable teacher is essential. To develop a "teaching-via-problem solving" and a "teaching-via-thinking" atmosphere, to "capture the opportunity of the moment" as Thornton and Tucker (1989, 21) suggest, a teacher has to be there to observe, to listen and to phrase as well as to place appropriate questions to students or groups of students. Delicate orchestration by a competent teacher is a necessary element of mathematics learning involving materials and models.

The proposals for appropriate settings these various authors make leave no doubt that teaching for number sense requires careful planning and a lot of effort. Because cooperative settings involve a lot of talk by students, teachers have to learn how to become good listeners. It is difficult to imagine how the development of mathematical power, which is central to the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and includes thinking, talking, connecting and problem solving, could be evaluated without making observations or without conducting interviews. Aligning the curriculum with methods and tasks for assessing students' learning implies that oral questions, oral presentations and student interviews become integral parts of these methods (ibid., 200–201). This alignment is required when the development of number sense and the development of mathematical power become an important part of the mathematics program.

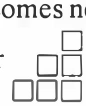
Assessment of development of mathematical power and number sense requires structured interview settings. Because interviews involve probing into how students think as well as how they think about their thinking, these settings represent research at the highest possible level. Teachers will require training and experience to conduct these interviews appropriately and efficiently (Liedtke 1991).*

The last part of this paper describes several activities to illustrate or reinforce a few of the points

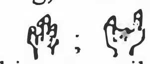
*Ten one-hour videotapes entitled "Diagnosis and Intervention in Mathematics" are available from Education Extension, Faculty of Education, University of Victoria, Box 3010, Victoria, B.C. V8W 3N4. Clinical interview tasks and settings are described, illustrated and analyzed. Intervention strategies and activities are also discussed.

that have been raised. A variety of examples are included in the *Arithmetic Teacher* focus issue (February 1989) on number sense.

Readiness for number sense, or the notion that mathematics deals with ideas or problems that do not have to have just one specific answer, can begin in the preschool setting or in the home. For example, when young children are asked to sort objects or pictures of objects (Carson and Lindsay 1972), they are often told the categories of classes that are to be considered such as food or clothing. It is simple to create a problem solving setting by asking children to respond to requests like Which of these do you think are in some way the same? Put them together and tell me how they are the same. Think of other ways of putting things together that are in some way the same. Try to guess how and/or why everyone else (the teacher) puts things that are in some way the same together. The intent would be to get children to find many common characteristics for a given group of objects and to enable them to discover new ways of thinking about and new ways of talking about these familiar objects. This flexibility should make it easier for children to move from a scheme of classes based entirely on perceptions to schemes based on other criteria such as number.

The idea of "there are many ways of *solving problems*," and Willoughby's (1990, 51) reminder that "there is generally more to be gained by solving a single problem in several ways than by solving several problems in a single way," can be permeated while examining patterns or ordered sequences. It would be advantageous for young children to realize that the questions, What comes next? for [red][red][blue][red][red][blue] or for  can be answered in more than one way. Young children need to realize that different number names can be chosen from 1, 2, 3, 4, 5, 6 . . . as a response to, What comes next? for 1, 2, 3, ☐ as long as a valid reason can be provided. (A teacher or an interviewer may have to judge whether or not the answer, "Six, because I am six years old" is a valid reason!) Listening to others justify their answers can initiate a search for other responses.

An examination or a study of numbers should include activities that lead students to conclude, for example, "Eight—You Are Beautiful!" (Liedtke 1983) or "Six—You Can Be Different." The idea is to present tasks and to initiate discussions to make young children realize that numerousness is independent of color, shape, size and arrangement. They need to realize that "as long as we can match a given number of fingers with an arrangement of objects, the same number name and symbol are assigned, no matter what characteristics these objects display" (ibid., 34).

Finger-flash activities can contribute to the development of number sense. For example, for numbers five and less, children are asked to say the number names (without counting) for displays of any combination of fingers—  . . . This can easily be changed into a tactile experience by having a student who has closed his/her eyes attempt to determine (without counting) how many fingers a partner is displaying.

The number of fingers on one hand represents an important benchmark for young children. For each of the possible arrangements of fingers on one hand, students should be able to recognize and state (without counting) the number shown as well as how many more it takes to show five, that is, for  three and two.

The same type of finger-flash activities that have been described for five and fewer fingers can be extended to the fingers on both hands. For any given arrangement of fingers, the students' goal should be to state how many are shown and then how many more fingers it would take to show ten.

After the numbers to 10 have been examined, young children could be faced with an activity where an arrangement of counters, such as 6, is briefly shown, either via an overhead projector or by showing a card of dots. The exposure is too brief to allow children to count. They are asked to indicate in some way, such as thumb up, thumb down (Reys, Suydam and Lindquist 1989) and closed hand, whether they think they saw more than, less than or about the same as a given number (that is, 5), respectively. Reys, Suydam and Lindquist (p. 77) suggest that numbers like 5 and 10 could be used as early benchmarks because they "are internalized from many concrete experiences, often accumulated over many years." A natural extension of this type of activity to benchmarks of 20 or 100 is possible in later years.

Thornton and Tucker (1989) discuss the contribution "lesson warm-ups" can make to the developing of number sense. One Grade 1 teacher begins many of her mathematics classes with a guessing game. For example, "I have recorded a number between five and twenty-five. Guess my number." Several simple ideas can add to the effectiveness of this type of activity:

- Projecting a hundred chart onto the chalkboard allows children to indicate their responses to What number could it be? and What number could it not be? by circling and crossing out numbers, respectively after each guess or response.
- Recording each student's guess, or a few key words of the guess on the chalkboard, will allow students to consider previous information as they attempt to phrase new guesses.
- Introducing the rule for posing questions that you may not say, Is it ______ (name of number)? will force children to think of and use some of the terminology they have encountered in the mathematics classroom.
- The rule that each question has to be of a different type from those asked previously can create a challenge, which initially can best be accommodated by having students in teams of two and then providing a little time to discuss possibilities prior to the posing of the question.

After students have learned to group by 10s and 1s to find the answer to, How many? a group of precounted objects greater than 10 is shown via an overhead projector. The projector is turned off before students can count the objects. They are asked to record 2 responses, an estimate to the nearest 10 (is about ______ 10s) and the other a lucky guess of how many they thought they saw. Although strategies are discussed and compared, students' responses with respect to these strategies are not evaluated or categorized (that is, as good, better or bad strategies). Neither are the estimates evaluated. The correct answer is announced, and the procedure is repeated with a different group of objects. One interesting observation consists of finding out how well some students are able to use the previous task(s) as a benchmark. Initially, a group of 10 could be included as a benchmark in the corner of the overhead display along with the objects to be considered.

Some authors who have written about number sense suggest that estimation contributes to number sense. It can be argued that without the presence of number sense, it would be very difficult, if not impossible, to teach students how to estimate. Without number sense, estimation is likely to be reduced to a procedural skill rather than becoming an important part of thinking. Perhaps it is safe to say that a certain degree of an understanding of number and the relationship between numbers is required before estimation can contribute to the continual development of number sense.

Thornton and Tucker (1989, 19) advise that, "lessons can be created that interweave learning about computation and number sense." As teachers plan, they need to ask themselves how it is possible, for example, to teach the basic facts, the algorithmic procedures for the four operations or measurement keeping in mind the importance of number sense. What kind of activities will contribute to the development of "operation sense" and "measurement sense"? What kind of activities and games can and will contribute to further the development of number sense? I hope that some of the ideas in this article will inspire you to provide an environment with a focus on the development of number sense and to actively search for strategies and activities to reach this goal. Such an environment will require special planning and an atmosphere that encourages children to talk and to take risks. It will require a setting where children feel good about themselves and feel good about the mathematics they know and the mathematics they are learning.

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Logowriter: The Real Story About the Tortoise and the Hare

Marilyn Hahn

I do not like numbers. I have never had an interesting or caring math teacher. In Grade 4, I remember receiving my first inklings that math was not for me. I remember that sinking feeling in my gut, that feeling of failure and hopelessness that would never go away—I was told I was in the lowest percentile in the Iowa "thing." Later, in Grade 7, Mr. K would get so angry at me for not "understanding something so simple" that spittle would hurtle across space and splatter my cheek. In Math 10 (I have no single recollection of the teacher), I received 35 percent on my final report card after much difficulty with the course. I was counseled to forget about my dream to be an architect or interior designer. In fact, I was told that university was out for me as Math 30 was a prerequisite.

So, at age 16, I quit school. (I still don't have my high school diploma, and just to show "those guys," I plan to get it after I finish my B.Ed. degree!)

How does this tie into Logo? Numbers terrify me. They bore me, frustrate me and leave me quite cold inside. The questions that needed to be asked and the problems posed with Logo were largely number oriented. So I froze. How can I even attempt to figure out that commonsense Logo circle when my stomach knots up, when that simple obvious numerical answer eludes me as my thought processes tumble and whirl in memories of horribly embarrassing geometry and algebra classes? How can I confidently approach problem solving when as a child I would gaze at question 3 on page 34 and just get sick because the process, much less the answer, just wasn't there? So, to me, Logo was a terrifying adventure into the past, an adventure where asking a math question and being exposed to a math problem were met with an irritated and impatient "here-let-me-show-you-quick-so-you-can-get-on-with-it" look.

As a result, I stared at the screen a long time before attempting to do it. I felt blank, empty, weak. I felt frustrated at my first attempts—it was a juvenile piece of work, quite devoid of any mature question asking and problem posing. I felt like someone was going to roll his or her eyeballs. Time was being wasted, and the turtle was stupid in the light of great essays to be written and little people to be encouraged and set free in the world of learning.

I started two projects before I settled on the final one. The first—after hours of work—had the flip-side accidentally erased. Oddly, I was so exhausted with it that it hardly raised the hair on the back of my neck. Fortunately, I'm old enough to know there is a tomorrow, so I started a different one. I then attempted a birthday card complete with cake, burning candles and blinking text. But . . . it was boring. Listen, I may not be a genius at Logo—or even show signs of potential—but I do know dull and I do know that I can do better . . . so I abandoned it.

I now envision these preliminary attempts as two wings enveloping me (sounds corny, eh?). I would lie awake, surrounded by them: 5 FD, RT 90 SETSH 34 STAMP ST HT. Waves filling in. Candles flickering. Turtles flying off the page. A blank page. On and on and on.

And then, in a moment, I saw the turtle chasing the rabbit! What?! Was it that third cup of coffee? Can I do that? And then there was a street, and a steep incline where the rabbit poops out and . . . and . . . a *snotty* turtle. And then there was a twist (every interesting writer throws a twist into the plot). Maybe the turtle didn't win the race. Or maybe there were consequences to his pride—maybe the race wasn't an easy "A." Then again, maybe the rabbit was very undeserving of his seemingly gracious loss. Maybe he had mob connections or a rich dad or an unhealthy attachment to his mother. Maybe . . . Oh! Is this called incubation?!? (Let's just say I'm too modest to call it inspiration.) I went for it.

Suddenly the math took a back seat to the story. I moved from the uncomfortable world of math to the more comfortable world of language. Even though I had to use numbers and logic to interpret the language, I felt more at ease. The project moved from a number exercise to a whole composition plan. I had a subject, a purpose, a thesis, preliminary notes (my two previous attempts) and some thoughts as to organization and development. I felt a playfulness emerge, and the tension slowly dissipated.

The command centre was my rough draft. The flipside became the paper whereon I wrote the good copy. The page was the illustration(s).

Sequential flow became effective sentence and paragraph construction; each procedure took on the nature of a sentence and paragraph: topical, ordered, adequately developed and organized. Conditional action—the "if statement"—took on the characteristics of logical writing: IF: ANSWER = [MOM] [PRINT[Right on!]STOP]. And looping was used for emphasis, variety and movement, while program calling formed the subplots and problems that all culminated in one grand solution.

I skimmed the list of primitives and saw my sentence phrases (PRINT), tense and moods (SHAPES), verbs (STAMP), adjectives (WAIT, FILL, SHADE), chronological orders (SETPOS) and characters (TELL [0 1 2 3])! This language I could understand!

So, I began to remind myself that just as a decent piece of writing doesn't just happen but takes mucking about, so does math. I started over, refined and reevaluated the Logo commands. I worked them. I had never done this with numbers before because I was always under the impression that numbering was immediate and that solutions were there to be had as fast as the math teacher could scribble them on the blackboard. Now I see that numbers were not religiously placed, never to be disturbed lest a curse come down upon thee. Rather, they were malleable and in fact, the most creative numbering all the more so! So, once I realized that Logo need not have been the stationary, numerical math monster I had first envisioned, I relaxed . . . and had fun . . . finally.

The obvious significance of this exercise is that despite my initial negative experiences and apprehension, I hung in there. I knew that this mathematics/ Logo activity wasn't impossible because others were doing it. I knew that successes in other areas came after a struggle and that this activity was really no more difficult than learning a second language or wearing a bathing suit after eight years. And I knew, despite the real anxieties that twisted my stomach into a ferocious knot, that grammar school failures are a thing of the past and that I can activate maturity and a "what the hell attitude" and lick this thing.

So, maybe it's not the greatest piece of work to flash across your screen and maybe you do find it elementary, but I see it as a thumb-to-the-nose gesture to all those counselors and math teachers who said it couldn't be done.

Problem Solving Ideas

Florence E. Fischer

Involve students as much as possible.

Challenge students always.

Amaze students just a little.

Frustrate students just enough.

Tell students as little as possible.

Encourage students for every step forward, no matter how small.

Accept ideas—even crazy ones.

Have fun—enjoy the problems yourself.

Involve students. Present problems to which the students can relate. Use familiar ideas and situations. Get every student to try the problem. Problems that can be solved by trying lots of examples are useful; every student can be working on a different example. Problems that use manipulative materials also invite all to take part.

Challenge students. Problems that are too easy can become boring. Students should have a sense of accomplishment when the problem is solved. This does not happen if there is no challenge. However, what challenges one student may not challenge another. Some students may be challenged by a list-making problem, others by geometrically oriented ones and others by complicated calculations. The teacher must choose problems appropriate for each student.

Amaze students. Using a "trick" to solve a question can be amazing to students. The problem then becomes to figure out what the trick is or how it works.

Frustrate students. This is probably the easiest thing for a teacher to do. The difficult part is to frustrate just enough to make the student take one step beyond the familiar. Small frustrations can stretch the mind; large ones can destroy it.

Tell students. Again, this is an area where teachers excel. Teachers by nature like to tell someone else how to do it. It is difficult to keep the mouth shut and let the student do most of the talking and thinking. Problems that are obvious to teachers are not so obvious to students. Keep quiet and let them think and work.

Encourage students. Students need to feel good about themselves. Teachers can create these good feelings by encouraging even the smallest progress. A student does not have to solve the problem to learn much about problem solving. Remember, the thinking skills developed are more important than the answer to the problem.

Accept ideas. Brainstorming is a viable technique to use in problem solving. Sometimes the wildest ideas can lead to solutions, and they do show creativity.

Have fun. If the teacher does not enjoy working on problems, there is little chance the students will. Let them know that the fun is in the process of the solution, not the solution itself.

Counting the Complement: The Probability of Consecutive Numbers in a Lotto Drawing

David R. Duncan and Bonnie H. Litwiller

Teachers are always looking for ways of incorporating ideas of probability into their mathematics classes. It is especially beneficial if a given probability situation can motivate further questions for study. We shall present one such situation.

Many states and provinces have introduced Lotto games in an attempt to raise revenue. Each of these Lotto games begins with an initial set of n numbers. To play the game, a player selects a subset of size r from the initial set. At specified intervals, the Lotto officials select and publicize a winning combination of r numbers. Players win various monetary prizes depending on the number of winning numbers that they match on tickets they have purchased.

An obvious problem for your class is to compute the probability of winning various prizes. This problem has been dealt with in numerous publications. The problem that we now wish to consider is one that arises from observing the winning sets of numbers over a span of time.

Frequently, the winning set contains at least two consecutive numbers. What is the probability that this will happen? We shall solve this problem using the specific Lotto game played in our location, in which six winning numbers are selected randomly from the set 1, 2, 3, . . . , 39. What is the probability that at least two of these winning numbers are consecutive? For example, the winning set might be 3, 6, 12, 13, 25, 31.

To solve this problem, we must proceed indirectly. First, calculate the number of ways the six winning numbers can be selected, whether or not they are consecutive. This can be done in

$C(39, 6) = \frac{39!}{6!\,33!} = 3{,}262{,}623$ ways.

Next, determine how many of these 3,262,623 combinations contain *no* consecutive numbers; this event is the complement of the event whose probability we are seeking. Then subtract the number of combinations in which no consecutive numbers appear from 3,262,623 to identify the number of combinations containing at least two consecutive numbers.

Proceed as follows to count the number of combinations that contain *no* consecutive numbers:

A. Identify the largest number to be selected in the winning set; call this number L.

B. For each case resulting from step A, determine the number of ways in which the remaining five numbers can be selected so that each is less than L and none of the six numbers is consecutive. To do so, visualize each of these five numbers as having a one integer "buffer" to its right on the natural number line. This buffer cannot be used for any of the other winning numbers. For example, if 13 were one of the winning numbers, then 14 cannot also be a winning number. Note that only a right-hand buffer is needed for each of the five smaller winning numbers; if 13 is a winning number then 12 cannot also be a winning number, because its right-hand buffer would then be 13.

The method of step B must be performed separately for each of the values of L as identified in step A.

Case 1

L (the largest winning number) is 39. This leaves 38 numbers apparently available for selection. However, each of the five smaller winning numbers has a right-hand buffer and consequently can be thought of as "using up" two spaces on the natural number line.

Thus, the 38 apparent available numbers now become only 33. The number of ways that five numbers can be selected from 33 is then $C(33, 5) = 237{,}336$. Specifically, this identifies the number of winning combinations having no consecutive numbers and for which the largest number is 39.

Case 2

L is 38. By the same reasoning of Case 1, the number of winning combinations having no consecutive numbers, and for which the largest number is 38, is $C(32, 5) = 201{,}376$.

Case 3

L is 37. Again using the reasoning of Case 1, the number of winning combinations having no consecutive numbers and for which the largest number is 37 is $C(31, 5) = 169{,}911$.

Following this technique for all possible values of L yields Table 1.

Table 1

L (Largest Winning Number)	Number of Winning Combinations with No Consecutive Numbers and Having Largest Value L	
39	$C(33, 5) =$	237,336
38	$C(32, 5) =$	201,376
37	$C(31, 5) =$	169,911
36	$C(30, 5) =$	142,506
35	$C(29, 5) =$	118,755
34	$C(28, 5) =$	98,280
33	$C(27, 5) =$	80,730
32	$C(26, 5) =$	65,780
31	$C(25, 5) =$	53,130
30	$C(24, 5) =$	42,504
29	$C(23, 5) =$	33,649
28	$C(22, 5) =$	26,334
27	$C(21, 5) =$	20,349
26	$C(20, 5) =$	15,504
25	$C(19, 5) =$	11,628
24	$C(18, 5) =$	8,568
23	$C(17, 5) =$	6,188
22	$C(16, 5) =$	4,368
21	$C(15, 5) =$	3 003
20	$C(14, 5) =$	2,002
19	$C(13, 5) =$	1,287
18	$C(12, 5) =$	792
17	$C(11, 5) =$	462
16	$C(10, 5) =$	252
15	$C(9, 5) =$	126
14	$C(8, 5) =$	56
13	$C(7, 5) =$	21
12	$C(6, 5) =$	6
11	$C(5, 5) =$	1
		1,344,904

We shall verify four of the entries of Table 1 by listing the combinations satisfying the "no consecutives" condition.

- L is 11. The only possible combination is 1, 3, 5, 7, 9, 11.
- L is 12. The six possible combinations are
  1, 3, 5, 7, 9, 12
  1, 3, 5, 7, 10, 12
  1, 3, 5, 8, 10, 12
  1, 3, 6, 8, 10, 12
  1, 4, 6, 8, 10, 12
  2, 4, 6, 8, 10, 12
- L is 13. The 21 possible combinations are
  1, 3, 5, 7, 9, 13
  1, 3, 5, 7, 10, 13
  1, 3, 5, 7, 11, 13
  1, 3, 5, 8, 10, 13
  1, 3, 5, 8, 11, 13
  1, 3, 5, 9, 11, 13
  1, 3, 6, 8, 10, 13
  1, 3, 6, 8, 11, 13
  1, 3, 6, 9, 11, 13
  1, 3, 7, 9, 11, 13
  1, 4, 6, 8, 10, 13
  1, 4, 6, 8, 11, 13
  1, 4, 6, 9, 11, 13
  1, 4, 7, 9, 11, 13
  1, 5, 7, 9, 11, 13
  2, 4, 6, 8, 10, 13
  2, 4, 6, 8, 11, 13
  2, 4, 6, 9, 11, 13
  2, 4, 7, 9, 11, 13
  2, 5, 7, 9, 11, 13
  3, 5, 7, 9, 11, 13
- L is 14. The 56 possible combinations are
  1, 3, 5, 7, 9, 14
  1, 3, 5, 7, 10, 14
  1, 3, 5, 7, 11, 14
  1, 3, 5, 7, 12, 14
  1, 3, 5, 8, 10, 14
  1, 3, 5, 8, 11, 14
  1, 3, 5, 8, 12, 14
  1, 3, 5, 9, 11, 14
  1, 3, 5, 9, 12, 14
  1, 3, 5, 10, 12, 14
  1, 3, 6, 8, 10, 14
  1, 3, 6, 8, 11, 14
  1, 3, 6, 8, 12, 14
  1, 3, 6, 9, 11, 14
  1, 3, 6, 9, 12, 14

1, 3, 6, 10, 12, 14
1, 3, 7, 9, 11, 14
1, 3, 7, 9, 12, 14
1, 3, 7, 10, 12, 14
1, 3, 8, 10, 12, 14
1, 4, 6, 8, 10, 14
1, 4, 6, 8, 11, 14
1, 4, 6, 8, 12, 14
1, 4, 6, 9, 11, 14
1, 4, 6, 9, 12, 14
1, 4, 6, 10, 12, 14
1, 4, 7, 9, 11, 14
1, 4, 7, 9, 12, 14
1, 4, 7, 10, 12, 14
1, 4, 8, 10, 12, 14
1, 5, 7, 9, 11, 14
1, 5, 7, 9, 12, 14
1, 5, 7, 10, 12, 14
1, 5, 8, 10, 12, 14
1, 6, 8, 10, 12, 14
2, 4, 6, 8, 10, 14
2, 4, 6, 8, 11, 14
2, 4, 6, 8, 12, 14
2, 4, 6, 9, 11, 14
2, 4, 6, 9, 12, 14
2, 4, 6, 10, 12, 14
2, 4, 7, 9, 11, 14
2, 4, 7, 9, 12, 14
2, 4, 7, 10, 12, 14
2, 4, 8, 10, 12, 14
2, 5, 7, 9, 11, 14
2, 5, 7, 9, 12, 14
2, 5, 7, 10, 12, 14
2, 5, 8, 10, 12, 14
2, 6, 8, 10, 12, 14
3, 5, 7, 9, 11, 14
3, 5, 7, 9, 12, 14
3, 5, 7, 10, 12, 14
3, 5, 8, 10, 12, 14
3, 6, 8, 10, 12, 14
4, 6, 8, 10, 12, 14

Recall that there are 3,262,623 possible winning combinations altogether, and that 1,344,904 of these combinations have no consecutive numbers. Therefore, 3,262,623 – 1,344,904 = 1,917,719 combinations have at least two consecutive numbers. The probability that at least two consecutive numbers will appear is then

$$\frac{1{,}917{,}719}{3{,}262{,}623} = 0.588 \approx 59 \text{ percent of the time.}$$

Cases in which the winning numbers contain at least two consecutive numbers should, therefore, be quite common.

To test this, we kept track of the local winning Lotto numbers during October, November and December 1991 and January 1992. For the 35 drawings that took place, 21 had at least two consecutive numbers. Thus in $\frac{21}{35}$ or $\frac{3}{5}$ or 60 percent of the time, "consecutives" occurred in the winning set of numbers. This is very close to the predicted 58.8 percent.

Challenges for readers and their students:

1. Redo the consecutive number problems for a Lotto game where you live or one which you play.
2. Verify your prediction which results from challenge problem 1 by checking actual Lotto results over a span of time.
3. For a given value of n (total number of numbers available), how large must r (number of numbers selected) be so that the probability of at least two consecutive numbers first exceeds $\frac{1}{2}$?
4. For a given value of r, how large must n be so that the probability of at least two consecutive numbers first exceeds $\frac{1}{2}$?
5. Use the computer to simulate problems of this type.

Checking Polynomial Arithmetic—Casting Out 9's Reincarnated

Murray L. Lauber

Casting out 9's is a centuries-old method for checking computations. With the introduction of calculators and computers, it has lost most of its value. But, as we shall see, it is not a completely antiquated procedure. A generalization of it can be used to check polynomial arithmetic, that is, computations involving adding, subtracting, multiplying and dividing polynomials. In addition, it has some affinities with a type of code currently used in detecting and correcting errors in messages transmitted by computers.

A Brief Review of Casting Out 9's

To understand the basis of casting out 9's, let us take a short excursion into the world of modular arithmetic. In the language and notation of modular arithmetic, an integer, a, is congruent to another integer, b, modulo n, n a counting number, if $a=b+kn$, or $a-b=kn$, where k is an integer. That is, a and b differ by a multiple of n. This is written $a \equiv b \pmod{n}$. For example, using $n=9$, we know that $33 \equiv 6 \pmod{9}$ because $33=6+(3*9)$. Another way of understanding this is to observe that 6 is the remainder when 33 is divided by 9. We will refer to the number less than 9 to which an integer is congruent modulo 9 as the mod 9 equivalent of the integer. For example, the mod 9 equivalent of 33 is 6. The process of finding the mod 9 equivalent of a number is appropriately referred to as casting out 9's because it subtracts a multiple of 9 from the number. For example, to find the mod 9 equivalent of 33, we subtract 3*9, or cast out three 9's, from 33.

Briefly, the method of checking computations with integers by casting out 9's is done as follows. Each number in the computation is changed to its mod 9 equivalent by casting out 9's to obtain an analogous problem modulo 9. In fact, computations can be checked as well by casting out 8's, 7's or any other counting number. For numbers represented in base 10, the particular beauty of casting out 9's is that it can be done simply by summing the digits of the numbers involved, recursively if necessary, to obtain single-digit numbers. Once the mod 9 equivalent of each of the numbers in the computation has been found, the computation is done using these single-digit numbers. The answer to this simple computation is then compared to the mod 9 equivalent of the answer to the original more complex computation, again using the method of summing digits. Some examples are given in Table 1.

A fuller exploration of casting out 9's is included in an article in *The Mathematics Teacher* (Lauber 1990). As noted in that article, casting out 9's has its analogues in other bases, for example, casting out 7's in base 8 arithmetic, casting out 4's in base 5 arithmetic, or, in general, casting out $(b-1)$'s in base b arithmetic. A variant of this general analogue, casting out $(x-1)$'s, is of particular use in checking polynomial arithmetic. We will focus specifically on integral polynomials, that is, polynomials with coefficients that are integers, but the analogue applies also to all real polynomials and even to those with complex coefficients.

What Is Casting Out $(x-1)$'s?

An integral polynomial in x may be thought of as a number base x. Consider, for example, the numbers 6347(base 10), 6347(base 8) and the polynomial $p(x) = 6x^3+3x^2+4x+7$. In expanded form, these may be written as follows.

6347(base 10) $= (6*10^3)+(3*10^2)+(4*10^1)+(7*10^0)$
6347(base 8) $= (6*8^3)+(3*8^2)+(4*8^1)+(7*8^0)$

In base 8, the number eight would be denoted by the digits 10. The digit 8 is used here to avoid confusion.

$p(x) = (6*x^3)+(3*x^2)+(4*x^1)+(7*x^0)$

Clearly, if $x = 10$, then $p(x)$ just becomes $(6*10^3)+(3*10^2)+(4*10^1)+(7*10^0)$ or 6347(base 10), and, if $x=8$, $p(x)$ becomes 6347(base 8). This illustrates the basic parallel between polynomials and numbers written in base 10 or some other base.

It should be clear, then, that casting out $(x-1)$'s from a polynomial in the variable x is the analogue to casting out 9's from a number in base 10. It should be apparent as well that the parallel in polynomial arithmetic to summing digits in base 10 arithmetic is summing the coefficients of the polynomials involved. To illustrate, consider the following examples.

Casting Out 9's

(a) By finding the remainder when dividing by 9:

6347 = (705*9)+2

(b) By summing digits recursively:

6347 → 6+3+4+7 = 20 → 2+0 = 2

Casting Out $(x-1)$'s

(a) By finding the remainder when dividing by $x-1$:

Synthetic division of $6x^3+3x^2+4x+7$ by $x-1$ yields a remainder of 20:

−1	6	3	4	7
	−0	−6	−9	−13
	6	9	13	20

(b) By summing the coefficients:

6+3+4+7 = 20

Table 2 gives three examples of checking computations with integral polynomials by casting out $(x-1)$'s, employing the method of summing the coefficients. The problem of addition of polynomials in (a) probably does not require further explanation. In (b), the product $(3x^3+2x^2-3x+6)(2x^2+7x-3)$ is checked. The sum of the coefficients of $3x^3+2x^2-3x+6$ is $3+2-3+6 = 8$. The sum of the coefficients of $2x^2+7x-3$ is $2+7-3 = 6$. The product of these sums, 8*6, is 48. The reader may check that the product of the two polynomials is $6x^5+25x^4-x^3-15x^2+51x-18$. The sum of the coefficients of this product is $6+25-1-15+51-18=48$. Thus the mod$(x-1)$ equivalent of the product of the two polynomials involved is equal to the product of the mod$(x-1)$ equivalents of the two polynomials. (It may be instructive to relate that, in my first attempt at finding this product, I quickly discovered, through casting out $(x-1)$'s, that the answer was incorrect.)

Checking division of polynomials is illustrated in (c) in Table 2. The reader may verify that when $3x^5-2x^4+x^3-4x^2+6x+9$ is divided by x^2-3x+7, the quotient is $3x^3+7x^2+15x-8$ and the remainder is $-75x+65$. This may be restated in terms of the division algorithm as follows.

$$3x^5-2x^4+x^3-4x^2+6x+9=$$
$$(3x^3+7x^2+15x-8)(x^2-3x+7)+(-75x+65).$$

The corresponding statement mod$(x-1)$, $13 = [(-39)*5] + 208$, is true. To ease computation, this statement could be further converted to its mod 9 equivalent using casting out 9's, but some care is needed in finding this mod 9 equivalent of −39 because it is negative. Its mod 9 equivalent may be found as follows: $-39 \rightarrow -(3+9) = -12 \rightarrow -(1+2) = -3$. But $-3 \equiv 6 \pmod 9$ because $-3=6+(-1)(9)$. The reader is invited to complete the construction of the mod 9 equivalent of the statement $13=[(-39)*5]+208$.

Why Casting Out $(x-1)$'s Works

There are several possible levels of mathematical justification for summing coefficients to check polynomial arithmetic. We will examine two of them. The first is direct, bypassing the ideas of modular arithmetic and casting out $(x-1)$'s. Consider, for example, the product of a third degree polynomial $p(x) = a_0+a_1x+a_2x^2+a_3x^3$ and a second degree polynomial $q(x) = b_0+b_1x+b_2x^2$. Then

$$p(x)q(x)=\{a_0+a_1x+a_2x^2+a_3x^3\}\{b_0+b_1x+b_2x^2\}$$

Using the commutative, associative and distributive properties, we obtain

$$\begin{aligned} p(x)q(x) &= \{a_0+a_1x+a_2x^2+a_3x^3\}b_0 \\ &\quad+\{a_0+a_1x+a_2x^2+a_3x^3\}b_1x \\ &\quad+\{a_0+a_1x+a_2x^2+a_3x^3\}b_2x^2 \\ &= \{a_0b_0+a_1b_0x+a_2b_0x^2+a_3b_0x^3\} \\ &\quad+\{a_0b_1x+a_1b_1x^2+a_2b_1x^3+a_3b_1x^4\} \\ &\quad+\{a_0b_2x^2+a_1b_2x^3+a_2b_2x^4+a_3b_2x^5\} \\ &= a_0b_0+\{a_1b_0+a_0b_1\}x+\{a_2b_0+a_1b_1+a_0b_2\}x^2 \\ &\quad+\{a_3b_0+a_2b_1+a_1b_2\}x^3 \\ &\quad+\{a_3b_1+a_2b_2\}x^4+a_3b_2x^5 \end{aligned}$$

The sum of the coefficients of $p(x)q(x)$ is

$$a_0b_0+(a_1b_0+a_0b_1)+(a_2b_0+a_1b_1+a_0b_2)$$
$$+(a_3b_0+a_2b_1+a_1b_2)+(a_3b_1+a_2b_2)+a_3b_2.$$

This sum is equal to $(a_0+a_1+a_2+a_3)(b_0+b_1+b_2)$, which is just the product of the sum of the coefficients of $p(x)$ and the sum of the coefficients of $q(x)$.

This demonstration that the sum of the coefficients of the product of $p(x)$ and $q(x)$ is equal to the product of the sums of their coefficients can be generalized to polynomials of any degree by induction. A similar argument can be constructed pertaining to the sum or difference of two polynomials. Thus the method of summing coefficients can be used to check polynomial arithmetic involving combinations of multiplication, addition and subtraction. Because by the division algorithm, division can be stated in terms of multiplication and addition, division of polynomials can also be checked by this method.

Another level of justification employs the notions of modular arithmetic. Its focus is initially on casting out $(x-1)$'s rather than summing coefficients. It is based on the following theorem:

Theorem: Let p(x) *and* q(x) *be polynomials with integral coefficients and* n *be a natural number. If* a, r *and* s *are integers such that* p(a) ≡ r(mod n) *and* q(a) ≡ s(mod n), *then*

(a) [p(a)+q(a)] ≡ (r+s)(mod n);

(b) p(a)q(a) ≡ *rs*(mod n).

A proof of (b) uses the definition of congruence modulo n given earlier along with the associative and distributive properties as follows:

$$p(a) \equiv r(\text{mod } n) \text{ and } q(a) \equiv s(\text{mod } n)$$
$$\Rightarrow \begin{cases} p(a)=r+kn \text{ for some integer } k, \text{ and} \\ q(a)=s+jn \text{ for some integer } j \end{cases}$$
$$\Rightarrow p(a)q(a)=(r+kn)(s+jn)$$
$$\Rightarrow p(a)q(a)=rs+rjn+kns+kjn^2$$
$$\Rightarrow p(a)q(a)=rs+(rj+ks+kjn)n$$
$$\Rightarrow p(a)q(a) \equiv rs(\text{mod } n)$$

The proof of part (a) is left to the reader.

This theorem may be restated in terms of remainders after dividing by n. Part (b), for example, says basically that the remainder when the product of $p(a)$ and $q(a)$ is divided by n is the same as the product of the remainders when $p(a)$ and $q(a)$ are each divided by n. That is, if we cast out n's before taking the product, we will get the same result as if we cast out n's after taking the product.

Putting $a=x$ and $n=x-1$ in the above theorem, and assuming x is an integer, justifies casting out $(x-1)$'s as a way of checking computations involving integral polynomials. What remains to be demonstrated is that casting out $(x-1)$'s from a polynomial, that is, finding the remainder when we divide it by $x-1$, is the same as summing its coefficients. This is quite easy to do. The remainder theorem guarantees that if a polynomial $p(x)$ is divided by $(x-1)$, then the remainder is $p(1)$. The only other observation needed is that $p(1)$ is just equal to the sum of the coefficients of $p(x)$. This follows because any power of 1 is just equal to 1. For example, if $p(x) = 3x^5-2x^4+x^3-4x^2+6x-9$, then $p(1) = 3-2+1-4+6-9$.

Each of these methods of justification has its own appeal. The latter is more general in one sense because it applies neatly to polynomials of any degree. But it has deficiencies as well. For example, it cannot be used to justify casting out $(x-1)$'s as a means of checking computations with nonintegral polynomials because, strictly speaking, the theory of modular arithmetic applies only to integral quantities. The first justification, though not so tidy, has the advantage of being generalizable to all real polynomials, and even to complex polynomials.

Polynomial Codes—An Outgrowth of Modular Arithmetic

Although it is too big a topic to cover in this article, it is of interest that the concept of quotient rings, a generalization of modular arithmetic, and employing some of the same basic notions as the procedure of casting out $(x-1)$'s, forms the basis for powerful error correcting/detecting codes for computer messages. There are some parallels, as well as some differences, between polynomial codes and casting out $(x-1)$'s. Polynomial codes do not have the capability of checking computations, but they are capable of detecting and correcting errors in the bits of "words" transmitted in computer messages. Polynomial codes are more powerful than casting out $(x-1)$'s in that they are capable of correcting as well as detecting errors. They are also of more practical significance because they serve a larger role in our technological society. For brief descriptions of the nature, along with some examples, of polynomial codes and other error detecting/correcting codes, refer to Laufer (1984, 1–61, 476–85), Lax (1991, 209–64), Biggs (1989, 375–98) and Gersting (1987, 339–67).

Conclusion

Casting out $(x-1)$'s has a lot of potential as a subtopic of polynomials in the high school mathematics curriculum. It could be a very useful tool in the mathematical repertoires of high school and college students because of its usefulness in checking polynomial arithmetic. It has the potential to illustrate a variety of methods of mathematical justification. It has value, as well, for illustrating how the centuries-old method of checking computations, casting out 9's, though now mostly obsolete because of calculators and computers, can be generalized into the still-useful tool of casting out $(x-1)$'s. In addition, it has significant affinities with the still-more-general notion of quotient rings that form the basis of polynomial codes. Perhaps through a brief study of casting out $(x-1)$'s, students' curiosity could be inspired to research the polynomial codes concept and thus begin to explore some notions usually reserved for college or university courses in abstract algebra. The progressive levels of generalization, from casting out 9's to casting out $(x-1)$'s to polynomial codes, could be used to illustrate how mathematics has developed historically, from particular to general, and to demonstrate the increasing power that accompanies the movement to higher levels of generalization.

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Biggs, N. L. *Discrete Mathematics. Rev. ed. New York: Oxford University Press, 1989.*

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Lauber, M. L. "Casting Out Nines—An Explanation and Extensions." *The Mathematics Teacher* (November 1990).

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Table 1 Examples of Casting Out 9's

	Problem	Summing Digits Recursively			Analogous Problem Modulo 9
a)	164	1+6+4 = 11	1+1 = 2		2
	x 27	2+7 = 9	0		x 0
	4428	4+4+2+8 = 18	1+8 = 9	0	0
b)	4389	4+3+8+9 = 24	2+4 = 6		6
	+ 2186	2+1+8+6 = 17	1+7 = 8		+ 8
	6575	6+5+7+5 = 23	2+3 = 5	5 = 1+4	14
c)	23.98	2+3+9+8 = 22	2+2 = 4		4
	x 4.31	4+3+1 = 8			x 8
	103.3538	1+0+3+3+5+3+8 = 23	2+3 = 5	5 = 3+2	32

Table 2 Examples of Casting Out $(x-1)$'s

	Problem	Sum of Coefficients	Analogous Problem Modulo $(x-1)$
a)	$4x^4 - 2x^3 + 10x^2 - 6x + 3$	$4 - 2 + 10 - 6 + 3 = 9$	9
	$+ \quad 7x^3 - 8x^2 - 2x + 6$	$7 - 8 - 2 + 6 = 3$	$+ 3$
	$4x^4 + 5x^3 + 2x^2 - 8x + 9$	$4 + 5 + 2 - 8 + 9 = 12$	12
b)	$3x^3 + 2x^2 - 3x + 6$	$3 + 2 - 3 + 6 = 8$	8
	$\times \quad 2x^2 + 7x - 3$	$2 + 7 - 3 = 6$	x 6
	$6x^5 + 25x^4 - x^3 - 15x^2 + 51x - 18$	$6 + 25 - 1 - 15 + 51 - 18 = 48$	48
c)	Dividend: $3x^5 - 2x^4 + x^3 - 4x^2 + 6x + 9$	$3 - 2 + 1 - 4 + 6 + 9 = 13$	13
	Divisor: $\div \quad x^2 - 3x + 7$	$1 - 3 + 7 = 5$	$\div$ 5
	Quotient: $3x^3 + 7x^2 + x - 50$	$3 + 7 + 1 - 50 = -39$	-39
	Remainder: $-151x + 359$	$-151 + 359 = 208$	208

OR

$$3x^5 - 2x^4 + x^3 - 4x^2 + 6x + 9 = (3x^3 + 7x^2 + x - 50)(x^2 - 3x + 7) + (-151x + 359)$$

is analogous to

$$13 = (-39)(5) + 208$$

[which is true]

Addition and Subtraction Puzzle

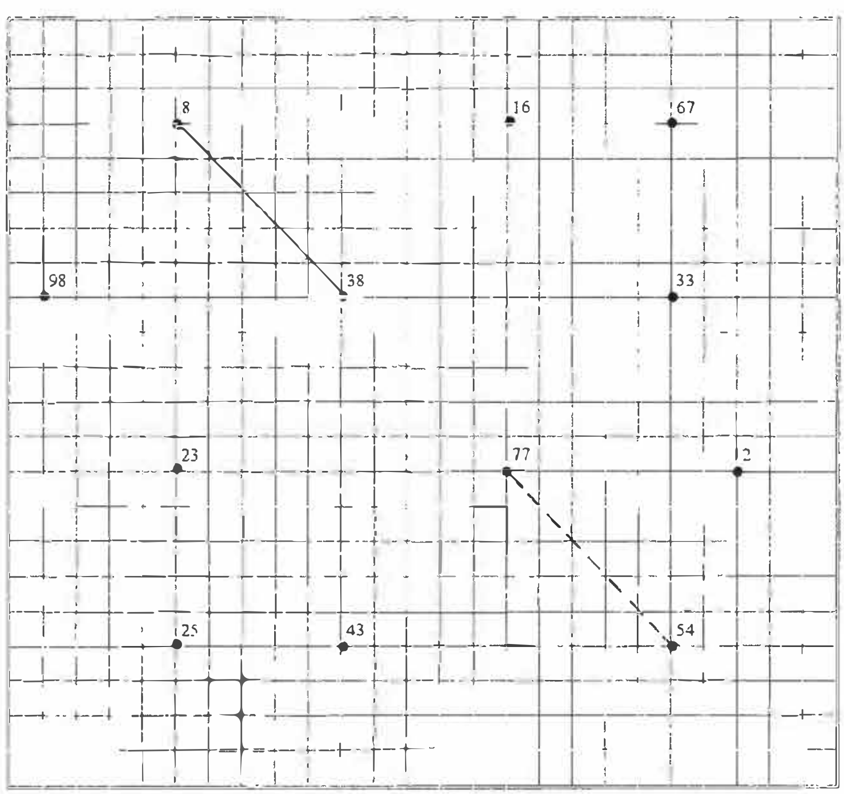
K. Allen Neufeld

1. On the grid below, 12 points are labeled with numbers. Join with a solid line pairs of points whose numbers are the first two members in each equation below. The first one has been done for you. The points labeled 38 and 8 have been joined with a solid line.

   a. 38 +8=46 b. __ +38=71 c. __+23=66
   d. __+16=24 e. __ +54=87 f. __+8=31
   g. __+43=81 h. __ +33=49 i. __+43=97

2. Next, join with a broken line pairs of points whose numbers are the first two members in each equation below. The first one has been done for you. The points labeled 77 and 54 have been joined with a broken line.

   a. 77 –54=23 b. __ –23=54 c. __ –16=61



3. If all instructions were correctly followed, you will have sketched a *regular hexahedron*. What is another name you might give to this figure? How many faces does it have? How many vertices?

Stocking Up in Mathematics: An Application

A. Craig Loewen

As mathematics teachers, we are often asked, When are we ever going to use this? The question is most disturbing because it implies that our students do not find our classes relevant and are probably frustrated and bored. But it is a fair question. As mathematics teachers, we must address questions regarding the relevancy of our subject discipline, that is, describe where mathematics is used. After all, mathematics did not just appear in a text one day. It was developed over thousands of years as people sought to provide descriptions and predictions of the world around them: mathematics is a process of generalization and an act of summarizing and describing the universe and the events within it. But how do we reflect this vision of mathematics to our students? One way to achieve an increased sense of relevance is through implementing activities, games and examples known as applications.

What Is an Application?

An application may be defined as any event, activity, description, problem or demonstration that illustrates how a mathematical property or definition occurs or is employed in an alternative context such as the *real world*. This definition has two important elements: (1) an application must address an identifiable mathematical concept, and (2) an application must show where mathematical concepts are found or used in a context outside of the classroom, preferably in the lives of our students. Simple examples of applications include the path that a football travels after kickoff (a parabolic path) and the application of estimation principles to predict our total at the grocery counter (no one likes to be caught short at the checkout.)

The strength of a true application lies in the fact that it provides a sense of genuine relevance to the study of mathematics, unlike the three traditional forms of relevance described by Haylock et al. (1985): artificial relevance, long-term relevance and vicarious relevance.

Artifical relevance is achieved through reference to some character or event that will capture the interest of the student. Implementing artificial relevance often provides an age-appropriate context to a problem. For example, the following problem employs artificial relevance:

> Each of the New Kids on the Block ate 7/8 of a pizza.
>
> How many whole pizzas were consumed?

In presenting this problem to students, it is hoped that reference to a particular music group will capture the students' interest and thus motivate the students to solve it.

Long-term relevance is achieved through a claim of usefulness in a future event or situation. For example, the teacher who responds to the student who asks, When am I going to use this? with You will use it on the exam next week! or You better know this when you get Miss Smith next year! is employing long-term relevance.

Finally, *vicarious relevance* is achieved through reference to another person who uses that principle or idea, or to another event in which the idea is employed. For example, if the teacher states: Farmers construct ratios like these when mixing herbicides and pesticides, then the teacher is employing vicarious relevance. Though vicarious relevance does provide a link to the real world, it remains somewhat unsatisfying to the student unless he or she *is* a farmer mixing herbicides and pesticides. None of these traditional forms of relevance constitutes genuine relevance. What do we mean by genuine relevance?

Genuine relevance is achieved through a perceived need for a given item of knowledge or process given that the lack of this knowledge or process serves as an impediment to some desired goal. For example, if a student wants to calculate the percentage he or she needs on the final exam to get a given grade in a course, then a lack of understanding of ratios, proportions and percents may impede the desired goal. The inherent difficulty with applications that express genuine relevance is that they are based on a *perceived individual* need, making them very difficult to develop and implement on a class-wide basis.

So, where does this leave the teacher? The teacher ultimately has three means to encourage and support a sense of relevance in the classroom:

1. Recognize and employ *teachable moments*. Teachable moments occur when students recognize they lack an important mathematical skill or item of information and thus are motivated to develop that skill or construct the needed knowledge.
2. Consistently employ *subject matter integration*. To integrate subject disciplines, the teacher should plan for and point out mathematical principles and properties whenever they appear during instruction in other disciplines such as science, physical education and social studies.
3. Involve the students in activities that *employ challenging decision making contexts* not unlike those found in the real world. The following activity is provided as an example of an application for the junior high mathematics classroom.

Stocking Up Game

This game is a simulation of the stock market where students work cooperatively in teams to construct a company, buy and sell a stock, and predict market trends so as to maximize profits. In this game, students are divided into groups of four. One student is the president of the company, one student is the stock analyst and two students serve as members of the board of directors.

Objectives

- Uses paper-and-pencil algorithms, estimation and calculators to perform computations.
- Records data in line graphs and interprets data from line graphs.
- Understnads and uses the terms *probability* and *chance*.
- Generalizes the probability of the occurrence of an event from a practical situation.

Materials

- Each team/company will require one copy of the Stocking Up chart (Figure 1), one six-sided die, one or more calculators.
- The teacher will require different spinners for each of the five trading sessions (Figure 2).

Procedure

- To begin the game, each team must select a company name and designate the following roles:
    1. The *president* is responsible for announcing the decisions of the company and for monitoring the company's cash and stocks.
    2. The *members of the board of directors* are responsible for determining market trends, assisting the president in all decisions, and keeping account of the company's cash and stocks.
    3. The *stock analyst* is responsible for charting the stock's progress through the five trading sessions (using the chart shown in Figure 1).
- Each company begins the game with $20,000. At the beginning of the game, each company is given a chance to purchase as many stocks as they would like at $50 per stock.
- On a turn, a company may elect to do one of (a) spin, (b) buy stocks or (c) sell stocks.
- If a company elects to spin, the teacher spins the appropriate spinner as shown in Figure 2, and the president of the company rolls a die. The spinner marked with a '1' should be used throughout the first trading session, while the spinner marked with a '2' should be used throughout the second trading session and so on. The result of the spin determines whether the value of the stock will go up or down while the result of the roll determines the amount the stock will change. Note: Spinners for the five trading sessions are hidden from the companies until the conclusion of the game.
- If a company elects to buy stocks, it must purchase the stocks at their current value. The number of stocks a company may purchase is determined by the company's president and board of directors. The number of stocks purchased is limited by the remaining cash.
- If a company elects to sell stocks, the stocks are converted to cash at their current stock market

value. The number of stocks sold is determined by the company's president and board of directors. The number of stocks sold is limited by the number of stocks owned.

- The turn continues to rotate between companies until all five trading sessions are completed.

Rules

- If a company cannot come to a quick consensus on a turn (as to whether the company will spin, buy or sell), the president will decide. All decisions of a president are final.
- If on any turn the value of the stocks reaches 0, all stocks for each company are lost. The stock is reset to a value of 50. Companies are given a chance to purchase new stocks with their remaining cash before play continues.
- If on any turn the value of stocks reaches 100, all stocks for each company are doubled. The stock value is reset to 50 and play continues.

Declaring the Winner

- At the end of the five trading sessions, all stocks are converted to cash. The company with the greatest cash total is the winning team.

Questions

- What trends in the stock value did you notice in each of the five trading sessions? Describe what might be found on the spinner used in each of the five sessions.
- Which of the spinners is likely to drive the value of the stocks up? Which of the spinners is likely to drive the value of the stocks down? Which of the spinners is likely to have the least effect on the value of the stocks? Explain.
- Which of the spinners express the same probability of driving the stock up or driving the stock down?
- If you knew what each of the spinners looked like before you started the game, how would you change your strategy? At what times in the game would you buy or sell to maximize your profits?
- Knowing what is found on each of the spinners, do the trends in any of the sessions surprise you? Why or why not?
- Create your own spinners the next time the game is played. Some spinners should drive the value of the stocks up, and some should drive the value of the stocks down.
- Create two different spinners that, on average, would simply maintain the value of the stock.

Adaptations

- Have students roll more than one die at a time—this adaptation will greatly increase fluctuation in the stock value.
- Introduce a second stock. When a company elects to spin, both stock values are changed using the process described above.
- Change the rules so that a team must spin on each turn. After spinning, the team may elect to buy, sell or pass the turn—this adaptation will shorten the time needed to play the game.

Suggestions to the Teacher

- Keep track of each company's stocks and cash on the blackboard, and assist students in making computations (at least initially).
- Help the students chart the first few fluctuations in stock value, and check with students regularly to ensure consistency of stock value between companies.

Reference

Haylock, D.W., et al. "Using Maths to Make Things Happen." *Mathematics in School* 14, no. 2 (1985): 32–34.

Figure 1

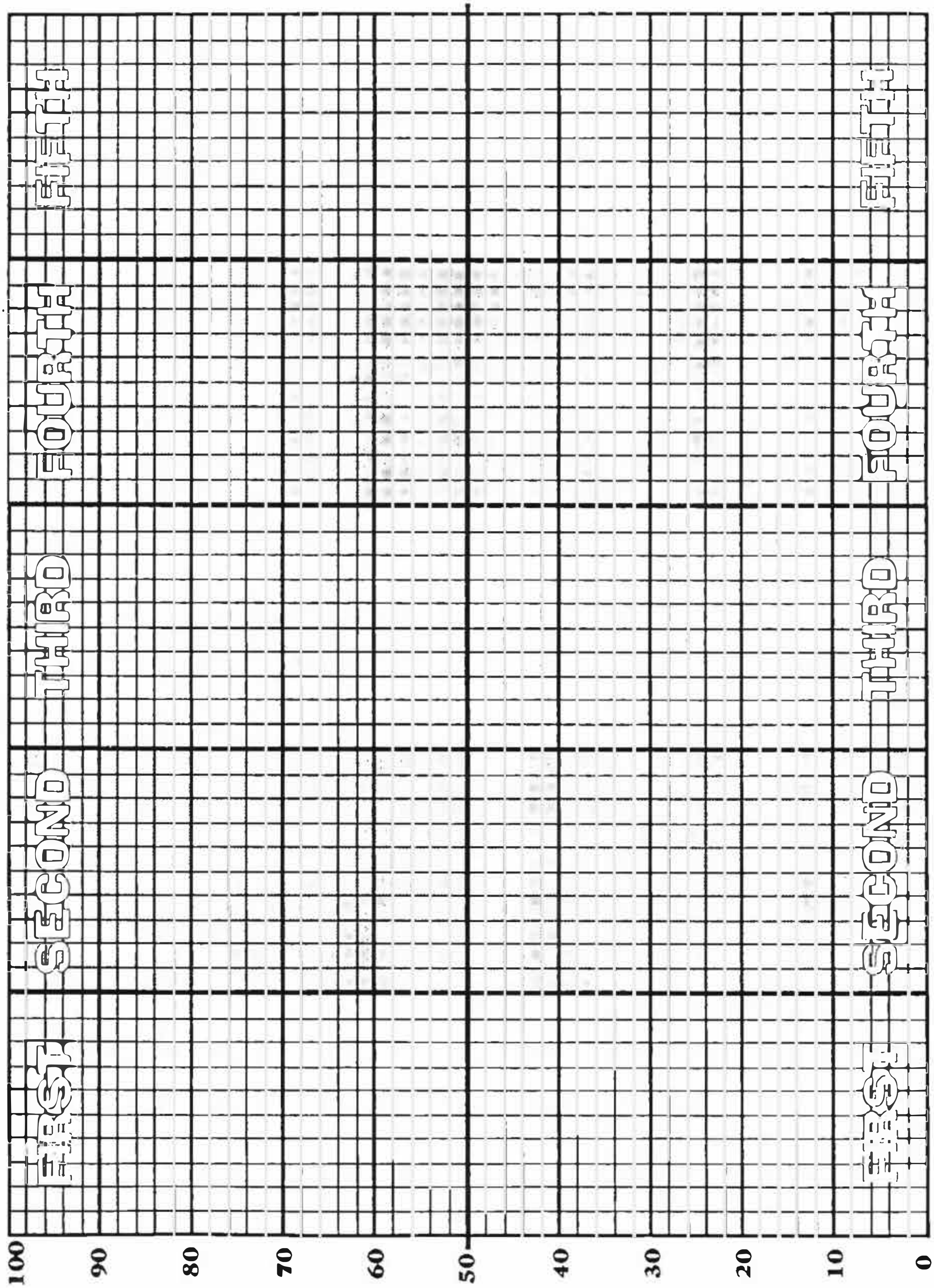
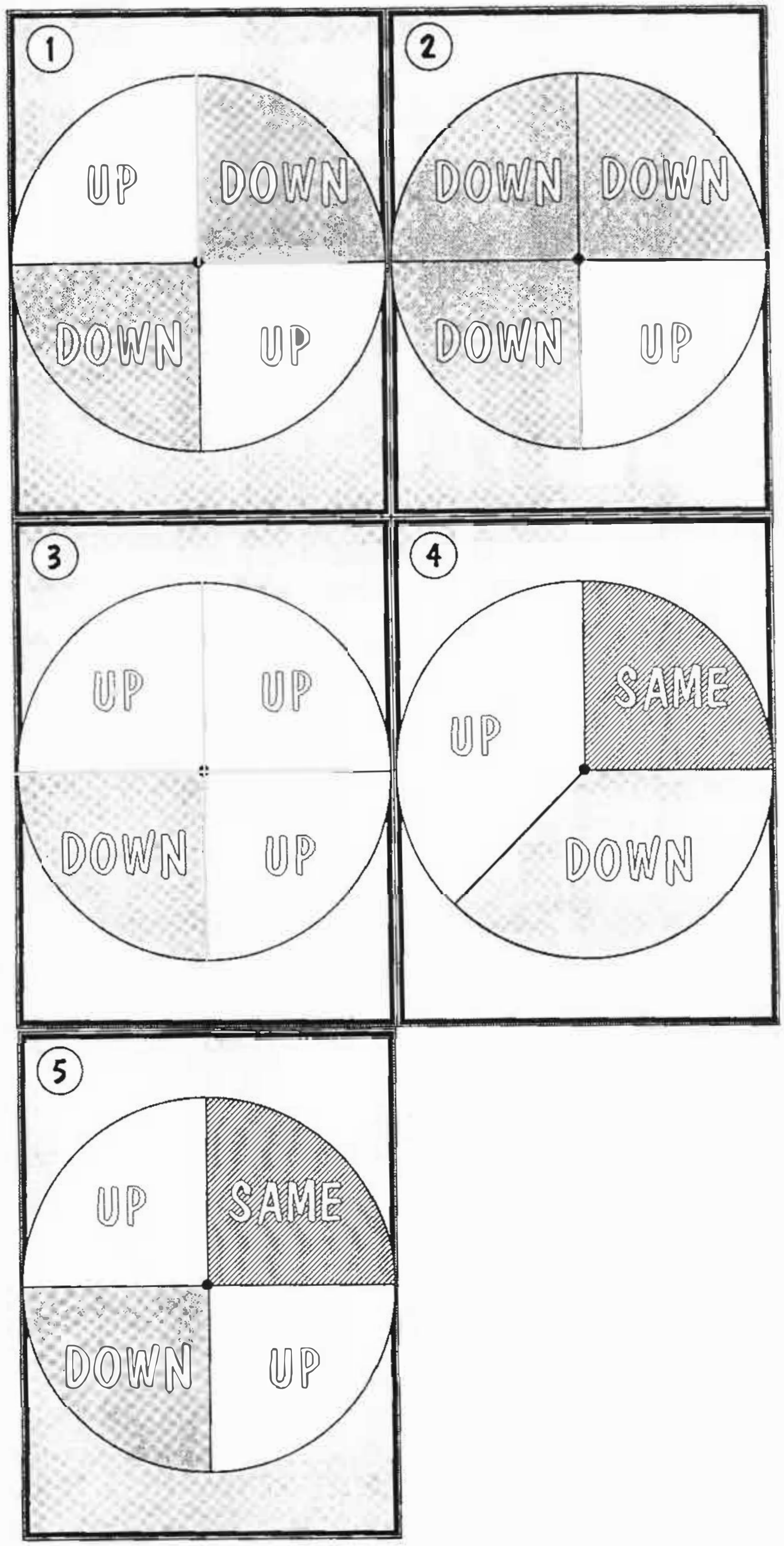


Figure 2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