Counting Card Combinations

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A poker player Jeanne notices that most of her five-card hands contain at least one heart. What is the probability that this result occurs; that is, that a given poker hand contains at least one heart?

We shall solve this problem in two ways. Each method uses the fact that the total number of possible five-card poker hands is

$$C(52,5) = \frac{52!}{5!47!} = 2{,}598{,}960.$$

Each of these approximately 2.6 million poker hands is equally likely to be dealt to Jeanne.

Method 1

We will count directly the number of hands having exactly one, two, three, four and five hearts.

Exactly One Heart

This type of hand would contain one heart and four cards chosen from the remaining 39. The number of ways in which the single heart can be chosen is $C(13,1) = 13$, while the number of ways in which the four non-hearts can be chosen is $C(39,4) = 82{,}351$.

By the Fundamental Principle of Counting, the number of poker hands that contain exactly one heart is then $(13)(82{,}251) = 1{,}069{,}263$.

Exactly Two Hearts

The number of ways of selecting the two hearts is $C(13,2) = 78$ while the number of ways of selecting the three non-hearts is $C(39,3) = 9{,}139$. Thus the total number of hands of this type is $(78)(9{,}139) = 712{,}842$.

Exactly Three Hearts

The number of ways of selecting the three hearts is $C(13,3) = 286$.

The number of ways of selecting the two non-hearts is $C(39,2) = 741$.

The total number of hands of this type is $(286)(741) = 211{,}926$.

Exactly Four Hearts

The number of ways of selecting the four hearts is $C(13,4) = 715$.

The number of ways of selecting the non-hearts is $C(39,1) = 39$.

The total number of hands of this type is $(39)(715) = 27{,}885$.

Exactly Five Hearts

The total number of ways of selecting five hearts and zero non-hearts is $C(13,5) \bullet C(39,0) = (1287)(1) = 1287$.

Altogether, the number of poker hands that contain at least one heart is $1{,}069{,}263 + 712{,}842 + 211{,}926 + 27{,}885 + 1{,}287 = 2{,}023{,}203$. The probability that Jeanne is dealt a hand containing at least one heart is then $\frac{2{,}023{,}204}{2{,}598{,}960} \approx 77.8\%$.

Method 2

The calculations of Method 1 involved many steps. A shorter and more elegant approach is to first consider the number of five-card poker hands that contain no hearts. Once this number is determined, the remaining hands can be found by subtracting from 2,598,960.

The number of poker hands containing *no* hearts is determined as follows:

1. Select the five cards from the 39 non-hearts; this can be done in C(39,5) = 575,757 ways.
2. Select the empty set from the 13 hearts. This can be done in (13,0) = 1 way.
3. The product of these two combinatorial results is 575,757; this is the total number of hands that contain no hearts.

Recall that there are 2,598,960 possible poker hands. If 575,757 hands contain no hearts, then the other hands must contain at least one heart. Therefore, 2,598,960 - 575,757 or 2,023,203 poker hands contain hearts. This is the same result as achieved by Method 1.

Note that exactly the same methods would apply if any of the other three suits (clubs, diamonds, spades) had been used instead of hearts. This leads to the following results:

P (at least one heart) = 77.8%
P (at least one diamond) = 77.8%
P (at least one club) = 77.8%
P (at least one spade) = 77.8%

Because one of these four cases must occur, why is the sum of the four probabilities not equal to 100 percent? This question could lead to an interesting class discussion. The teacher can reemphasize that probabilities can be summed only if the events in question are mutually exclusive.

Challenges for the reader and his/her students:

1. Redo the problem of this article (finding the probability that a poker hand contains at least one card from a certain category) for the categories of kings, face cards and red cards.
2. Redo the original problem (at least one heart) and the three additional problems of challenge 1 for a 13-card bridge hand. The reader should be aware that the size of the numbers involved in the computations may cause their calculators to round and thus yield only approximate answers. Alternative computing algorithms may be used to eliminate this rounding problem.