Subtracting Infinities

Sandra M. Pulver

Consider the example $\int_{-1}^{2} \frac{dx}{x^3}$, a plain and simple integral on the face of it. But this integral is improper because the integrand is discontinuous on the interior of the interval, that is, at $x = 0$. So, to solve, we must break it up.

$$\int_{-1}^{2} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}$$

$$= \lim_{b \to 0^-} \int_{-1}^{b} x^{-3}\,dx + \lim_{b \to 0^+} \int_{b}^{2} x^{-3}\,dx$$

$$= \lim_{b \to 0^-} \left.\frac{x^{-2}}{-2}\right]_{-1}^{b} + \lim_{b \to 0^+} \left.\frac{x^{-2}}{-2}\right]_{b}^{2}$$

$$= \lim_{b \to 0^-} \left.\frac{-1}{2x^2}\right]_{-1}^{b} + \lim_{b \to 0^+} \left.\frac{-1}{2x^2}\right]_{b}^{2}$$

$$= \lim_{b \to 0^-} \left(\frac{-1}{2b^2} + \frac{1}{2(1)}\right) + \lim_{b \to 0^+} \left(\frac{-1}{2(4)} + \frac{1}{2b^2}\right)$$

$$= \frac{1}{(-2)(-\text{very small})^2} + \frac{1}{2} \qquad + \text{ second part}$$

$$= \frac{1}{(-2)(-\text{very very small})^2} + \frac{1}{2} \qquad + \text{ second part}$$

$$= -\infty + \frac{1}{2} \qquad + \text{ second part}$$

and the limit of the first part does not exist. So, this integral diverges and we are finished.

But look what can happen when students use $-\infty$ as a concrete limit and *do* go further.

Working on the second part of the limit, students obtain

$$+\left(-\frac{1}{8}\right) + \lim_{b \to 0^+} \frac{1}{2b^2} =$$

$$+\left(-\frac{1}{8}\right) + \frac{1}{2(+\text{very very small})} =$$

$$-\frac{1}{8} + \infty$$

Students then proceed further, saying that the limit of the whole example is therefore $= -\infty + \frac{1}{2} - \frac{1}{8} + \infty$.

Now the infinities "cancel" each other, leaving $+\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$ as a final result!

Here is an excellent time and place to explain to a student, once again, the concepts of an infinite limit. If a limit is found to be infinite, this means that the limit simply does not exist, and to continue to the second part of the example is foolish. The limit of the whole example cannot exist if the limit of the first part does not exist!