Generating Geometric Sets

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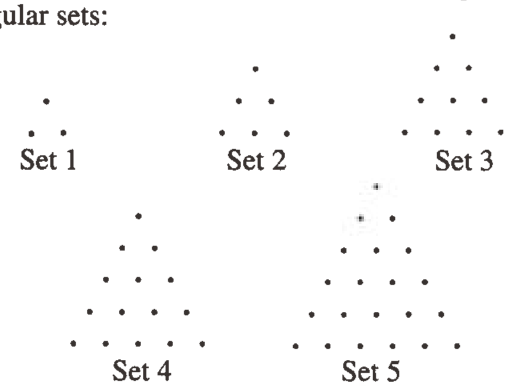
Geometric shapes can be used to represent numbers. For example, consider the following five triangular sets:



For each of these triangular sets, three types of points will be counted:

1. The points on the set boundary
2. The points in the set interior
3. The points in the entire set

Table 1 reports the numbers of points in these categories:

Table 1

Triangular Set	**1**	**2**	**3**	**4**	**5**
Points on boundary	3	6	9	12	15
Points in interior	0	0	1	3	6
Total points in triangular set	3	6	10	15	21

Square sets and hexagonal sets can also be pictured and counted in the same ways as for triangular sets.

Square Sets

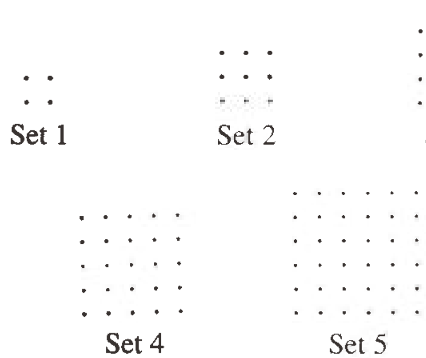


Table 2

Square Set	**1**	**2**	**3**	**4**	**5**
Points on boundary	4	8	12	16	20
Points in interior	0	1	4	9	16
Total points in square set	4	9	16	25	36

Hexagonal Sets

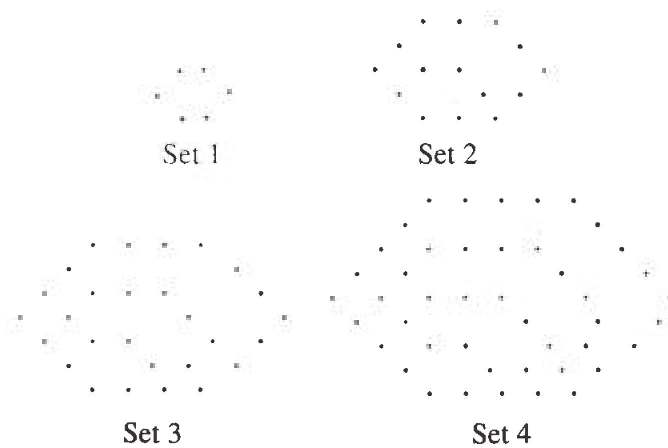


Table 3

Hexagonal Set	**1**	**2**	**3**	**4**
Points on boundary	6	12	18	24
Points in interior	0	3	10	21
Total points in hexagonal set	6	15	28	45

A study of Tables 1, 2 and 3 suggests a variety of patterns. The most obvious of these patterns is that the *boundary* totals for triangular, square and hexagonal sets are consecutive natural number multiples of 3, 4 and 6, respectively. Other patterns are seen on extended Tables 1, 2 and 3; Table 3 has also been augmented with the last row of Table 1:

Table 1 (Extended)

Triangular Set	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Points on boundary	3	6	9	12	15	18	21	24	27
Points in interior	0	0	1	3	6	10	15	21	28
Total points in triangular set	3	6	10	15	21	28	36	45	55

The circled numbers are *three* columns apart.

Table 2 (Extended)

Square Set	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Points on boundary	4	8	12	16	20	24	28	32	36
Points in interior	0	1	4	9	16	25	36	49	64
Total points in square set	4	9	16	25	36	49	64	81	100

The circled numbers are *two* columns apart.

Table 3 (Extended)

Set Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Points on boundary of hexagonal set	6	12	18	24	30	36	42	48	54
Points in interior of hexagonal set	0	3	[illegible]	21	36	55	78	105	136
Total points in hexagonal set	6	15	28	45	66	91	120	153	190
Total points in triangular set	3	6	10	15	21	28	36	45	55

The numbers of interior points (beginning with 3) are *every other* triangle set total (3, 6, 10, 15, 21, 28, 36, 45, 55, . . .).

Remarkably, the three patterns (triangular, square and hexagonal) that relate interior and total points are all different. The first two (triangles and squares) were similar in that they involved only numbers of the type being studied, while the hexagonal pattern involved hexagonal and triangular sets.

The patterns which have been described can be communicated by the use of notation.

Let T_n, S_n and H_n be the total number of points in the nth triangular, square and hexagonal sets; TI_n, SI_n and HI_n be the number of interior points in the nth triangular, square and hexagonal sets; TB_n, SB_n and HB_n be the number of boundary points in the nth triangular, square and hexagonal sets.

The patterns are as follows:

$TB_n = 3n,\ SB_n = 4n,\ HB_n = 6n$

$TI_n = T_{n-3}$ (for $n \geq 4$)

$SI_n = S_{n-2}$ (for $n \geq 3$)

$HI_n = T_{2n-3}$ (for $n \geq 2$)

(for example, for $n = 4$, $HI_4 = 21 = T_{2(4)-3} = T_5$)

Additional patterns can also be deduced, based on the preceding patterns and general observation that, for each figure, the boundary and the interior numbers sum to the total numbers in the sets:

$TB_n + TI_n = T_n$

Thus,

$TB_n = T_n - TI_n$

$3n = T_n - T_{n-3}$ (for $n \geq 4$)

$SB_n + SI_n = S_n$

$SB_n = S_n - SI_n$

$4n = S_n - S_{n-2}$ (for $n \geq 3$)

$HB_n + HI_n = H_n$

$HB_n = H_n - HI_n$

$6n = H_n - T_{2n-3}$ (for $n \geq 2$)

Challenges

1. Construct other polygonal sets (for example, pentagonal, heptagonal and octagonal).
2. Justify the patterns of this article by writing proofs.
3. How could the meanings of T_n, S_n and H_n be extended so that the results are valid if any integer value of n is used?