Children Solve Mathematical Problems with Multiple Solutions

K. Allen Neufeld

Problem solving has received a major focus in elementary mathematics curricula during the '80s, and emphasis on this major area will grow as curricula are planned for the '90s. More than ever, elementary students need to be equipped with the ability to solve the various problems they encounter in school as well as outside the classroom. "Good problem solvers possess a broad understanding of the concepts involved in a problem" (Irons and Irons 1989).

Attitudes to problem solving are also of considerable importance. "Many students have rather intense effective reactions to mathematical problem solving" (McLeod 1988). Unless teachers establish a positive climate, children will be reluctant to meet the challenge of the unknown. "The problem solver needs enthusiasm to proceed with the solution. Enthusiasm signifies the willingness to accept a challenge or set one's own challenge" (House, Wallace and Johnson 1983).

Teachers can play a significant role promoting problem-solving skills. Many students are the products of earlier curricula when problem solving didn't receive the emphasis it does today. Also, many students had elementary teachers who themselves had negative reactions to the solution of all but the most routine of problems.

> In my work with teacher-interns, I encounter very few who are willing or able to take this approach at the outset. Even though they are college graduates, many of them math majors, they, too, search for the infallible equation and are just as afraid as their students will be of appearing foolish. Teacher educators, then, have to model and encourage general problem solving behavior. (Noddings 1989)

Problems with Multiple Solutions

Many education students encountered standard-textbook problems when they were in elementary school. Often all the problems of a particular set focused on a recently studied computational process featuring the use of a particular equation format. Students were seldom challenged to solve process problems. "A process problem is one that cannot be simply solved by translating the one or more number sentences. Process problems emphasize the thinking processes for obtaining problem solutions" (Charles and Lester 1984).

A positive attitude toward problem solving is important for teachers *and* their pupils. Of equal importance is the willingness to accept a challenge. Pupils who are reluctant to solve problems may be encouraged by the fact that several possible solutions exist, and that finding even one of them is rewarding. Other pupils are challenged to find all the solutions, and they feel rewarded when they use a strategy that will ensure complete success.

> Situations that allow students to experience problems with "messy" solutions or too much or not enough information or that have multiple solutions, each with different consequences, will better prepare them to solve problems they are likely to encounter in their daily lives. (Romberg 1989)

> Problems that have more than one solution tend to foster a problem-posing mindset because they are not as limited as one-answer problems. (Moses, Bjork and Goldenberg 1990)

Children will find mathematics rewarding and challenging only if their teachers are equipped to meet the needs of incorporating excellent problems in their lesson plans. "Because excellent problem-solving questions are seldom created 'on the spot,' teachers will benefit from writing lesson plans that include questions they can use at crucial moments" (Cemen 1989). Of equal importance is the teacher's understanding of the importance of problem solving in the mathematics curriculum.

> If a teacher conceives of mathematics solely in terms of speed, accuracy and one right answer, then it is unlikely that such a teacher will stimulate students to monitor their solution processes,

estimate answers, search for alternate solution methods, pose problems, or engage in similar worthwhile activities. (Grouws and Good 1989)

The factor of multiple solutions is related to the strategy factor. Problems having only one solution may or may not be more difficult than problems having multiple solutions. However, if a student is asked to find all solutions, a natural question is "How can I be sure that I have all solutions?" The effect of multiple solutions on the difficulty of the problem needs to be further researched. (LeBlanc, Proudfit and Putt 1980)

Example Problems and Extensions

In this section, three problems will be presented, one featuring each of the pictorial, symbolic and manipulative modes. Following the examination of the several solutions for each of the problems, ways to revise the problems will be suggested. Solution patterns will also be included for the revised problems.

1. Find all the squares, each of a different area, that can be made on a 4 by 4 (25-dot region) area.

Diagram 1 shows the eight possible solutions if one is restricted to a whole-number solution for each area. The dotted lines within some of the squares are included as an aid to verify the area.

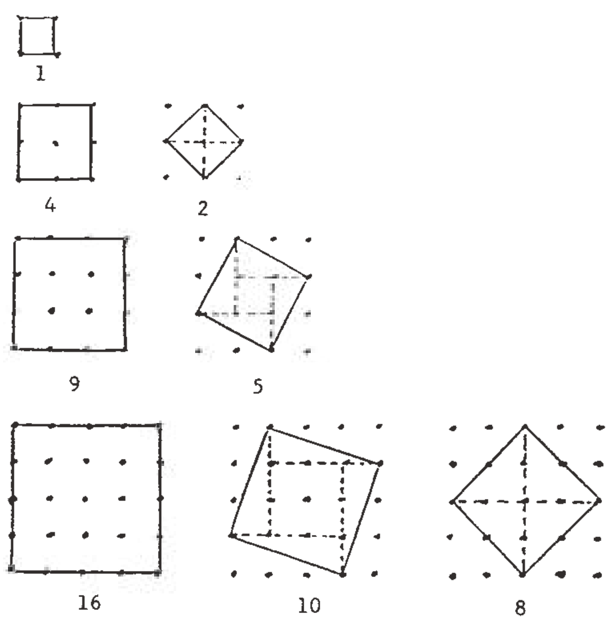

Diagram 1

If the initial problem was revised to feature a 3 by 3 array, the solutions would be indicated by the top five squares in Diagram 1. Similarly, a 2 by 2 array would yield the top three squares in Diagram 1. Diagram 2 shows the seven additional squares that can be made on a 6 by 6 array. A 5 by 5 array would generate all the squares in Diagram 1 as well as the top three squares in Diagram 2.

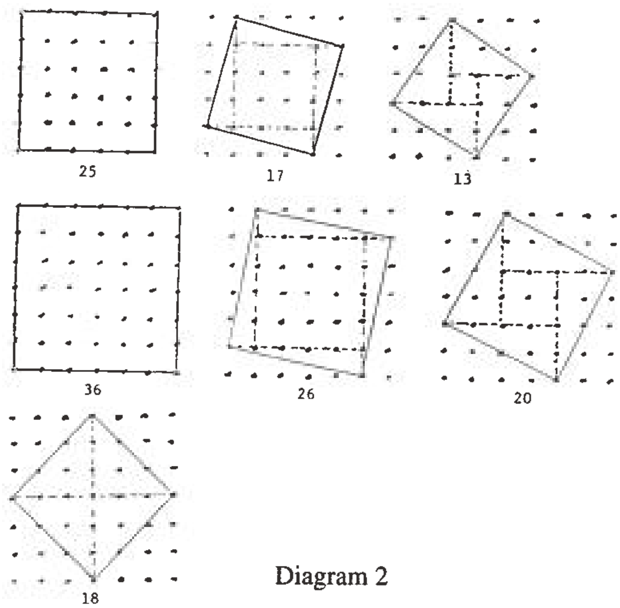

Diagram 2

The following table provides a numerical summary of all the squares that can be made on arrays of ever-increasing dimensions. Students can be encouraged to look for patterns and to predict the number and types of squares for arrays of larger dimensions.

Size of array	**Number of squares**	**Total number of squares**
1 x 1	1	1
2 x 2	1 + 2	3
3 x 3	3 + 2	5
4 x 4	5 + 3	8
5 x 5	8 + 3	11
6 x 6	11 + 4	15
7 x 7	15 + 4	19
8 x 8	19 + 5	24
9 x 9	24 + 5	29
10 x 10	29 + 6	35

2. Find all the ways that bicycles, tricycles and wagons can be made from 20 wheels if there is at least one of each kind and no wheels are left over.

The following chart shows the four possible solutions. The suggested strategy is to begin with one wagon (the vehicle with the largest number of wheels) and to consider the required number of each of the other vehicles to reach the final total of 20 wheels.

Bicycles	Tricycles	Wagons
5	2	1
2	4	1
3	2	2
1	2	3

A simple way to revise this problem would be to change the total number of wheels. The following table shows the possible solutions for numbers of wheels from 16 to 23. The problem featuring 16 wheels has only two solutions, while the problem with 23 wheels has 8, the largest number of solutions.

Wheels	Bicycles	Tricycles	Wagons	Vehicles
16	3	2	1	6
16	1	2	2	5
17	5	1	1	7
17	2	3	1	6
17	3	1	2	6
18	4	2	1	7
18	1	4	1	6
18	2	2	2	6
19	6	1	1	8
19	3	1	1	7
19	4	1	2	7
19	1	3	2	6
19	2	1	3	6
20	5	2	1	8
20	2	4	1	7
20	3	2	2	7
20	1	2	3	6
21	7	1	1	9
21	4	3	1	8
21	1	5	1	7
21	5	1	2	8
21	3	1	3	7
22	6	2	1	9
22	3	4	1	8
22	4	2	2	8
22	2	2	3	7
23	8	1	1	10
23	5	3	1	9
23	2	5	1	8
23	6	1	2	9
23	3	3	2	8
23	4	1	3	8
23	1	3	3	7
23	2	1	4	7

Another variation could be made in the kinds of vehicles chosen. What might be the results if each vehicle had an odd number of wheels (for example, unicycles, tricycles and five-wheeled pentikes)? Students should also be encouraged to look for patterns in the numerical charts.

3. Make a square with the two largest tangram pieces. Find how many ways a square of the same size can be made with other arrangements of tangram pieces. Sketch each way by drawing lines around the chosen sets of pieces. Each way must use a different set of pieces.

Students attempting this problem would benefit by having a sheet of paper containing a number of congruent squares, the size of each to accommodate the square made by the two largest tangram pieces. Diagram 3 shows the five possible congruent squares.

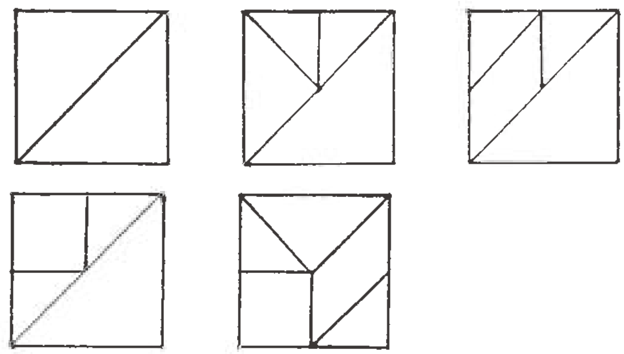

Diagram 3

A variation of this problem would be to ask students to use tangram pieces to make all the different right-angled isosceles triangles. Diagram 4 shows the 12 possible triangles. Interestingly, five different sizes of these triangles exist. Children would probably benefit by having a sheet of paper containing outlines of the 12 triangles.

An extension of either version of the tangram problem featuring the symbolic mode could be accomplished by having students draw each of the five squares and the twelve right-angled isosceles triangles on squared paper. Suppose that the measure of the edges of each square is four units and the measures of the triangles are as follows:

A	hypotenuse	4 units
B—D	equal edges	4 units
E—I	hypotenuse	8 units
J—K	equal edges	6 units
L	equal edges	8 units

Students can then calculate the areas of the squares and triangles as well as the areas of the individual tangram pieces in each illustration. These measures could then be compared with the measures of the actual plastic tangram pieces and combinations.

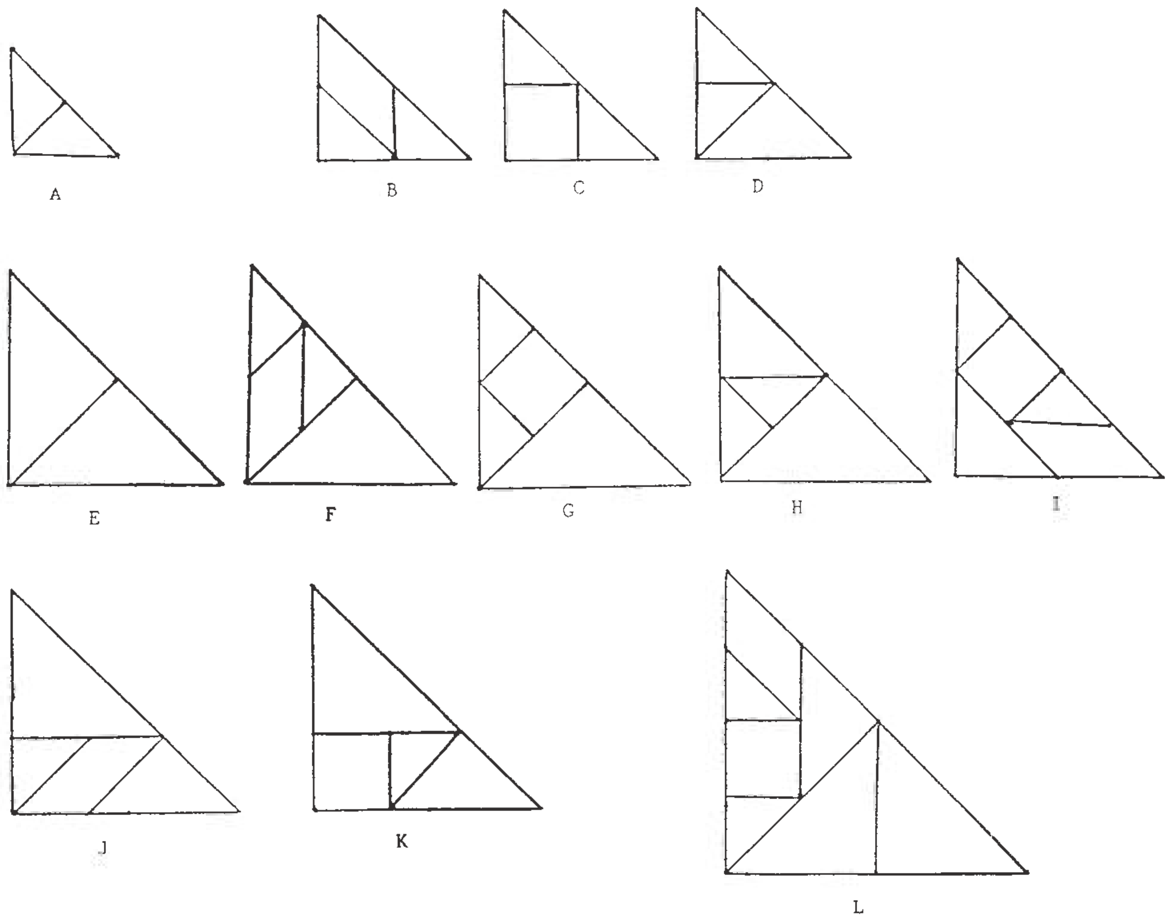

Diagram 4

References

Cemen, P. "Developing a Problem-Solving Lesson." *The Arithmetic Teacher* 37, no. 2 (October 1989).

Charles, R., and F. Lester. "An Evaluation of a Process-Oriented Instructional Program in Mathematical Problem Solving in Grades 5 and 7." *Journal for Research in Mathematics Education* 15, no. 1 (January 1984).

Grouws, D., and T. Good. "Issues in Problem Solving Instruction." *The Arithmetic Teacher* 36, no. 8 (April 1989).

House, P.A., M.L. Wallace and M. A. Johnson. "Problem Solving as a Focus: How? When? Whose Responsibility?" In *The Agenda in Action*, 1983 Yearbook of the National Council of Teachers of Mathematics, edited by G. Shufelt and J.R. Smart. Reston, Va.: NCTM, 1983.

Irons, R., and C. Irons. "Language Experiences: A Base for Problem Solving." In *New Directions for Elementary School Mathematics*, 1989 Yearbook of the National Council of Teachers of Mathematics, edited by P. Trafton and A. Shulte. Reston, Va.: NCTM, 1989.

LeBlanc, J., L. Proudfit and I. Putt. "Teaching Problem Solving in the Elementary School." In *Problem Solving in School Mathematics*, 1980 Yearbook of the National Council of Teachers of Mathematics, edited by S. Krulik and R. Reys. Reston, Va.: NCTM, 1980.

McLeod, D. "Affective Issues in Mathematical Problem Solving: Some Theoretical Considerations." *Journal for Research in Mathematics Education* 19, no. 2 (March 1988).

Moses, B., E. Bjork and E. Goldenberg. "Beyond Problem Solving: Problem Posing." In *Teaching and Learning Mathematics in the 1990's*, 1990 Yearbook of the National Council of Teachers of Mathematics, edited by T. Cooney and C. Hirsch. Reston, Va.: NCTM, 1990.

Noddings, N. "Preparing Teachers to Teach Mathematical Problem Solving." In *The Teaching and Assessing of Mathematical Problem Solving*, Research Agenda for Mathematics Education Volume 3, edited by R. Charles and E. Silver. Reston, Va.: NCTM, 1989.

Romberg, T. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.