Playing with *c x (1 - x)*

J. Dale Burnett

I have had fun playing with the function *c x (1 - x)*. The activities have at least a three-fold nature: one is to explore some characteristics of this function; another is to demonstrate various computer software packages useful for facilitating such explorations; and finally there are the educational issues involving what occurs and does not occur in school classrooms.

Exploration of the function *c x (1 - x)* breaks into two components: one examines the mathematics, the other develops an appreciation of the contexts where one might expect to find such a function. The contexts also split into parts: science, mathematics and education; similarly with the exploration: graphing, algebra and dynamic properties are all of interest. This splitting, and splitting again, is itself a metaphor for this article and for the ways in which one might conceive knowledge. Part of the intent is to make this metaphor explicit, arguing that such explicitness is an important pedagogical principle. Let's share the secrets with our students. The investigation is divided into three sections. The first section examines the function using a variety of computer software. The second section looks at the same function from a biological perspective, and the third section builds on some features that were noted during the study of population dynamics. This latter section opens up an entire new universe of mathematical topics.

I emphasize that this is written not about a thorough analysis of this function. Rather it is about some suggestions for mathematics.

Section 1 Quadratic Functions

Let's begin with some simple algebra and graphing.

Let $y = c\,x\,(1 - x)$

What does the graph of this equation look like?

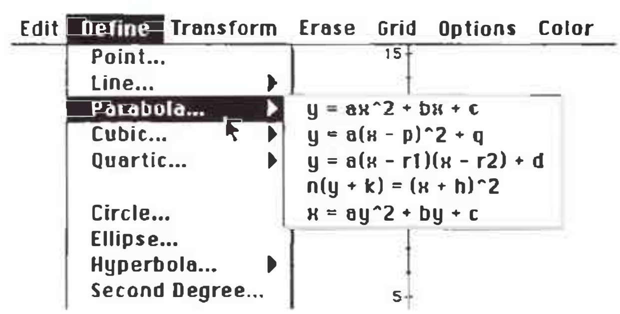
Even this question can immediately lead to alternative conceptions and approaches. One approach is to simply find out. Thus, at least to begin with, one doesn't even try to imagine what the graph might be. Let's just let the technology show us.

The most likely activity for many students and teachers would involve use of a graphing calculator. Fair enough. Numerous workshops have been offered and many articles written about the use of such devices in mathematics classrooms. I will explore other possibilities, possibilities that use more powerful tools. These tools already exist, but their use in classrooms is relatively rare. I am referring to the laptop computer, supplemented by sophisticated software packages. I will use four such software packages in the remainder of this paper: Zap-a-Graph, Microsoft Excel, Mathematica and STELLA.

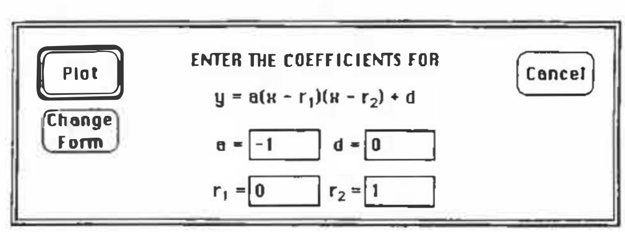
Zap-a-Graph

Zap-a-Graph is a software package for graphing mathematical functions. It is relatively easy to use since usually the user need only change the values of a set of predefined parameters. Here is an example using our equation.

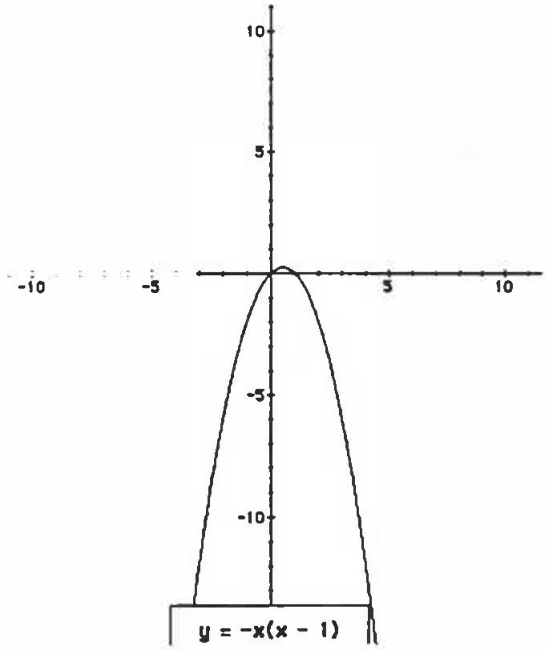
Using the pull-down menu for Zap-A-Graph gives us the following choices:



The third choice corresponds to the form *c x (1 - x)*, although even this recognition requires a certain level of symbol sophistication. Here is the dialogue box after I select that choice, with the values that I require now entered into the appropriate boxes.



Selecting the Plot option results in the following display:



Another approach is to play a bit with the algebra of the equation and see what develops. Multiplying the factors gives

$y = cx - cx^2$

A quick diversion. The above consideration of $y = c\,x\,(1 - x)$ and $y = cx - cx^2$ indicates that different canonical forms can serve different purposes. The first form is useful for noting the roots of the equation. Clearly, y is 0 when x is 0 and 1. The second form indicates the polynomial nature of the expression, emphasizing in this case that we are considering a polynomial of degree 2. Are there other forms that have other interesting properties? Note the different forms used by Zap-a-Graph in the various pull-down menus. It is interesting to speculate on what might happen if a teacher asked a class to examine the different forms provided by Zap-A-Graph and recommend when each form might be used.

Microsoft Excel

In addition to using graphing software, another approach is to think of using spreadsheets to explore mathematical topics. Spreadsheets may be used to obtain tables and graphs for a function. Using Excel, for example, one can quickly set up the necessary formulas. We will begin by setting $c = 1$ (a simple first choice—the value is stored in cell A2) and let x vary from -10 to +10 (another simple choice). Let's see what happens. Here are the cell formulas:

Worksheet1

	A	B	C	D	E
1	c	x	cx	cx^2	y = cx - cx^2
2	1				
3		-10	=$A$2*B3	=C3*B3	=C3-D3
4		=B3+1			

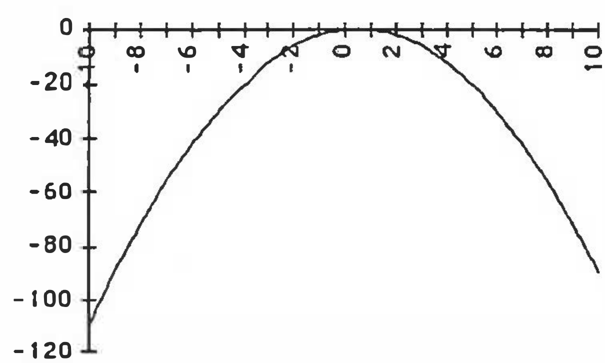
Using the fill-down command, one quickly obtains these values:

Worksheet1

	A	B	C	D	E
1	c	x	cx	cx^2	y = cx - cx^2
2	1				
3		-10	-10	100	-110
4		-9	-9	81	-90
5		-8	-8	64	-72
6		-7	-7	49	-56
7		-6	-6	36	-42
8		-5	-5	25	-30
9		-4	-4	16	-20
10		-3	-3	9	-12
11		-2	-2	4	-6
12		-1	-1	1	-2
13		0	0	0	0
14		1	1	1	0
15		2	2	4	-2
16		3	3	9	-6
17		4	4	16	-12
18		5	5	25	-20
19		6	6	36	-30
20		7	7	49	-42
21		8	8	64	-56
22		9	9	81	-72
23		10	10	100	-90

This provides a tabular representation of the function, and it deserves examination because it shows the arithmetic detail of the function. A discussion of the relative merits of the equation, the table and the graph might prove interesting because they are simply different representations of the same idea.

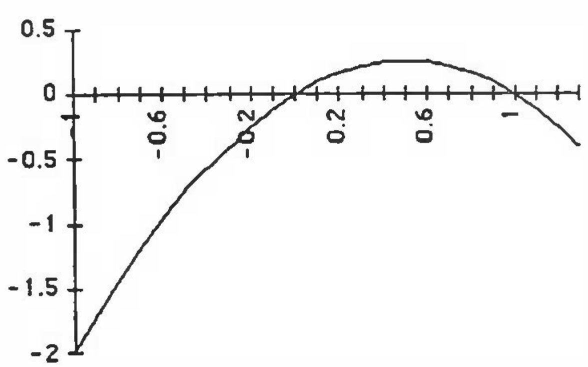
It is relatively easy to obtain the corresponding graph:



Yes, it is a parabola. It is always reassuring to get the same result using different approaches. Much of science has progressed using this principle, and it is a critical feature of most work involving computers. One should always ask, "How do I know that the result that the computer has displayed is correct?"

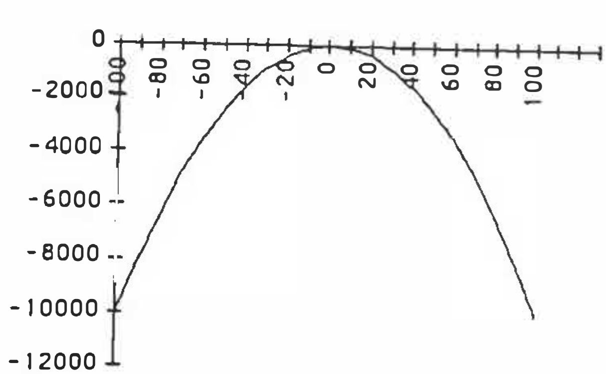
Obtaining the same result using different software packages goes a long way toward providing a satisfactory answer to the question. This is rarely emphasized in school where, quite often, only one software package is available.

One could pose many questions at this point. One such question could involve obtaining a clearer picture of what happens near the x-axis. Ideally, at least from my perspective, the students will learn to ask most of these questions themselves, with the teacher acting more as a gentle catalyst and as a resource. Notice how the emphasis shifts from getting answers to asking questions.



One might also wonder, what are the roots of this function? A close look at the graph suggests that these would be $x = 0$ and $x = 1$. These both check by substituting back into the original equation. It is not always clear why one might want to know these values, except that we have a formula for obtaining them and thus feel that we should use it whenever we can. Why do we care what the roots are?

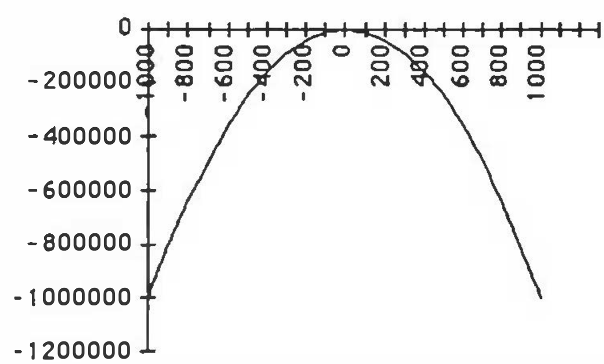
We can also explore the behavior of the function for a wider range of x values. Let's try this. For openers, let's look at the graph when x varies from -100 to +100 in steps of 10.



The table gives more accurate values:

	A	B	C	D	E
1	c	x	cx	cx^2	y = cx - cx^2
2	1				
3		-100	-100	10000	-10100
4		-90	-90	8100	-8190
5		-80	-80	6400	-6480
6		-70	-70	4900	-4970
7		-60	-60	3600	-3660
8		-50	-50	2500	-2550
9		-40	-40	1600	-1640
10		-30	-30	900	-930
11		-20	-20	400	-420
12		-10	-10	100	-110
13		0	0	0	0
14		10	10	100	-90
15		20	20	400	-380
16		30	30	900	-870
17		40	40	1600	-1560
18		50	50	2500	-2450
19		60	60	3600	-3540
20		70	70	4900	-4830
21		80	80	6400	-6320
22		90	90	8100	-8010
23		100	100	10000	-9900

Let's try -1,000 to +1,000 (in steps of 100).



Yet appearances can be deceiving (Goldenberg 1988). The scale of both axes has been changing. All graphs should be thought of as being drawn on rubber. The "shape" of the graph is in large part a property of the scales of the two axes. That is, the equation determines the "essential" features (for example, a parabola opening down), but the shape of this figure, in terms of how narrow or wide it is, is a property of the scale of the axes.

Another issue that deserves early mention is notational conventions. The idea of a standard, to be used worldwide by all mathematicians, has had a seductive lure to it. Teachers of mathematics, and their students, have also wished for such a standard. We are not even close to such an ideal. For example, there is not even widespread agreement on how to write the numerals: zero, one and seven are written differently in many European countries from the conventions of North America. Early computers exacerbated the difficulties. It was common to see the asterisk (*) to represent multiplication and ** to represent exponentiation because there was no provision for superscripting. Spreadsheets added more difficulties, because their notational conventions, while making sense within a spreadsheet environment, represented further departure from textbook and handwritten standards. Some view this as a weakness. I prefer to think of it as a strength (Burnett 1987).

We should become comfortable with a variety of conventions. As more software tools appear, the number of ways that we decide to represent commands, procedures and ideas will increase. In the past, different conventions were the result of different cultural factors and evolved on a time scale consistent with change in that culture. This is still true today, but now culture also includes the effects of technology, which not only places new demands on conventions but also has accelerated the time scale for accommodating these changes. One of our tasks as educators is to help students gain flexibility in handling different symbol systems. Another task is to help them deal with rapidly changing situations.

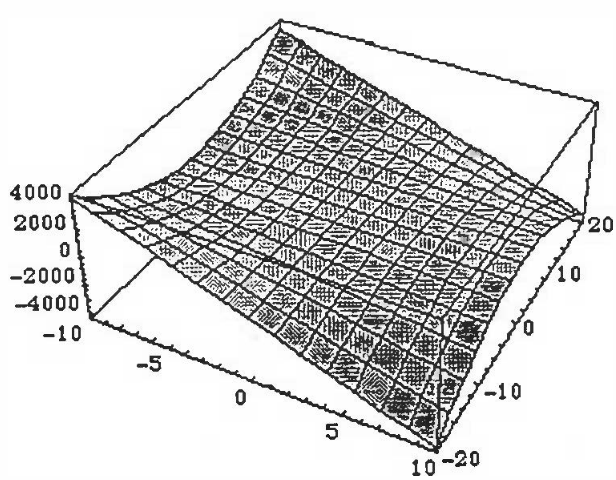
Mathematica

Recall that we have been looking at the equation $y = c\,x\,(1 - x)$. So far, we have restricted ourselves to the special case of $c = 1$.

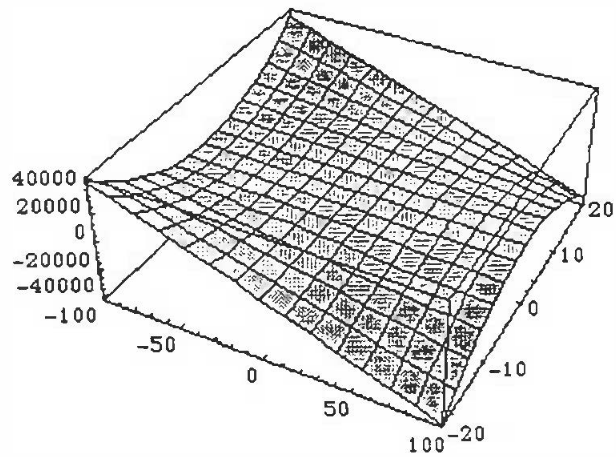
In the terminology of mathematics, x (and y) are called variables, c is called a parameter. What is the difference between a parameter and a variable? They both vary.

Here are some graphs of $y = c\,x\,(1 - x)$ where y is the vertical axis, c is the axis along the left side of the base and x is the axis along the right side of the base.

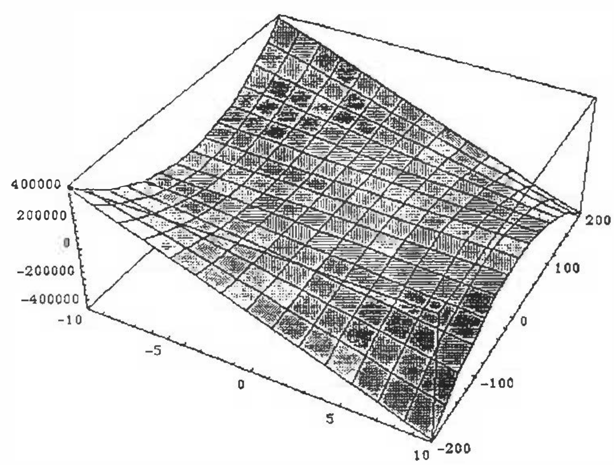
For the first figure, c varies from -10 to +10 and x varies from -20 to +20:



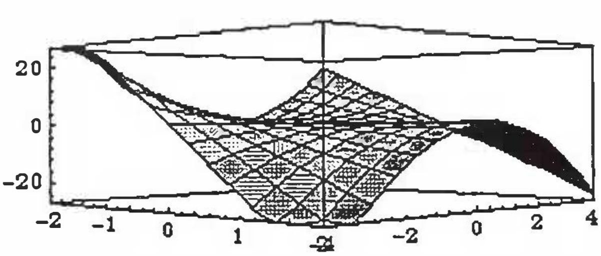
Now let's look at the case where c varies from -100 to +100:



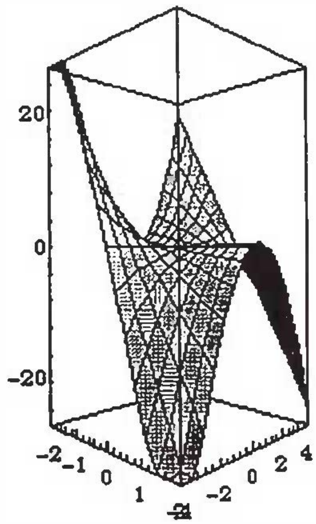
In the next figure, c varies from -10 to +10 and x varies from -200 to +200:



Now imagine that we are at "eye level" and that c varies from -2 to +2 and x varies from -4 to +4:



Here is the same figure with the axes drawn to a different scale:



Essentially what we see in each figure is an infinite number of parabolas, some opening down (when c is positive) and some opening up (when c is negative). Each previous parabola drawn using Zap-a-Graph or Excel is simply one possible vertical "slice" from the surface. The three-dimensional representation is a much more comprehensive and powerful way of envisioning this mathematical situation, namely as a surface rather than as a curve. Technology makes this possible.

What do we see? Open up, open down, roots, steepness, curvature. Let's play with the notions of steepness and curvature for a bit: calculus.

What is the derivative of this surface? What does it look like? One can ask Mathematica to compute the derivative of an algebraic expression. If we define a function F by typing in the following expression:

F[x_]: = c x (1 - x)

then type

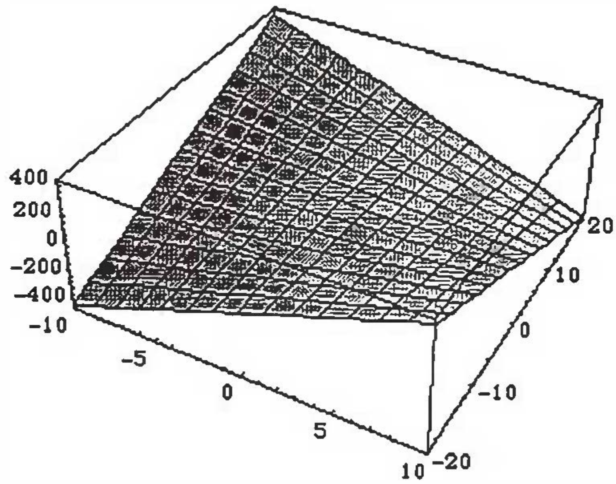
D[F[x],x]

we receive the following result:

c (1 - x) - c x

Once again, we must use a slightly different set of notational conventions for specifying a function. These conventions are necessary because one must take a different set of factors into account when designing a software environment from those that one must consider when using pencil and paper.

We can then plot this function, obtaining the following:



In this case, c varies from -10 to +10 and x from -20 to +20. What do you see? Does this figure make sense?

The above activities give the complete picture for the equation $y = c\,x\,(1 - x)$ for all possible values of x and c. It is also easy to imagine rotating this figure dynamically, such as many CAD programs illustrate.

Let's stop to catch our breath. What has happened so far?

First, we have used the technology to do all the labor. It has drawn the graphs and even computed the derivative.

Second, the technology has permitted us to think at a higher level of abstraction. Instead of thinking about a single quadratic expression, we have examined a continuous family of possibilities, moving from a consideration of two-dimensional graphs to three-dimensional graphs.

Third, to use the different software packages, we have had to learn the notational conventions of each.

Fourth, the emphasis has been on visualization, particularly "deep visualization" with its strong emphasis on understanding. There has been no attention to manual skills. Mathematics education must include activities that help us to "see," to understand and interpret what we see. *Envisioning Information* (Tufte 1990) exemplifies this concern for meaningful representation. This emphasis represents a significant departure from the current curriculum that still devotes a major amount of effort to the development of manual skills, skills that may no longer be necessary (Burnett 1992).

But technology is not the answer. It is a tool for exploring ideas. The tool still requires a human operator. Exploration requires an active mind. What does one "see" when one views the above surfaces? What new questions come to mind?

Mathematics is about asking questions. Learning is about asking questions. Have you asked any good questions lately?

Now that we are rested, let's explore this equation in a couple of other ways. One is to note its use in nonmathematical contexts, particularly biology and ecology. Students often ask, "Why are we studying this?" Section 2 provides an introductory answer to this question for the case of the function $f(x) = c\,x\,(1 - x)$. Another approach is explore the mathematics of this equation in a different light, the light of iteration. Such explorations require technology to provide the labor-intensive calculations. Section 3 provides an entry into a whole new world of mathematics, one currently of interest to many of the world's leading mathematicians.

Section 2 Biology

I remember reading that biology was the study of dead things. It was a cynical attack on science education, but one that had an aspect of truth to it. Perhaps there is some comfort in the realization that other parts of the school curriculum are also under pressure to change. What many school students fail to realize is that the study of biology also involves the study of artificial things, things that were never living or dead (Levy 1992). Enter the world of mathematical modeling and simulation.

A sound strategy for entering a new field is to begin with something simple. Let's consider the population dynamics of a single species, for example, rabbits.

Let $P(n)$ represent the population of rabbits at time n. Clearly, the actual value of $P(n)$ depends on many factors: weather, availability of food supply, number of predators, disease and so forth. It also depends on the number of rabbits around at time $n - 1$. A very simple first approximation to this situation is the following function:

$$P(n) = c\,P(n - 1)$$

Notationally, this expression can be a bit confusing: do the brackets symbolize the argument of the function P, or multiplication? Thus it is usual to switch to a form of subscript notation. Notational conventions are not just a function of technology, they have always been with us:

$$P_n = c\,P_{n-1}$$

At the same time, it is a good idea to change the meaning of P from a whole number representing the total number of rabbits to a proportion of some arbitrary large upper limit of rabbits. Thus P now takes values between 0 and 1. Proportions are often easier to deal with because we all know what a proportion of 0.8 means (that is, 80 percent of the maximum possible population), whereas a value of 3 million rabbits still leaves a person with a sense of "So what?" Is that a large number, a small number or a typical number? This same principle applies to many figures we receive when the media discuss the economy and our present preoccupation with budgets. A department is asked to trim $2 billion from its budget. Is this a large number, a small number or a reasonable number?

If c is 1, the population remains constant, since $P_n = P_{n-1}$.

If c is larger than 1, it is fairly easy to see that the population grows without bound. This may make sense for a while, but, at some point, other factors such as a dwindling food supply should start to kick in. The model does not seem realistic. Indeed, at some point, the proportions will exceed a value of 1, which is meaningless. If c lies between 0 and 1, the population continues to dwindle and eventually will be fractional. Negative values of c make no sense in this context. Thus the model needs to be adjusted to prevent the case of runaway growth. We need a term that reduces the growth as P gets large. One approach is to add a second multiplicative term, $(1 - P)$:

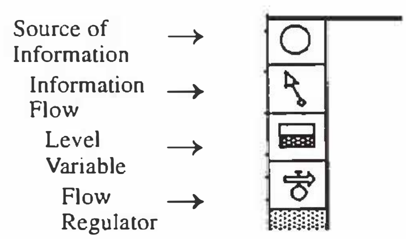
$$P_n = c\,P_{n-1}\,(1 - P_{n-1})$$

This is the same equation that we have been discussing in the preceding sections.

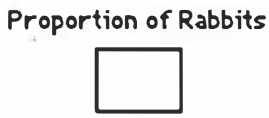
We have approached the question of simple population dynamics from an algebraic perspective. We might just as easily have approached it from a geometric perspective. What we are looking for is a graph that increases for a while and then decreases. One such curve is a parabola that opens downward. Once again, we can end up with the same equation.

Let's look at a third approach using a simulation modeling package called STELLA II. STELLA is an acronym for Systems Thinking, Experiential Learning Laboratory, with Animation. It takes a particular approach to modeling, known as system dynamics. The original Club of Rome report (Meadows et al. 1972), one of the first documents to warn us of the dangers of unlimited industrial expansion, used this approach. The current version of STELLA uses a small set of icons that can be placed anywhere on the screen and joined to other icons to create a flow model.

Two of the four icons represent movement of information; the other two represent movement of some conceptual quantity.



First, we create a level variable (like a bathtub) that contains the level of proportion of rabbits. We simply select the Level Variable icon, drag it onto the screen and type a name to give the icon an identifying label.

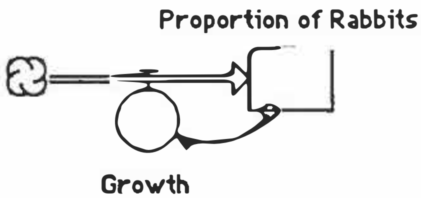


It is simply a box that will show how full it is over time. Opening the box (by double clicking on it) brings up a dialogue box where the user specifies the initial value for this level. I will start with it half full:

INITIAL(Proportion_of_Rabbits) = ...

.5

Next, we need to specify the rate of increase in rabbits. This is accomplished by means of a flow indicator:

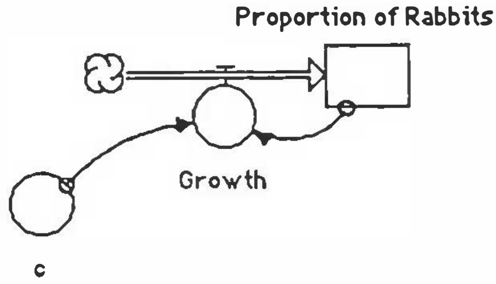


The icons are intended to represent a tap (regulator) that controls the rate of growth of the rabbits. The "cloud" symbol on the left simply means that there is a source of rabbits to begin with. The feedback connector arrow joining the level indicator to the tap is the heart of the model. This permits one to use the current level as a variable controling the tap for the next cycle. Opening the tap calls up another dialogue box where the equation can be typed in:

Growth = ...

Proportion_of_Rabbits*(1-Proportion_of_Rabbits)

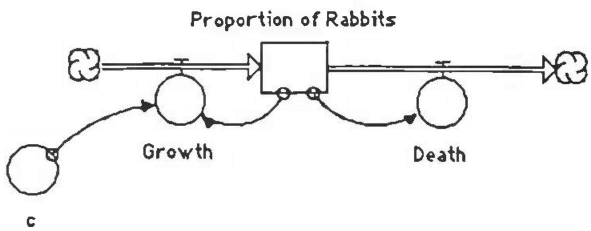
We still need to include the constant *c* which is intended to take all external factors into a global growth constant:



Opening up the circle icon permits me to specify the initial value of *c* (as 2).

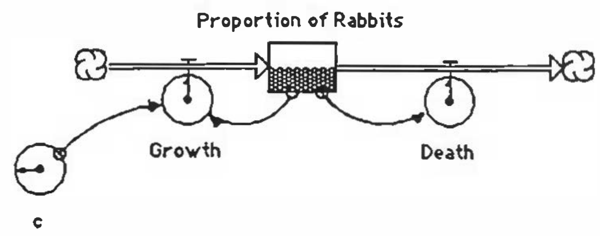
A brief digression on acceptable values for *c* is appropriate. Clearly, *c* must be greater than 0, since negative values make no sense in this context (all proportions must remain positive). But if *c* is too large, the proportions may rise above 1, which also is nonsense. The product of $P(1 - P)$ is a maximum when P is 0.5. This maximum is 0.25. Because c times 0.25 cannot exceed 1, *c* cannot exceed 4. Thus we may substitute any value of *c* between 0 and 4 into the model. A discussion of the boundary conditions of any mathematical situation is an important aspect of mathematical understanding.

There is one final step. The model as currently specified represents growth without end. We also need to include a "death" factor. This is also represented by another tap, indicating the outflow.

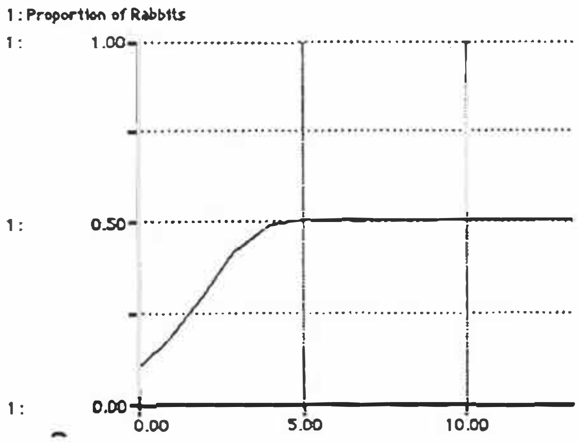


Now that the model is constructed, we need to run the simulation. STELLA maintains a graph of the values as the simulation is run over a specified time (for example, 20 cycles).

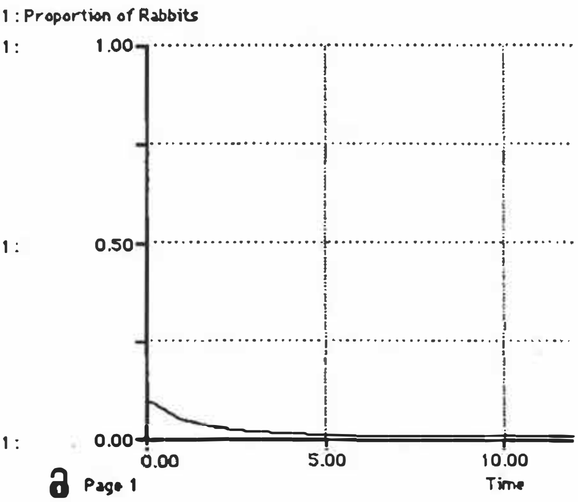
With *c* set to 2 and the initial value for the Proportion of Rabbits set at 0.1, we obtain the following graph, where the population level quickly approaches a value of 0.5. The final display looks like this:



It is important to realize that the viewer watching the computer screen sees the level rise to the halfway level and then remain constant (it is also possible to ask for a graph that shows how the level changed over time):

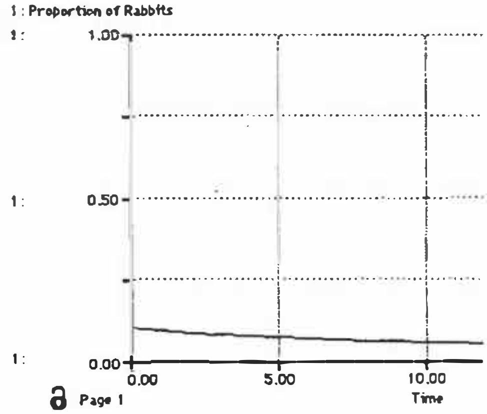


Leaving the structural equations unaltered, one can play with different initial values for the proportion of rabbits, and with the value of c, the constant. Let's see what happens if we play with different values of c. Here is the graph when c is 0.5.

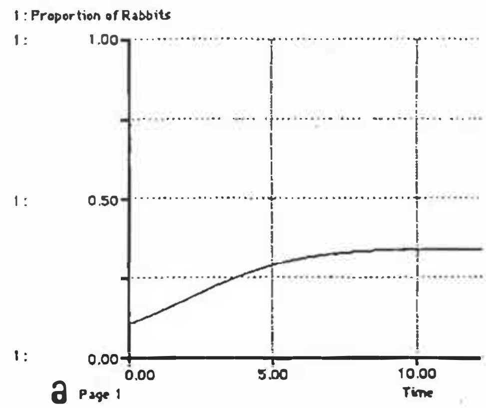


With such a low value of c, even a proportion of 0.1 is not sustainable.

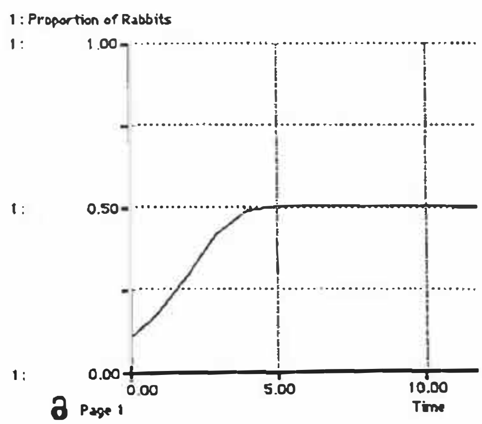
Setting $c = 1$ yields the following graph:



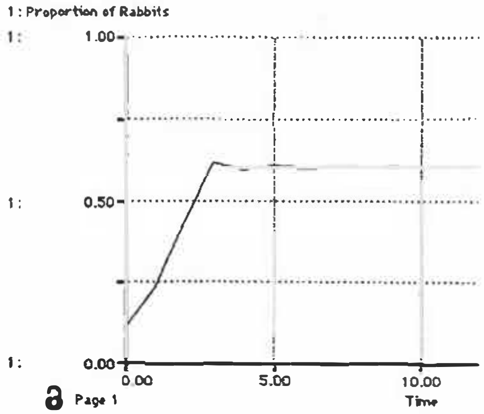
This is a little better, but the proportion of rabbits is still declining. Let's continue up in steps of 0.5. The next case is for $c = 1.5$:



For $c = 1.5$, the proportion of rabbits increases steadily to a value close to 0.3. Recall the graph for $c = 2$:

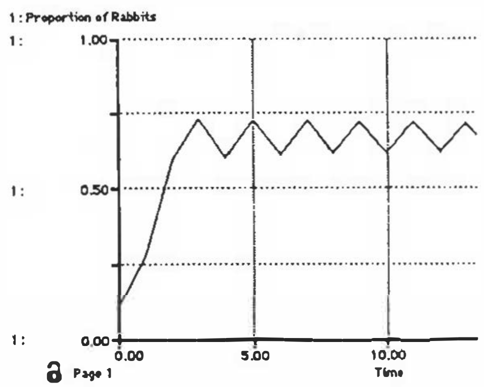


Here the proportion of rabbits increases steadily to a value of 0.5. Now consider the case where $c = 2.5$:



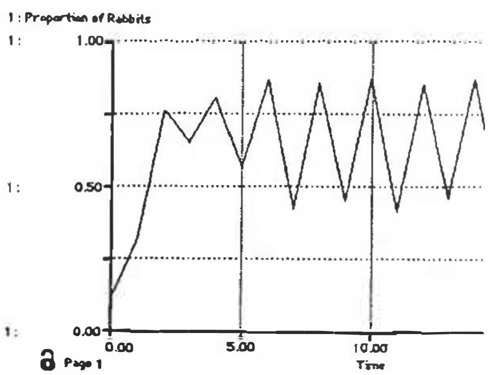
The pattern here is slightly different: there is a mild form of oscillation during the first few cycles before the proportion stabilizes at a value near 0.6.

Here is the graph for $c = 3.0$:

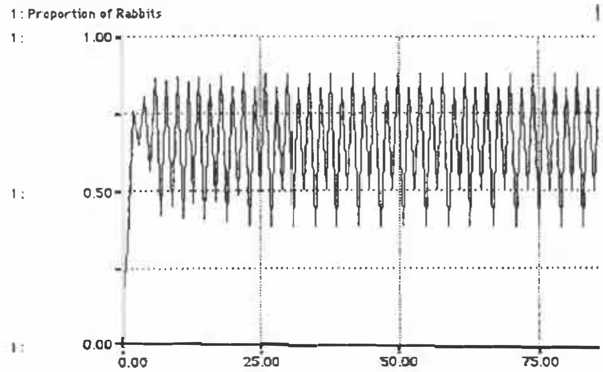


The pattern here is dramatically different. The population of rabbits appears to cycle through a set of values, without converging toward a limiting value.

Here is $c = 3.5$:



Expand the number of cycles from 10 to 100 to get a better picture of what happens in the long run:

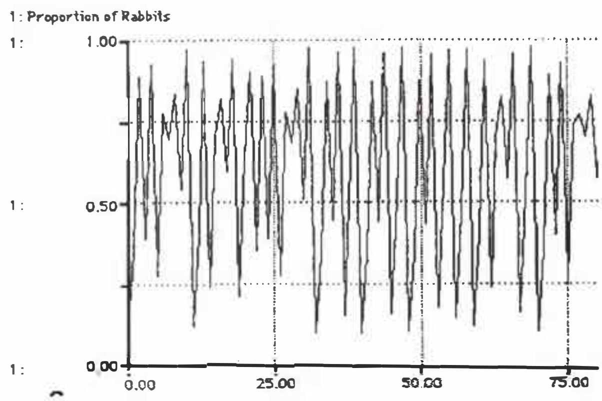


This appears cyclical, but it is difficult to tell by just looking at the graph. The software also provides a corresponding table. Here is a section of the table for the periods from 70 to 78:

Time	Proportion of Rabbits
70.00	0.87
71.00	0.38
72.00	0.83
73.00	0.50
74.00	0.87
75.00	0.38
76.00	0.83
77.00	0.50
78.00	0.87

Examination of the table reveals a cycle of period 4.

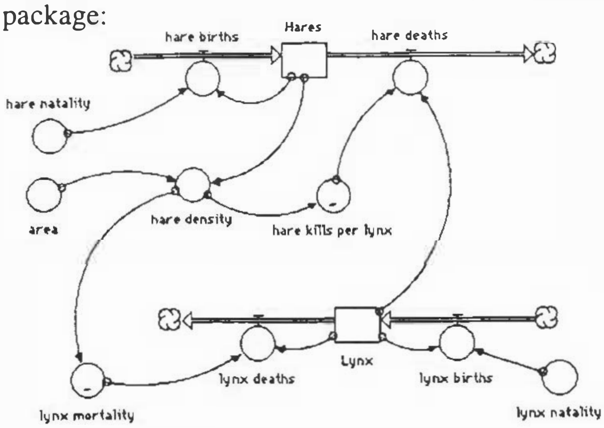
Finally, let's look at the graph and a portion of the table for $c = 3.9$:



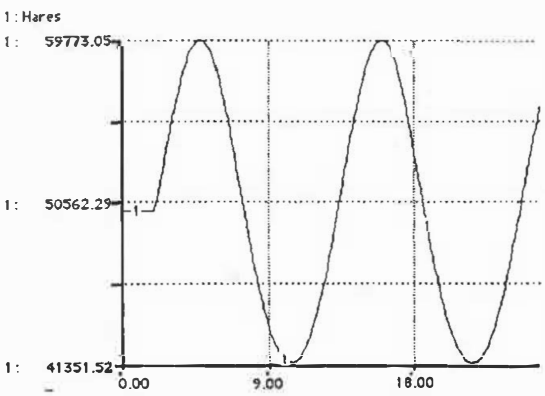
Time	Proportion of Rabbits
70.00	0.10
71.00	0.35
72.00	0.89
73.00	0.39
74.00	0.93
75.00	0.25
76.00	0.73
77.00	0.77
78.00	0.70
79.00	0.82
80.00	0.57
81.00	0.96

There is no discernible pattern at all! The population dynamics do not settle down, as in the other cases. The only parameter that has been altered is the value for c. Yet the mathematics of this model leads to chaotic behavior as c approaches 4. The chaos is inherent in the mathematics; it is not due to complications in the model caused by other factors.

It is also possible to include other factors, notably the lynx, to make the model more realistic. Here is a sample model that comes with the STELLA software package:



Here is a sample graph run for this model:



Although the model is considerably more complex, the graph is more regular and periodic than in the simpler case of $c\,x\,(1 - x)$ when c was set to 3.9.

One must distinguish between the complexity of the model, which is a biological issue, and the complexity of the output, which may be due to the complexity of the model or it may due to the inherent complexity of the mathematics. Because we have little experience with iterating functions, we have yet to acquire a sophisticated intuition of what to expect under different conditions. Certainly, most people, including most mathematicians, did not suspect that iterating a function as simple as $c\,x\,(1 - x)$ could lead to such chaotic results.

I will take one more quick look at the mathematics underlying the simple model where c took on different values between 0 and 4. Such an investigation leads into a new topic—the study of chaos.

Section 3 The Unexpected

This section is a brief introduction to the mathematics of chaos. Let's return to the function

$f(x) = c\,x\,(1 - x)$

and examine its behavior under iteration. We have two tools at our disposal, Microsoft Excel and Mathematica. Let's begin with a spreadsheet approach and then see if Mathematica can provide some additional insights.

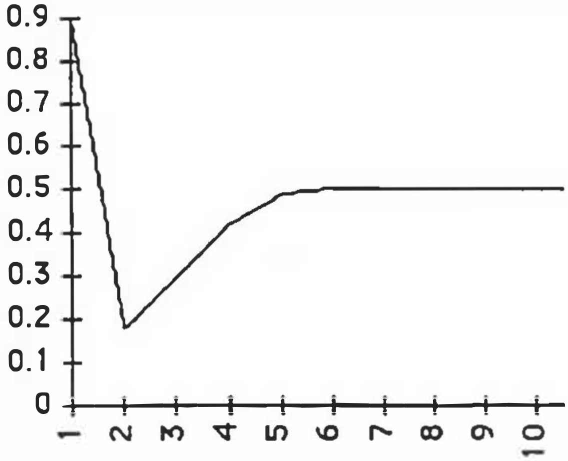
The only parameter we will change will be c. In the following examples, we will begin with an initial x value of 0.9. We will examine three different functions:

$f(x) = 2\,x\,(1 - x)$
$f(x) = 3.5\,x\,(1 - x)$
$f(x) = 3.9\,x\,(1 - x)$

Although these functions only differ by a small amount, their behavior under repeated iteration is unexpected.

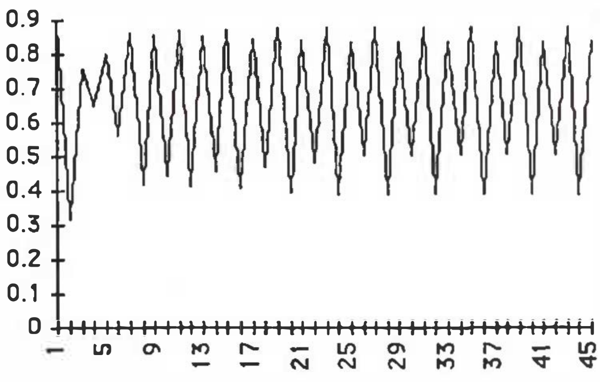
Setting $c = 2$ in Cell A2, setting the formula =$A$2*B3*(1-B3) in cell C3, setting 0.9 in cell B3, setting the formula =C3 in cell B4, and then filling down results in the following table and corresponding graph:

	A	B	C
1	c	x	cx(1-x)
2	2		
3	1	0.9	0.18
4	2	0.18	0.30
5	3	0.30	0.42
6	4	0.42	0.49
7	5	0.49	0.50
8	6	0.50	0.50
9	7	0.50	0.50
10	8	0.50	0.50
11	9	0.50	0.50
12	10	0.50	0.50
13	11	0.50	0.50



First, it is important to note that we are no longer dealing with parabolas. We are looking at the behavior of a function under repeated iteration, where the value of the function at cycle n becomes the argument of the function at cycle $n + 1$. In fact, this is the same graph that we obtained earlier when we were using STELLA. As a reminder, because $c = 2$, we are looking at the function $f(x) = 2x(1 - x)$. The second point to note is that we began the iteration with the initial x value of 0.9. The above graph is called the orbit of the point 0.9 for the function $2x(1 - x)$.

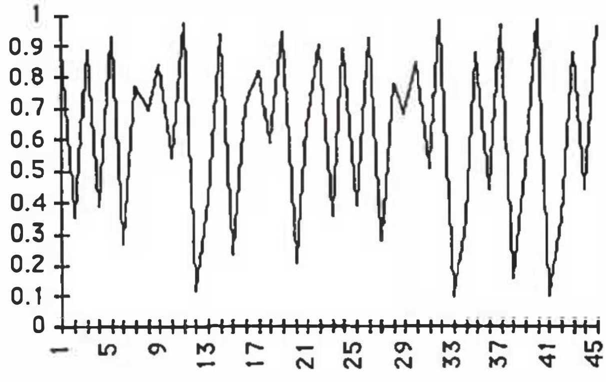
Let's look at two more examples using Excel. First is the case where $c = 3.5$, and the first value of x is 0.9. Here is the orbit of the point 0.9 for the function $3.5x(1 - x)$:



It is dramatically different from the orbit for the function $2x(1 - x)$, yet the only difference between the two functions is the leading coefficient.

A review of the table indicates that the iterations converge toward a cycle that repeats itself every fourth time (that is, 0.87 - 0.38 - 0.83 - 0.50 - and so on).

If we change c to 3.9, we obtain the following chart:

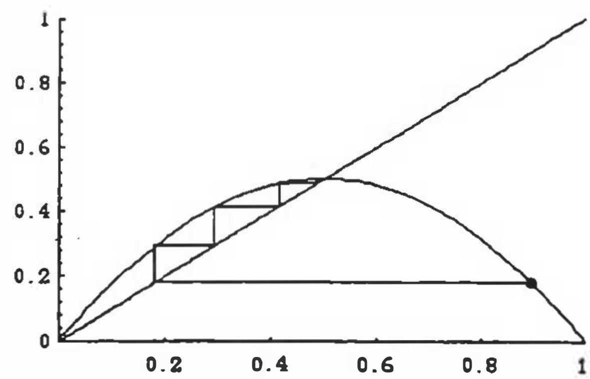


This time, there is no recurring pattern. The graph is said to be chaotic.

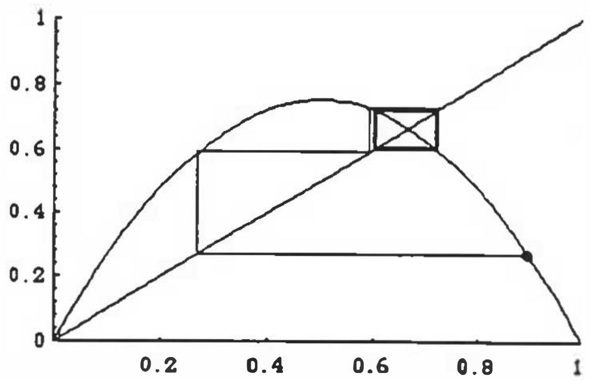
It is possible to explore any function to see its behavior under iteration. Devaney (1990) provides a rich sampling of problems for the beginner, examples easily within the range of high school students.

There is also a graphical procedure for examining the behavior of any function under iteration. The idea is fairly straightforward. Draw both the function of interest, call it $f(x)$, and the function $g(x) = x$ on the same grid. Then select any point x that you wish to begin the iterations with. Locate this point on the line $y = x$. This will be the point (x,x). Draw a vertical line joining this point with the graph of $y = f(x)$. This will be the point $(x, f(x))$. Now draw a horizontal line joining this point to the graph of $g(x)$. This will be the point $(g(x), g(x))$. Now repeat the process. Here is an example:

Consider the function $f(x) = 2x(1 - x)$.
Suppose we begin with $x = 0.9$. Then $f(0.9) = 2 \times 0.9 \times 0.1 = 0.18$. This represents the starting point of the iteration. Here is a graph showing the path of the graphical analysis and produced using Mathematica. Note that the path quickly converges to a value of 0.5. This is another way of viewing the situation that we previously examined using Excel.

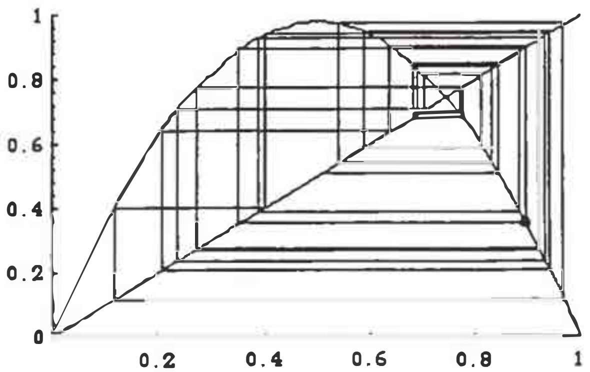


Now let's re-examine the function $f(x) = 3.5x(1 - x)$. Once again, we will begin with $x = 0.9$:



This time the orbit shows a cycle of period 4.

Finally, let's look at $f(x) = 3.9x(1 - x)$, beginning with $x = 0.9$:



The orbit fails to settle down. This is another view of mathematical chaos.

Summary

Pagels (1988) provides a thought-provoking analysis of developments in science over the last decade, using the phrase "sciences of complexity" to capture this new perspective and identifying a number of themes to characterize this new approach to science. His first theme is "the importance of the computer. . . . One of the ways that future science will progress is by a combination of precise observations of actual systems followed by computer modeling of those systems" (p. 43). He goes on to include "the computational viewpoint in mathematics" and then "the rise of computational biology" and "the study of nonlinear dynamics" and concludes with "the study of complex systems" as other themes characteristic of new approaches to science. The book ends with the sentence, "The future, as always, belongs to the dreamers." What are we doing in education to encourage such dreaming?

There is much concern from many interest groups about the present state of our classrooms, and mathematics courses take their fair share of the spotlight. Suggestions for improvement fall into three main categories:

1. How can we improve our present pedagogy?
2. What should we delete, to have more time for an in-depth exploration of what is left?
3. What should we add, because the new topics represent an important part of our evolving knowledge?

This article falls into the first and third categories. While it is true that I have used fairly sophisticated software packages as an integral part of my pedagogy, the main pedagogical thrust has been an open-ended exploratory approach with a conscious effort to understand this function as much as possible. Thus I have attempted to provide a metacognitive perspective to my own investigations. The article might also be viewed as a form of portfolio, a record of my explorations to date. This leads naturally into considerations surrounding portfolio assessment and authentic assessment. Perkins (1992) has written a stimulating book that attempts to address the issue of educational reform. His basic claim is that we need to focus on the curriculum and that our criterion should be deep, meaningful learning. I would like to think that this article represents a tentative beginning toward such an approach.

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STELLA II Version 1.00. High Performance Systems, Inc., Hanover, N.H.

Zap-a-Graph Version 4.2. Brain Waves Software, Fitzroy Harbour, Ont.