Fractal Cards

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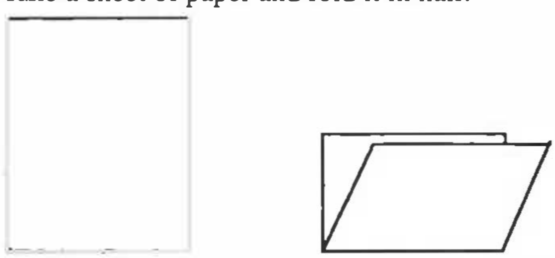
Fractal cards are beautiful "pop-up" cards that are reasonably easy to make and full of mathematics. Activities simple enough for middle school students or challenging enough for secondary and post-secondary students can be developed around fractal cards. Constructing a fractal card is quite simple, yet the mathematics is quite elegant. Some mathematics of fractal cards includes the following: measure, number patterns, sequences, series, limits, fractals and geometry.

Creating Fractal Cards

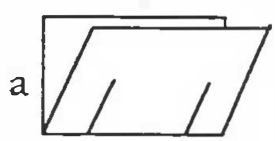
To make a fractal card, you define a simple rule (which includes a fold and some cuts) and then repeat that rule until the physical properties of the paper prevent you from continuing at smaller scales (Figure 1). Once you have cut the fractal, you must "open" it up and "pop" it out. The resulting object consists of a series of cuboids stacked one on top of the other (Figure 2).

Figure 1
Directions for Making a Fractal Card

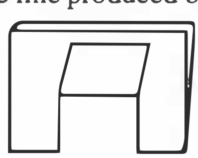
1. Take a sheet of paper and fold it in half.



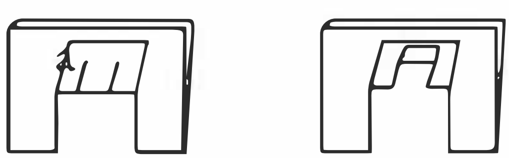
2. Make two cuts, each half the length of *a* and a quarter of the way in from each edge.



3. Fold along the line produced by the two cuts.



4. Repeat Steps 2 and 3 until it is too difficult to cut or fold the paper.



5. Once you stop cutting, you must open the folds and push out the fractal.

Figure 2

Fractal Card

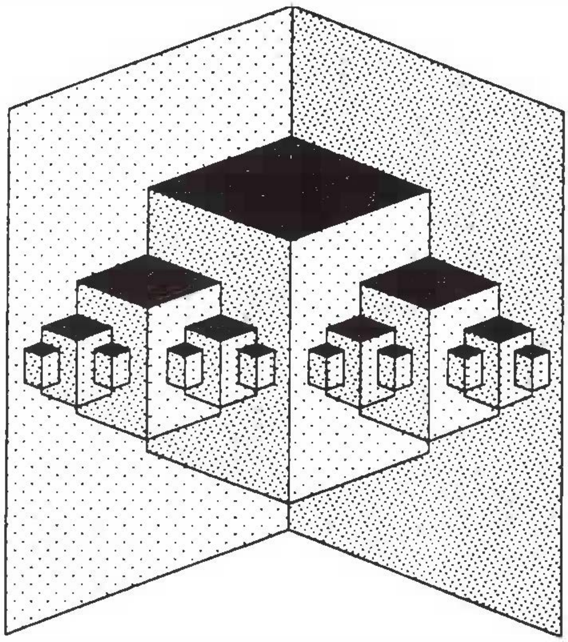


Table 1

Growth Pattern of Cuboids

Stage	Number of Cuboids
1	1
2	3
3	7
4	15
5	31
6	?
...	
n	$2^n - 1$

The Mathematics of Fractal Cards

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) calls for mathematics teaching in which concrete materials are used to help students understand mathematical concepts and make connections between them. Specifically, the standards recommend an increased emphasis in the content areas of discrete mathematics and geometry and methodological changes for learning mathematics that include pattern making and recognition, conjecturing, testing and generalizing, problem posing and problem solving. Fractal cards provide the mathematics teacher with the opportunity to attend to all these recommendations.

The growth pattern is a good place to begin an exploration of the mathematics of fractal cards. Middle school and secondary school students can count the cuboids, tabulate their observations and predict growth at various stages (Table 1). They might observe that, to determine the number of cuboids at any stage, you need only double the number at the previous stage and add 1. Senior high school students should be challenged to generalize this growth sequence and determine that the total number of cuboids (T) at any stage (n) is $T_n = 2^n - 1$. A further challenge for senior high school students is to find the limit of this series as n goes to infinity.

Measurement is another area for investigation. If the fractal card is cut out of graph paper (the card doesn't look as nice), determining the surface area of the card can be done by counting. Senior high school students should be able to investigate the area by assigning a value to the first cuboid and then determining that the successive cuboids have areas of half the area of the previous cuboid. The student must be observant to note that you do not include the whole cuboid after the first stage since only half of the cuboid contributes to increasing the surface area of the whole fractal (prove this to yourself). The formula which defines the growth in surface area is

$$SA = 2\Sigma\, ^1\!/_4{}^{\,i-1}$$

Here is an opportunity for students to use discrete mathematics to help them solve the measurement problem.

The concept of a limit can also be discussed in this context. Once again, notice that this can be done at many levels. For some students, discussing the finite nature of the surface area can be restricted to using the students' intuition about the surface area of the paper and therefore the surface area of the fractal. That is, the surface area must be finite (converge) because the fractal is cut out of a piece of paper with finite area. Students interested in using technology can approximate the limit using a computer program (Tables 2 and 3). Calculus students can be asked to find the limit of this sum.

Fractal cards are a rich space for mathematical explorations. For example, take the fractal card you just made and hold it sideways to look at the image from end to end. A beautiful pattern emerges. Now many more questions arise about that pattern, such as found in the following list of discussion items and questions pertaining to fractal cards:

- Discuss the language associated with fractals and their construction.
- Describe the growth pattern of fractals.

Table 2
Program to Calculate Surface Area
(Written for Texas Instruments calculator TI-81)

```
Prgm 1: SA
: 1→Y
: 0→Z
: Lbl 2
: 1 / (4 ^ (Y - 1))→X
: Y + 1→Y
: X + Z→Z
: Y - 1→U
: DISP U
: 2 * Z→V
: DISP V
: PAUSE
: If Y < 11
: Go To 2
: END
```

Table 3
Output from Program in Table 2
Approximated Limit of the Surface Area

Iteration	Surface Area (units^2)
1	2.00
2	2.50
3	2.625
4	2.65625
5	2.6640625
6	2.666015625
7	2.666503906
8	2.666625977
9	2.666656494
10	2.666664134

- Investigate what happens to the number of cuboids as the number of iterations goes to infinity.
- Write a sequence to describe the growth pattern of the fractal.
- How many cuts are there at the tenth iteration (for example)?
- What is the surface area of the fractal generated?
- What is the volume contained by the fractal?
- How far away is the farthest cuboid from the first cuboid?
- How far away can the farthest possible cuboid be from the centre?
- Investigate the number pattern you generate if you assign numbers to the cuboids based on iterations.
- Create your own fractal card using fractal cuts and then find the mathematics in your fractal.

As you can see, this activity provides a wide open space for students to explore some very interesting mathematics.

Acknowledgment

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References

National Council of Teachers of Mathematics (NCTM)—Commission on Teaching Standards for School Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

Uribe, D. *Fractal Cuts*. Norfolk, U.K.: Tarquin, 1993.