delta-k

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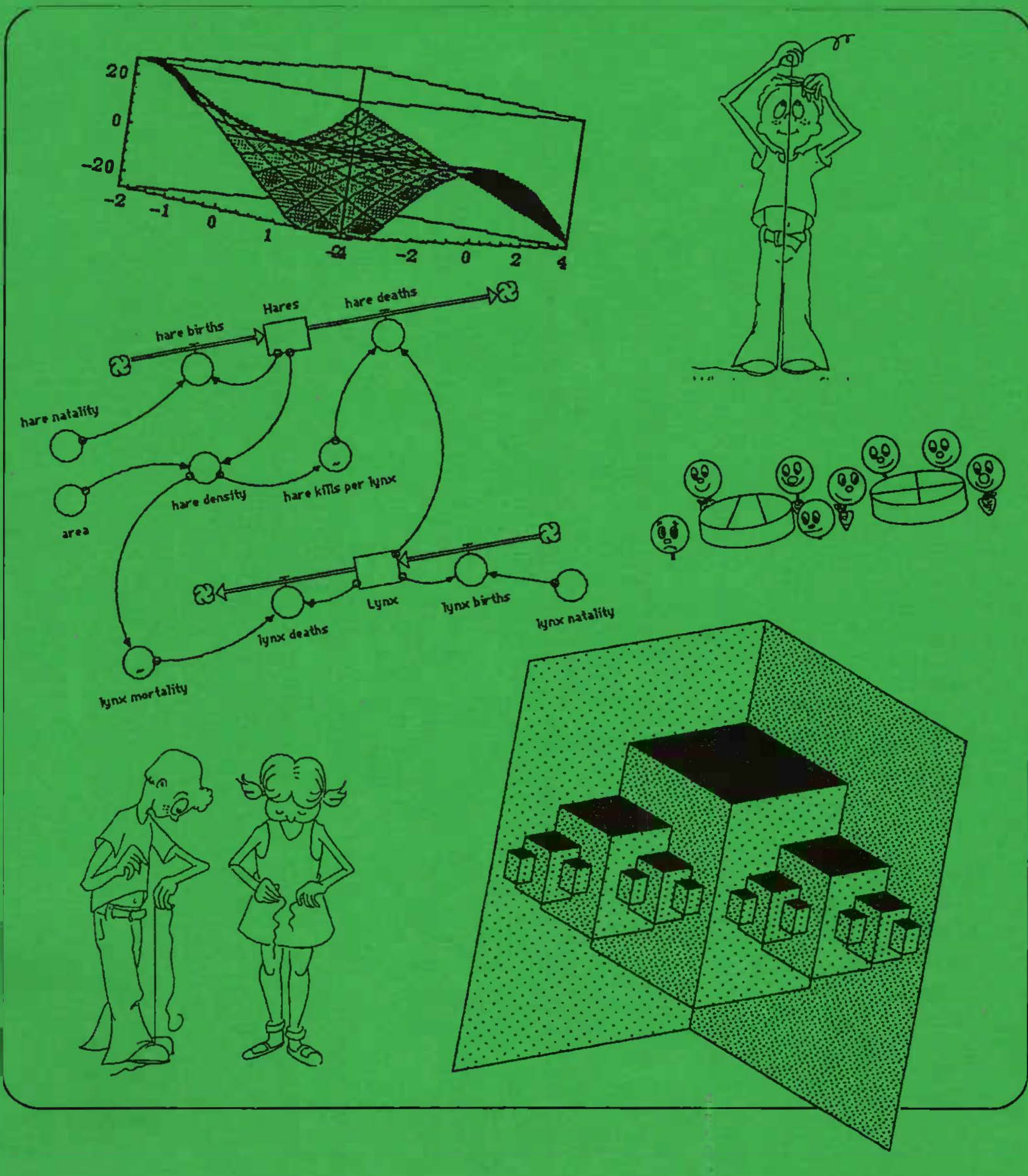

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Warm-Up

If all consonants cost $1 and all vowels (including *y*) cost $2, how much are your first and last names worth?

(taken from Ginn *Interactions*)

EDITORIAL

I dedicate this issue of the *Journal* to A. Craig Loewen and John Percevault, who for so many years have worked diligently in gathering and assembling articles and activities for *delta-K*. Through their hard work, they created a journal recognized far and wide by mathematics educators for its high quality.

This issue of *delta-K* contains a potpourri of articles, so I hope there will be something here to interest all readers.

To quote from Craig's last editorial:

> We continue to seek articles for upcoming issues of *delta-K*. We are always interested in a variety of articles, but we are particularly interested in innovative teaching ideas to help make the learning of mathematics more interesting, meaningful and enjoyable. Please give some serious thought to what you could contribute to your colleagues across the province. Your ideas are needed and important!

I would even appreciate a cartoon, a joke or a "thought for the day" to include. Submissions can be made to me at the address listed in the executive list on the back inside cover.

I express my personal appreciation to all contributors to this issue. Special thanks go to the Gifted and Talented Education Council for its position paper on the gifted and talented. I have often felt that gifted and talented students have not received the attention they deserve in the area of mathematics education. Certainly, the paper provides some food for thought.

Arthur Jorgensen

delta-K is the official journal of the Mathematics Council (MCATA) of The Alberta Teachers' Association (ATA).

The objective of the *Journal* is to assist MCATA to achieve its objective of improving teaching practices in mathematics by publishing articles that increase the professional knowledge and understanding of teachers, administrators and other educators involved in teaching students mathematics. The *Journal* seeks to stimulate thinking, to explore new ideas and to offer various viewpoints. It serves to promote MCATA's convictions about mathematics education.

Position Paper of the Gifted and Talented Education Council of The Alberta Teachers' Association, September 1994

The Gifted and Talented Education Council

The Gifted and Talented Education Council's Statement of Position

Since its inception in 1986, the Gifted and Talented Education Council's mandate has been to facilitate the professional development of teachers from ECS to Grade 12 and to improve the quality of educational practices for gifted and talented students in Alberta. A statement of position is presented to provide direction for effective teaching practices for these learners. This position

I. recognizes the *nature of giftedness* and the *characteristics and needs* of gifted and talented students, and their right to programming matched to their individual needs;

II. identifies effective *educational practices* that meet the diverse characteristics and needs of gifted and talented learners; and

III. identifies characteristics of effective *professional development* specific to the education of the gifted and talented.

Gifted and talented education is an important area of specialization; however, gifted and talented education must be seen as a part of appropriate programming for all students and, as such, supports the premise that all learners need to be challenged to maximize their potential.

I. Nature of Giftedness, Characteristics and Needs

The Gifted and Talented Education Council believes

- giftedness is the demonstration of a learner's potential to order and interpret the world in new and different ways, through a dynamic interaction between an individual's advanced abilities, personality traits, and his/her environment;
- gifted and talented learners have characteristics and needs that are both similar to and different from all other learners and, like all learners, have a right to have their needs both recognized and met; and
- there is a diversity of characteristics, talents and needs both within and among gifted and talented learners. Given this diversity gifted and talented learners have special educational needs. It is a basic responsibility of education to provide for these needs.

Contemporary conceptions of giftedness, built upon a broadened understanding of human talents and abilities, view giftedness as the multifaceted, multidimensional *potential* for creative productivity, considering a person's accomplishments or attainments over a sustained period of time. Evidence indicates that learners who display the potential for the development of gifted behaviors require differentiated educational programs and services beyond those provided by general education in order to realize their contribution to self and society.

Current understandings of the nature and diversity of human abilities emphasize that recognition practices should focus on a broad range of student needs. This facilitates the development of appropriate curriculum and instructional strategies rather than merely categorizing or labelling the student. Therefore, it is important that the characteristics and needs of gifted and talented learners be recognized. It is these varied combinations of characteristics, often to a greater degree and dimension, that makes some learners markedly different from other learners. These varied human capacities require distinctive educational responses. Gifted education is a response to these different characteristics and needs.

The chart on the following page describes some typical characteristics and related educational needs.

Evidence indicates that, while there is no singular profile of an able learner, gifted and talented learners display at least some of the following characteristics or traits in varying degrees and combinations. These traits require distinctive educational responses as indicated below:

Characteristics	*Learner's Educational Need*
• ability to learn quickly, efficiently; rapid pace of information processing	• progress through content at individual pace or developmental rate
• ability to move quickly through the stages of cognitive development, and at the same time be more advanced than chronological peers at each stage of development	• requires less introduction and practice of skills and is able to spend more time on application, synthesis, and evaluation of ideas
• unusual capacity for perceiving, processing and producing ideas and solutions to problems	• explore content and processes of learning at a level commensurate with abilities
• advanced ability to see abstractions, readily make connections to new contexts, and work at varying levels of complexity	• explore ideas in greater depth and breadth
• advanced ability to use self-regulatory or metacognitive processes to guide thinking	• initiate, plan and direct personal learning and engage in independent study
• capacity for perseverance; able to sustain long periods of concentration and attention	• allow flexible scheduling for indepth and long-term studies
• capacity for high levels and widely eclectic interests; enthusiasm, fascination and intense involvement in a particular problem, area of study, or form of human expression	• pursue topics or problems which pique interest, and to be exposed to a wide spectrum of ideas and issues
• heightened sensitivity and intuitiveness; high degree of awareness; keenly observant; acute sense of justice and values	• set realistic goals from which to judge one's own progress and that of others; understand self in relationship to others; and explore ethical and moral issues
• sophisticated facility for expression	• communicate in various forms to various audiences
• advanced ability to analyze, evaluate, and hypothesize ideas	• apply higher level abstract thinking skills
• propensity for inventive versatile thought	• reconceptualize existing knowledge and create new knowledge in an area of study

II. Effective Educational Practices

Along with a broadened conception of giftedness and the recognition of students with complex, dynamic and varied characteristics comes the need for an educational delivery system that provides diverse and varied options, flexible pacing, and activities and strategies that extend, supplement, or replace learnings in the core curriculum. Effective educational practices for gifted and talented learners include the following:

- A Supportive Learning Environment
  A school community works together to recognize the strengths, interests, learning styles and needs of individual gifted and talented learners. Students become an active part of this partnership.
- Effective Systems of Delivery
  A variety and range of programming options and services must be available to meet the diverse characteristics and needs of gifted and talented learners. Services for gifted learners must begin in the regular classroom and reach beyond to part-time or full-time special classes and in extreme cases to special settings. Resources for the development and implementation of programs and services for the gifted and talented should be both increased and distributed equitably across the province.
- Appropriate Curriculum Matched to the Learner
  Curriculum, instructional strategies, resources and evaluation practices must be adapted to meet the diverse needs of gifted and talented learners. This involves matching the content, process and products of learning to learner needs. Evaluation practices must also recognize the characteristics and needs of these learners.
- Effective Teaching and Learning Strategies
  Teachers require a variety of teaching and learning strategies to shorten the introduction and

practise of skills and to provide more time for application and connecting learning to other disciplines and real life experiences. Classroom strategies to increase student participation in directing and evaluating their learning are important.

The [following] chart details the specifics of effective educational practices.

The Gifted and Talented Education Council recognizes and supports the following principles for designing effective programming for gifted and talented learners:

1. A Supportive Learning Environment is

responsive	- recognizes and respects the right of gifted and talented learners to an appropriate education and ensures that all students have access to programming which appropriately challenges their abilities and talents
individualized	- provides for the assessment of and programming for the individual student with reference to strengths, interests, learning style preferences and related cognitive and affective needs
interactive	- provides opportunities to interact with intellectual peers and to learn with individuals who have different abilities and interests
cooperative	- recognizes that learning is a partnership that involves student, teacher, school, parent and the community
self-directed	- provides opportunity for choices, learning how to learn, decision making, growth of personal autonomy and self-discipline

2. Effective Systems of Delivery are

economical	- compact and streamline the organization of curriculum to match learning needs of gifted students by building on mastered skills and content
flexible in tempo and pacing	- provide an appropriate time span for learning activities that is consistent with characteristics of gifted learners; allow students to learn at a level and pace matched to their abilities and skills
blended and balanced	- provide for individual, small group and large group instruction; limited range and multi-age groupings; enrichment and acceleration
comprehensive	- incorporate a blend of community resources and school-based support services in program development and delivery
flexible in scheduling	- vary time and learning settings with opportunities for cross graded projects, individual and group projects
integrated	- recognize the need to balance homogeneous and heterogeneous groupings based on the ongoing assessment of student strengths, interests and needs

3. Appropriate Curriculum Matched to the Learner is

holistic	- provides for the learner's intellectual, academic, vocational, ethical and social/emotional development
differentiated	- places an emphasis on varying the content, process, and products of learning
complex	- provides for exposure to and interaction with major systems of thought, incorporating advanced concepts, generalizations, principles, and theories related to significant issues and problems; allows for exploration of content in greater depth and breadth
interdisciplinary	- presents content related to broad-based themes, issues and problems; integrates multiple disciplines into an area of study; provides for the transfer of learning to other domains of knowledge, new or unusual situations
challenging	- develops complex, abstract, investigative and higher level thinking processes and integrates these processes with basic skills
productive	- encourages the development of creative products that challenge existing ideas or produce new ideas
personalized	- provides opportunities to explore and develop personal strengths and interests

continuous	- provides an on-going, comprehensive and sequential progression of skills and processes to be learned across the curriculum as well as across the grades
4. Effective Teaching and Learning Strategies are	
eclectic	- provide for a variety of teaching/learning strategies
multiple and varied	- provide for exposure to and use of a variety of advanced and specialized resources including primary and secondary sources, as part of the learning process
facilitative	- promote self-initiated, self-directed and sophisticated investigative activities
self-evaluative	- provide opportunities for students to design and/or participate in the development of procedures for the evaluation of their work
performance-based	- provide opportunities to display learning outcomes in varied and individual ways
relevant	- shorten introduction and practice, as appropriate, to provide more time for application

III. Effective Professional Development

Teachers are responsible for providing appropriate learning opportunities for all students including those who are gifted and talented and thus continually seek to improve the implementation of the curriculum through effective strategies of organization, instruction, assessment and evaluation. Initial teacher training and ongoing professional development opportunities in the area of gifted and talented education are essential for student success.

Diversity of professional development opportunities is necessary to recognize different levels of teacher development and background experience, mandated curriculum changes and the changing nature of education. The characteristics of the able learner, as well as effective teaching and learning practices in the area of gifted education have implications for the changing role of the teacher. Many of these practices will improve learning for all students.

Effective teacher training and professional development should vary in format and include opportunities to broaden the understanding of gifted and talented learners and to develop expertise in curriculum differentiation and effective instructional practices. Opportunities should include building of professional networks, developing leadership skills and collaborating with other stakeholders in gifted education.

The Gifted and Talented Education Council recognizes and supports teacher training and professional development activities which

- require personal goal-setting and long-range planning;
- foster active involvement;
- provide opportunity for reflection, assessment and growth over time;
- acilitate collegial collaboration;
- encourage teacher research;
- respect the teacher's need for practical skills and strategies;
- promote the continuous improvement of curriculum and instruction;
- inform the teacher of current knowledge, research, and methodologies in the field and in general education;
- promote the pursuit of personal excellence and professional responsibility; and
- facilitate change.

In conclusion, by recognizing the inherent worth and uniqueness of all individuals and the right to equity of educational opportunity, the Gifted and Talented Education Council of The Alberta Teachers' Association emphasizes the value of learning to assist students to realize their full potential. The council dedicates its efforts to support teachers to improve the quality of educational practices for gifted and talented students in Alberta.

Distributed with the knowledge and consent of The Alberta Teachers' Association.

Fractal Cards

Elaine Simmt

Fractal cards are beautiful "pop-up" cards that are reasonably easy to make and full of mathematics. Activities simple enough for middle school students or challenging enough for secondary and post-secondary students can be developed around fractal cards. Constructing a fractal card is quite simple, yet the mathematics is quite elegant. Some mathematics of fractal cards includes the following: measure, number patterns, sequences, series, limits, fractals and geometry.

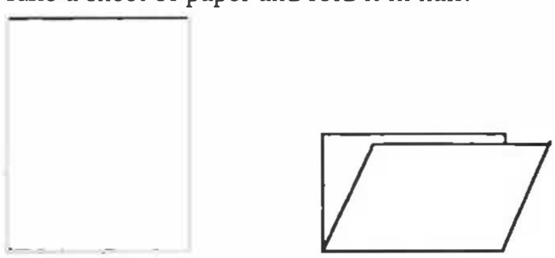
Creating Fractal Cards

To make a fractal card, you define a simple rule (which includes a fold and some cuts) and then repeat that rule until the physical properties of the paper prevent you from continuing at smaller scales (Figure 1). Once you have cut the fractal, you must "open" it up and "pop" it out. The resulting object consists of a series of cuboids stacked one on top of the other (Figure 2).

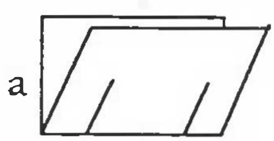
Figure 1

Directions for Making a Fractal Card

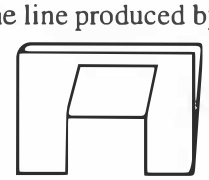
1. Take a sheet of paper and fold it in half.



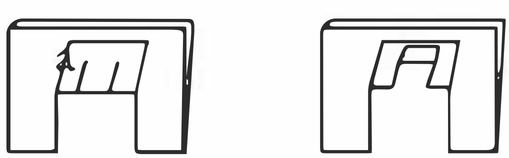
2. Make two cuts, each half the length of a and a quarter of the way in from each edge.



3. Fold along the line produced by the two cuts.



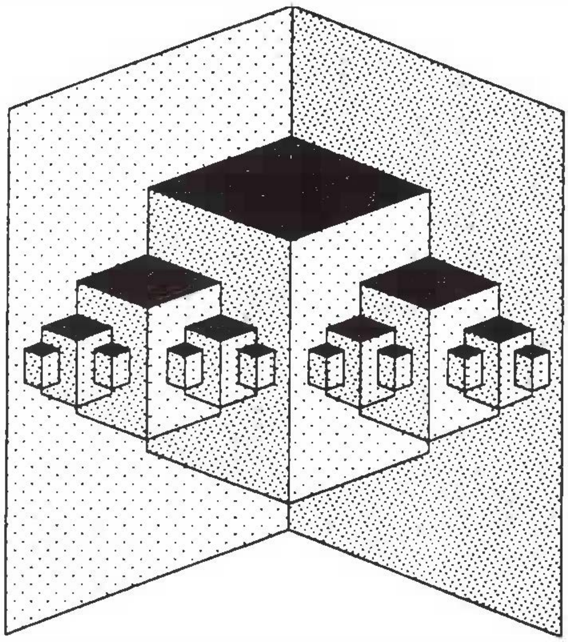
4. Repeat Steps 2 and 3 until it is too difficult to cut or fold the paper.



5. Once you stop cutting, you must open the folds and push out the fractal.

Figure 2

Fractal Card



The Mathematics of Fractal Cards

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) calls for mathematics teaching in which concrete materials are used to help students understand mathematical concepts and make connections between them. Specifically, the standards recommend an increased emphasis in the content areas of discrete mathematics and geometry and methodological changes for learning mathematics that include pattern making and recognition, conjecturing, testing and generalizing, problem posing and problem solving. Fractal cards provide the mathematics teacher with the opportunity to attend to all these recommendations.

The growth pattern is a good place to begin an exploration of the mathematics of fractal cards. Middle school and secondary school students can count the cuboids, tabulate their observations and predict growth at various stages (Table 1). They might observe that, to determine the number of cuboids at any stage, you need only double the number at the previous stage and add 1. Senior high school students should be challenged to generalize this growth sequence and determine that the total number of cuboids (T) at any stage (n) is $T_n = 2^n - 1$. A further challenge for senior high school students is to find the limit of this series as n goes to infinity.

Table 1

Growth Pattern of Cuboids

Stage	Number of Cuboids
1	1
2	3
3	7
4	15
5	31
6	?
...	
n	$2^n - 1$

Measurement is another area for investigation. If the fractal card is cut out of graph paper (the card doesn't look as nice), determining the surface area of the card can be done by counting. Senior high school students should be able to investigate the area by assigning a value to the first cuboid and then determining that the successive cuboids have areas of half the area of the previous cuboid. The student must be observant to note that you do not include the whole cuboid after the first stage since only half of the cuboid contributes to increasing the surface area of the whole fractal (prove this to yourself). The formula which defines the growth in surface area is

$$SA = 2\Sigma \tfrac{1}{4}^{i-1}$$

Here is an opportunity for students to use discrete mathematics to help them solve the measurement problem.

The concept of a limit can also be discussed in this context. Once again, notice that this can be done at many levels. For some students, discussing the finite nature of the surface area can be restricted to using the students' intuition about the surface area of the paper and therefore the surface area of the fractal. That is, the surface area must be finite (converge) because the fractal is cut out of a piece of paper with finite area. Students interested in using technology can approximate the limit using a computer program (Tables 2 and 3). Calculus students can be asked to find the limit of this sum.

Fractal cards are a rich space for mathematical explorations. For example, take the fractal card you just made and hold it sideways to look at the image from end to end. A beautiful pattern emerges. Now many more questions arise about that pattern, such as found in the following list of discussion items and questions pertaining to fractal cards:

- Discuss the language associated with fractals and their construction.
- Describe the growth pattern of fractals.

Table 2
Program to Calculate Surface Area
(Written for Texas Instruments calculator TI-81)

```
Prgm 1: SA
: 1→Y
: 0→Z
: Lbl 2
: 1 / (4 ^ (Y - 1))→X
: Y + 1→Y
: X + Z→Z
: Y - 1→U
: DISP U
: 2 * Z→V
: DISP V
: PAUSE
: If Y < 11
: Go To 2
: END
```

Table 3
Output from Program in Table 2
Approximated Limit of the Surface Area

Iteration	Surface Area (units^2)
1	2.00
2	2.50
3	2.625
4	2.65625
5	2.6640625
6	2.666015625
7	2.666503906
8	2.666625977
9	2.666656494
10	2.666664134

- Investigate what happens to the number of cuboids as the number of iterations goes to infinity.
- Write a sequence to describe the growth pattern of the fractal.
- How many cuts are there at the tenth iteration (for example)?
- What is the surface area of the fractal generated?
- What is the volume contained by the fractal?
- How far away is the farthest cuboid from the first cuboid?
- How far away can the farthest possible cuboid be from the centre?
- Investigate the number pattern you generate if you assign numbers to the cuboids based on iterations.
- Create your own fractal card using fractal cuts and then find the mathematics in your fractal.

As you can see, this activity provides a wide open space for students to explore some very interesting mathematics.

Acknowledgment

The author acknowledges the skills of Brian Brecka (who recently graduated from the University of Alberta in secondary education) and thanks him for the image of the fractal card in Figure 2.

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Celebrate Mathematics: Everybody's Heritage and Everybody's Future

Mary Montgomery Lindquist

This is a summary of comments made at the opening session at the NCTM Canadian Regional Conference held in Edmonton in October 1994.

Welcome to the celebration of the 75th anniversary of the National Council of Teachers of Mathematics (NCTM). Before I address the theme of this year's celebration, let's look back at the beginning to that day in February 1920 when 127 teachers from 20 states met in Cleveland, Ohio. It must have been a decisive group that capitalized on the experience of local mathematics teachers' groups and the need for unity. That very day, they founded the Council. The first president, C.M. Austin, described the need for such a group:

> During the same period [1910–1920] high school mathematics courses have been assailed on every hand. So-called educational reformers have tinkered with the courses, and they, not knowing the subject and its values, in many cases have thrown out mathematics altogether or made it entirely elective. The individual teachers and local organizations have made a fine defense to be sure, but there could be no concerted action. (NCTM 1970, 194)

Reflect on the goals of the Council as set forth (below) by Austin. Do we still have the same needs? How has the situation changed? Two changes are immediately notable: the Council now includes Canada, and its membership includes more than secondary mathematics teachers.

> *First*, it will at all times keep the values and interests of mathematics before the educational world. Instead of continual criticism at educational meetings, we intend to present constructive programs, by the friends of mathematics. We prefer that curriculum studies and reforms and adjustments come from the teachers of mathematics rather than from the educational reformers.
>
> *Second*, it will furnish a medium through which teachers in one part of the country may know what is going on in every other part of the country.
>
> *Third*, the Council through its journal will furnish a medium of expression for all of the teachers of the country.
>
> *Fourth*, the Council will help the progressive teacher to be more progressive. It will also arouse the conservative teacher from his satisfaction. . . . (NCTM 1970, 195)

The theme of the 75th anniversary is "Mathematics: Everybody's Heritage, Everybody's Future." Throughout the year, we are celebrating teaching and the teachers of mathematics. In the following, I would like to share with you what may have been some of the headlines in our mathematics education heritage in NorthAmerica.As we look at these headlines, think about what implications they may have for today's reform efforts.

1556: First Mathematics Book Published in the Americas

Author: Juan Diez Freyle
Place: Mexico City
Title: *Summario Compendioso de las quentas de plata y ore*

This first book published on the North American continent dealt with the gold equivalencies to other currencies. It chiefly called for use of ratios and proportions. There were several other Spanish books published prior to 1700, all of which dealt with the mathematics of the practical.

As we honor our heritage, we need to remember all the cultures that have contributed to the subject that some consider a universal language. As we relate to the people of diverse cultures, we will help our students realize that mathematics is a human endeavor to which we all can contribute.

1729: First English-Language Mathematical Book Published

Author: Isaac Greenwood
Title: *Arithmetick, Vulgar and Decimal*

This was the first book written and published in the English language that dealt solely with mathematics. And, yes, our students today still may think of fractions as vulgar! But the students of 1729 were not school children but adults studying for business and trade. Arithmetic was not even permitted in the early schools.

1807: Harvard: Arithmetic for Admission

Over 200 years after the establishment of Harvard, arithmetic will be required of all students entering college in 1807. It is expected by the year 1816 that the "whole of arithmetic" will be required, not just the whole number operations, reduction and the rule of three. By 1820, Harvard will require algebra for admission.

This headline reminds us how young mathematics education actually is in North America. It took Harvard 200 years to require arithmetic for admission, and less than 200 years have passed since that 1807 watershed. We've come a long way!

1861: Both Puzzled

But, Sir, if wanst nought be nothin', then twice nought must be somethin', for it's double what wanst nought is." (NCTM 1970, xviii)

This picture and caption were published in the centennial history of the Ontario Educational Association and remind us that our central mission is to help students understand and use mathematics. Have we made any progress in helping our students learn about 0?

1890: What Algebra?

The Cajori study of algebra reform released its findings today. Teachers of algebra were asked what reforms are needed in the teaching of algebra?

Some typical answers were:

- More of the spirit and reason and less of mechanical solution.
- Rattle the bones of the algebraic skeleton, as exhibited in this country, and show it in its living, breathing continuity and beauty of form.
- Anything to make it less a collection of dry bones and more a living and beautiful science. (NCTM 1970, 160)

We are still asking today about algebra. Do the responses of teachers in 1890 surprise you?

1932: Canadian View of Math Today?

Come to high school, young people. . . . True, the material we shall teach you will be completely useless to most of you. . . . We shall teach it to you anyhow, even if we make incurable bluffers out of you in the process. Our duty in this new day is to train you for citizenship, and we propose to perform that duty by administering to you exactly the same training as was given to an entirely different class for an entirely different purpose a generation ago. (NCTM 1970, 413)

These were the words of a Canadian mathematics teacher. They remind us that we always need to change and always need to keep the needs of our students in mind.

1954: Crisis in Math

The present crisis in mathematics is due not to any deterioration in the work of mathematics teachers, but to an urgent national need for more and better mathematics at a time when administrators and the public have for years slighted mathematics and, indeed, discouraged all vigorous mental effort in the high schools. . . . Excellence in scholarship is permitted, but mediocrity is considered more democratic. (May 1954, 303)

As we moved into this era of reform in the 1950s, the role of the mathematics teacher shifted. To me, the greatest change in the present reform is the centrality of the teacher. Perhaps we have come to the true intent of the NCTM founders. We have also come to realize that it is not an either-or situation. It will take all of us to make the vision of the Standards become a reality.

1974: Four Bones in Teaching

> Wishbones—those that say, "If only somebody would do something!"
> Jawbones—those that do much talking about what ought to be done
> Knucklebones—those that knock no matter what is done
> Backbones—those that do the work
> (Smith 1975, 529)

Yes, we have much to do so that mathematics is every student's future. It will take all of us. In his 1974 presidential address, Gene Smith described teachers in terms of bones. What type of bone are you?

Now, let's celebrate. But before we leave, take the time to remember one teacher who made a difference to you. And take one step, so one more student will remember you as the teacher with backbone, the one who makes it work.

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Playing with *c x (1 - x)*

J. Dale Burnett

I have had fun playing with the function *c x (1 - x)*. The activities have at least a three-fold nature: one is to explore some characteristics of this function; another is to demonstrate various computer software packages useful for facilitating such explorations; and finally there are the educational issues involving what occurs and does not occur in school classrooms.

Exploration of the function *c x (1 - x)* breaks into two components: one examines the mathematics, the other develops an appreciation of the contexts where one might expect to find such a function. The contexts also split into parts: science, mathematics and education; similarly with the exploration: graphing, algebra and dynamic properties are all of interest. This splitting, and splitting again, is itself a metaphor for this article and for the ways in which one might conceive knowledge. Part of the intent is to make this metaphor explicit, arguing that such explicitness is an important pedagogical principle. Let's share the secrets with our students. The investigation is divided into three sections. The first section examines the function using a variety of computer software. The second section looks at the same function from a biological perspective, and the third section builds on some features that were noted during the study of population dynamics. This latter section opens up an entire new universe of mathematical topics.

I emphasize that this is written not about a thorough analysis of this function. Rather it is about some suggestions for mathematics.

Section 1 Quadratic Functions

Let's begin with some simple algebra and graphing.

Let *y = c x (1 - x)*

What does the graph of this equation look like?

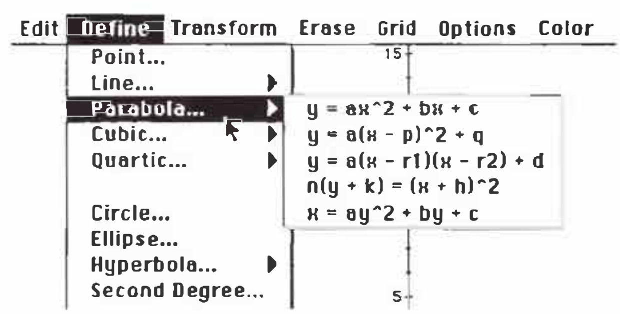
Even this question can immediately lead to alternative conceptions and approaches. One approach is to simply find out. Thus, at least to begin with, one doesn't even try to imagine what the graph might be. Let's just let the technology show us.

The most likely activity for many students and teachers would involve use of a graphing calculator. Fair enough. Numerous workshops have been offered and many articles written about the use of such devices in mathematics classrooms. I will explore other possibilities, possibilities that use more powerful tools. These tools already exist, but their use in classrooms is relatively rare. I am referring to the laptop computer, supplemented by sophisticated software packages. I will use four such software packages in the remainder of this paper: Zap-a-Graph, Microsoft Excel, Mathematica and STELLA.

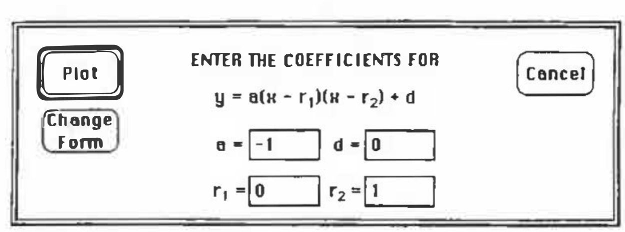
Zap-a-Graph

Zap-a-Graph is a software package for graphing mathematical functions. It is relatively easy to use since usually the user need only change the values of a set of predefined parameters. Here is an example using our equation.

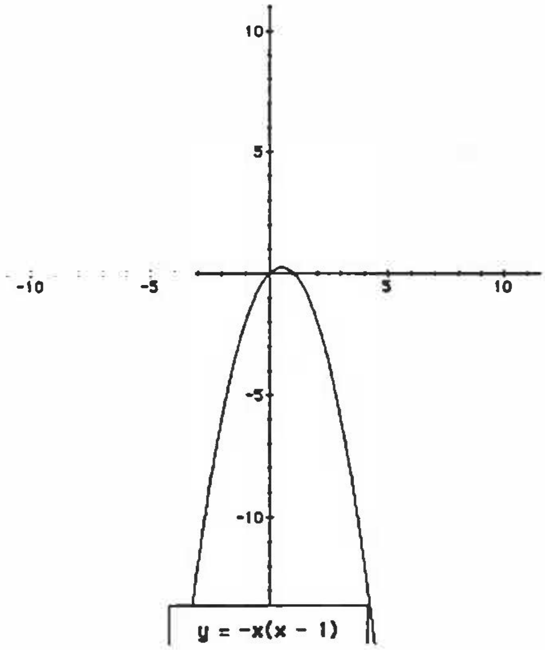
Using the pull-down menu for Zap-A-Graph gives us the following choices:



The third choice corresponds to the form *c x (1 - x)*, although even this recognition requires a certain level of symbol sophistication. Here is the dialogue box after I select that choice, with the values that I require now entered into the appropriate boxes.



Selecting the Plot option results in the following display:



Another approach is to play a bit with the algebra of the equation and see what develops. Multiplying the factors gives

$y = cx - cx^2$

A quick diversion. The above consideration of $y = c\,x\,(1 - x)$ and $y = cx - cx^2$ indicates that different canonical forms can serve different purposes. The first form is useful for noting the roots of the equation. Clearly, y is 0 when x is 0 and 1. The second form indicates the polynomial nature of the expression, emphasizing in this case that we are considering a polynomial of degree 2. Are there other forms that have other interesting properties? Note the different forms used by Zap-a-Graph in the various pull-down menus. It is interesting to speculate on what might happen if a teacher asked a class to examine the different forms provided by Zap-A-Graph and recommend when each form might be used.

Microsoft Excel

In addition to using graphing software, another approach is to think of using spreadsheets to explore mathematical topics. Spreadsheets may be used to obtain tables and graphs for a function. Using Excel, for example, one can quickly set up the necessary formulas. We will begin by setting $c = 1$ (a simple first choice—the value is stored in cell A2) and let x vary from -10 to +10 (another simple choice). Let's see what happens. Here are the cell formulas:

Worksheet1

	A	B	C	D	E
1	c	x	cx	cx^2	y = cx - cx^2
2	1				
3		-10	=$A$2*B3	=C3*B3	=C3-D3
4		=B3+1			

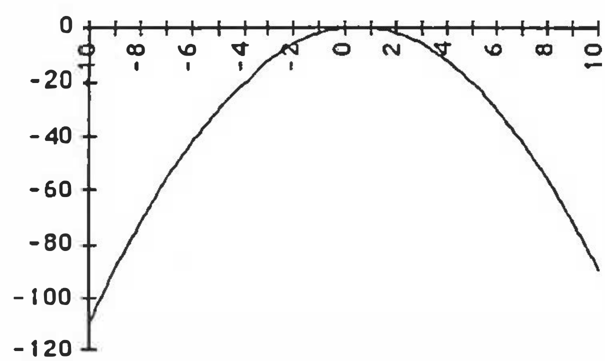
Using the fill-down command, one quickly obtains these values:

Worksheet1

	A	B	C	D	E
1	c	x	cx	cx^2	y = cx - cx^2
2	1				
3		-10	-10	100	-110
4		-9	-9	81	-90
5		-8	-8	64	-72
6		-7	-7	49	-56
7		-6	-6	36	-42
8		-5	-5	25	-30
9		-4	-4	16	-20
10		-3	-3	9	-12
11		-2	-2	4	-6
12		-1	-1	1	-2
13		0	0	0	0
14		1	1	1	0
15		2	2	4	-2
16		3	3	9	-6
17		4	4	16	-12
18		5	5	25	-20
19		6	6	36	-30
20		7	7	49	-42
21		8	8	64	-56
22		9	9	81	-72
23		10	10	100	-90

This provides a tabular representation of the function, and it deserves examination because it shows the arithmetic detail of the function. A discussion of the relative merits of the equation, the table and the graph might prove interesting because they are simply different representations of the same idea.

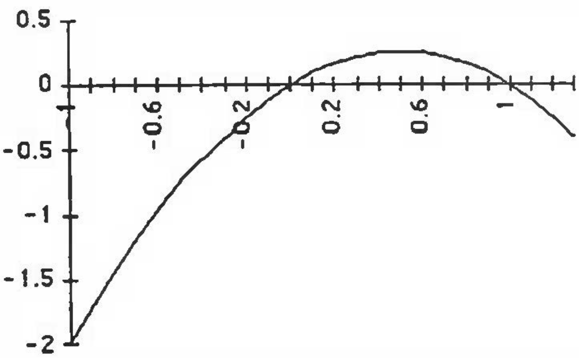
It is relatively easy to obtain the corresponding graph:



Yes, it is a parabola. It is always reassuring to get the same result using different approaches. Much of science has progressed using this principle, and it is a critical feature of most work involving computers. One should always ask, "How do I know that the result that the computer has displayed is correct?"

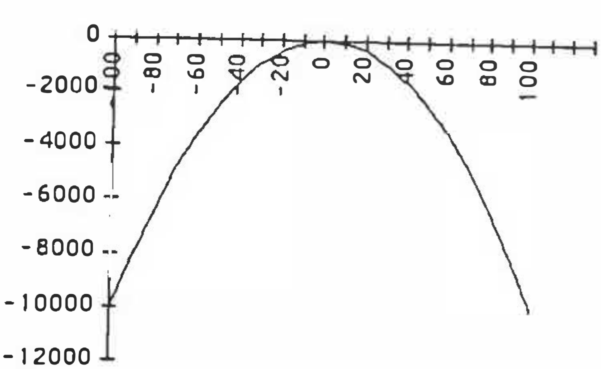
Obtaining the same result using different software packages goes a long way toward providing a satisfactory answer to the question. This is rarely emphasized in school where, quite often, only one software package is available.

One could pose many questions at this point. One such question could involve obtaining a clearer picture of what happens near the x-axis. Ideally, at least from my perspective, the students will learn to ask most of these questions themselves, with the teacher acting more as a gentle catalyst and as a resource. Notice how the emphasis shifts from getting answers to asking questions.



One might also wonder, what are the roots of this function? A close look at the graph suggests that these would be $x = 0$ and $x = 1$. These both check by substituting back into the original equation. It is not always clear why one might want to know these values, except that we have a formula for obtaining them and thus feel that we should use it whenever we can. Why do we care what the roots are?

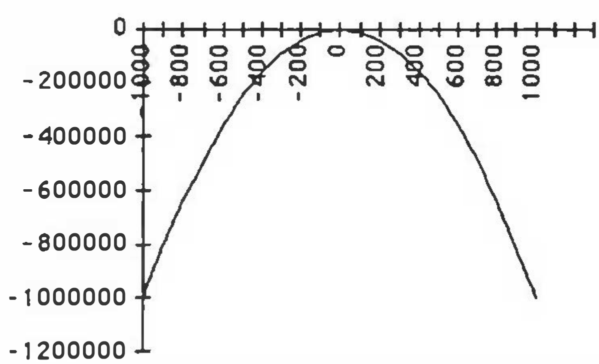
We can also explore the behavior of the function for a wider range of x values. Let's try this. For openers, let's look at the graph when x varies from -100 to +100 in steps of 10.



The table gives more accurate values:

	A	B	C	D	E
1	c	x	cx	cx^2	y = cx - cx^2
2	1				
3		-100	-100	10000	-10100
4		-90	-90	8100	-8190
5		-80	-80	6400	-6480
6		-70	-70	4900	-4970
7		-60	-60	3600	-3660
8		-50	-50	2500	-2550
9		-40	-40	1600	-1640
10		-30	-30	900	-930
11		-20	-20	400	-420
12		-10	-10	100	-110
13		0	0	0	0
14		10	10	100	-90
15		20	20	400	-380
16		30	30	900	-870
17		40	40	1600	-1560
18		50	50	2500	-2450
19		60	60	3600	-3540
20		70	70	4900	-4830
21		80	80	6400	-6320
22		90	90	8100	-8010
23		100	100	10000	-9900

Let's try -1,000 to +1,000 (in steps of 100).



Yet appearances can be deceiving (Goldenberg 1988). The scale of both axes has been changing. All graphs should be thought of as being drawn on rubber. The "shape" of the graph is in large part a property of the scales of the two axes. That is, the equation determines the "essential" features (for example, a parabola opening down), but the shape of this figure, in terms of how narrow or wide it is, is a property of the scale of the axes.

Another issue that deserves early mention is notational conventions. The idea of a standard, to be used worldwide by all mathematicians, has had a seductive lure to it. Teachers of mathematics, and their students, have also wished for such a standard. We are not even close to such an ideal. For example, there is not even widespread agreement on how to write the numerals: zero, one and seven are written differently in many European countries from the conventions of North America. Early computers exacerbated the difficulties. It was common to see the asterisk (*) to represent multiplication and ** to represent exponentiation because there was no provision for superscripting. Spreadsheets added more difficulties, because their notational conventions, while making sense within a spreadsheet environment, represented further departure from textbook and handwritten standards. Some view this as a weakness. I prefer to think of it as a strength (Burnett 1987).

We should become comfortable with a variety of conventions. As more software tools appear, the number of ways that we decide to represent commands, procedures and ideas will increase. In the past, different conventions were the result of different cultural factors and evolved on a time scale consistent with change in that culture. This is still true today, but now culture also includes the effects of technology, which not only places new demands on conventions but also has accelerated the time scale for accommodating these changes. One of our tasks as educators is to help students gain flexibility in handling different symbol systems. Another task is to help them deal with rapidly changing situations.

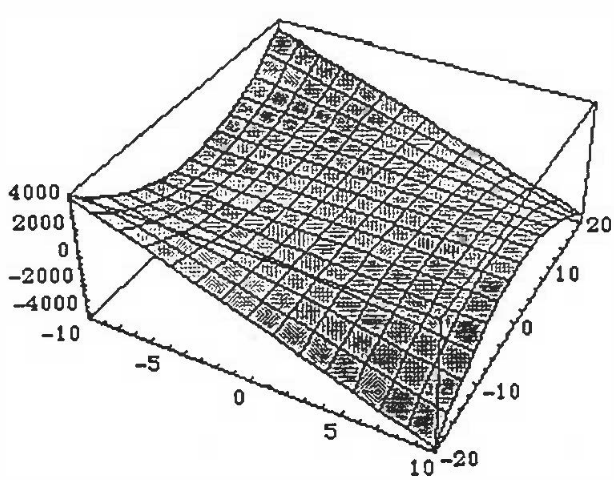
Mathematica

Recall that we have been looking at the equation $y = c x (1 - x)$. So far, we have restricted ourselves to the special case of $c = 1$.

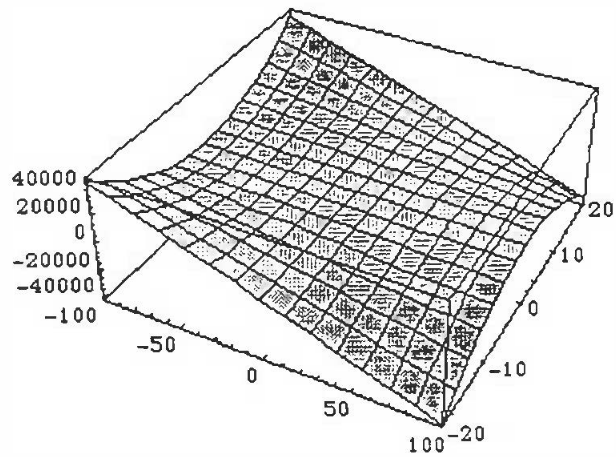
In the terminology of mathematics, x (and y) are called variables, c is called a parameter. What is the difference between a parameter and a variable? They both vary.

Here are some graphs of $y = c x (1 - x)$ where y is the vertical axis, c is the axis along the left side of the base and x is the axis along the right side of the base.

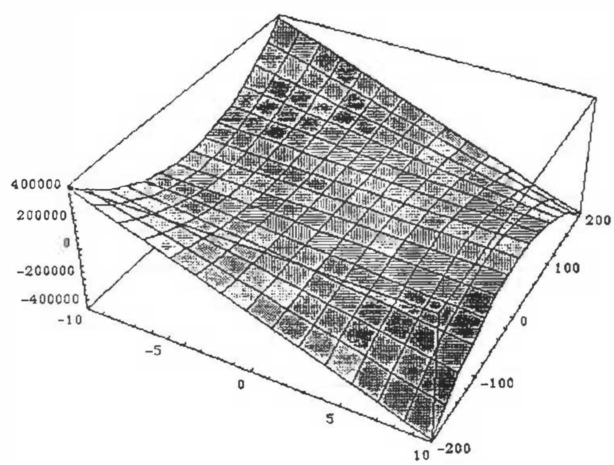
For the first figure, c varies from -10 to +10 and x varies from -20 to +20:



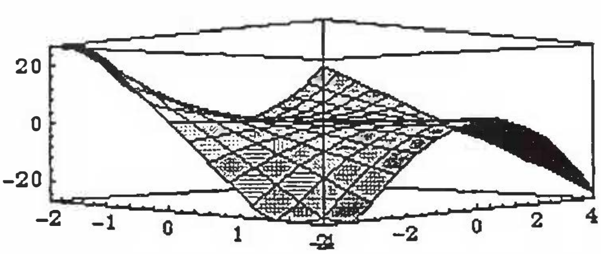
Now let's look at the case where c varies from -100 to +100:



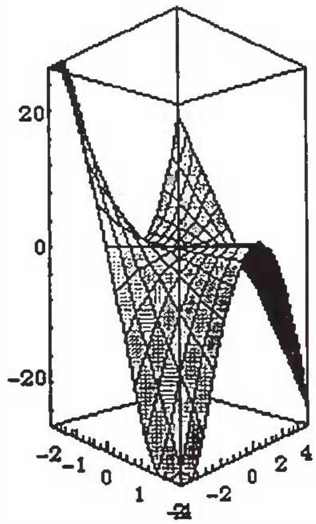
In the next figure, c varies from -10 to +10 and x varies from -200 to +200:



Now imagine that we are at "eye level" and that c varies from -2 to +2 and x varies from -4 to +4:



Here is the same figure with the axes drawn to a different scale:



Essentially what we see in each figure is an infinite number of parabolas, some opening down (when c is positive) and some opening up (when c is negative). Each previous parabola drawn using Zap-a-Graph or Excel is simply one possible vertical "slice" from the surface. The three-dimensional representation is a much more comprehensive and powerful way of envisioning this mathematical situation, namely as a surface rather than as a curve. Technology makes this possible.

What do we see? Open up, open down, roots, steepness, curvature. Let's play with the notions of steepness and curvature for a bit: calculus.

What is the derivative of this surface? What does it look like? One can ask Mathematica to compute the derivative of an algebraic expression. If we define a function F by typing in the following expression:

F[x_]: = c x (1 - x)

then type

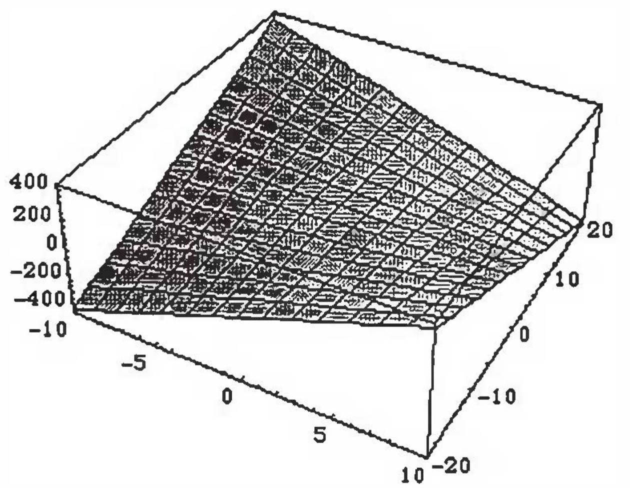
D[F[x],x]

we receive the following result:

c (1 - x) - c x

Once again, we must use a slightly different set of notational conventions for specifying a function. These conventions are necessary because one must take a different set of factors into account when designing a software environment from those that one must consider when using pencil and paper.

We can then plot this function, obtaining the following:



In this case, c varies from -10 to +10 and x from -20 to +20. What do you see? Does this figure make sense?

The above activities give the complete picture for the equation $y = c\,x\,(1 - x)$ for all possible values of x and c. It is also easy to imagine rotating this figure dynamically, such as many CAD programs illustrate.

Let's stop to catch our breath. What has happened so far?

First, we have used the technology to do all the labor. It has drawn the graphs and even computed the derivative.

Second, the technology has permitted us to think at a higher level of abstraction. Instead of thinking about a single quadratic expression, we have examined a continuous family of possibilities, moving from a consideration of two-dimensional graphs to three-dimensional graphs.

Third, to use the different software packages, we have had to learn the notational conventions of each.

Fourth, the emphasis has been on visualization, particularly "deep visualization" with its strong emphasis on understanding. There has been no attention to manual skills. Mathematics education must include activities that help us to "see," to understand and interpret what we see. *Envisioning Information* (Tufte 1990) exemplifies this concern for meaningful representation. This emphasis represents a significant departure from the current curriculum that still devotes a major amount of effort to the development of manual skills, skills that may no longer be necessary (Burnett 1992).

But technology is not the answer. It is a tool for exploring ideas. The tool still requires a human operator. Exploration requires an active mind. What does one "see" when one views the above surfaces? What new questions come to mind?

Mathematics is about asking questions. Learning is about asking questions. Have you asked any good questions lately?

Now that we are rested, let's explore this equation in a couple of other ways. One is to note its use in nonmathematical contexts, particularly biology and ecology. Students often ask, "Why are we studying this?" Section 2 provides an introductory answer to this question for the case of the function $f(x) = c\,x\,(1 - x)$. Another approach is explore the mathematics of this equation in a different light, the light of iteration. Such explorations require technology to provide the labor-intensive calculations. Section 3 provides an entry into a whole new world of mathematics, one currently of interest to many of the world's leading mathematicians.

Section 2 Biology

I remember reading that biology was the study of dead things. It was a cynical attack on science education, but one that had an aspect of truth to it. Perhaps there is some comfort in the realization that other parts of the school curriculum are also under pressure to change. What many school students fail to realize is that the study of biology also involves the study of artificial things, things that were never living or dead (Levy 1992). Enter the world of mathematical modeling and simulation.

A sound strategy for entering a new field is to begin with something simple. Let's consider the population dynamics of a single species, for example, rabbits.

Let $P(n)$ represent the population of rabbits at time n. Clearly, the actual value of $P(n)$ depends on many factors: weather, availability of food supply, number of predators, disease and so forth. It also depends on the number of rabbits around at time $n - 1$. A very simple first approximation to this situation is the following function:

$$P(n) = c\,P(n - 1)$$

Notationally, this expression can be a bit confusing: do the brackets symbolize the argument of the function P, or multiplication? Thus it is usual to switch to a form of subscript notation. Notational conventions are not just a function of technology, they have always been with us:

$$P_n = c\,P_{n-1}$$

At the same time, it is a good idea to change the meaning of P from a whole number representing the total number of rabbits to a proportion of some arbitrary large upper limit of rabbits. Thus P now takes values between 0 and 1. Proportions are often easier to deal with because we all know what a proportion of 0.8 means (that is, 80 percent of the maximum possible population), whereas a value of 3 million rabbits still leaves a person with a sense of "So what?" Is that a large number, a small number or a typical number? This same principle applies to many figures we receive when the media discuss the economy and our present preoccupation with budgets. A department is asked to trim $2 billion from its budget. Is this a large number, a small number or a reasonable number?

If c is 1, the population remains constant, since $P_n = P_{n-1}$.

If c is larger than 1, it is fairly easy to see that the population grows without bound. This may make sense for a while, but, at some point, other factors such as a dwindling food supply should start to kick in. The model does not seem realistic. Indeed, at some point, the proportions will exceed a value of 1, which is meaningless. If c lies between 0 and 1, the population continues to dwindle and eventually will be fractional. Negative values of c make no sense in this context. Thus the model needs to be adjusted to prevent the case of runaway growth. We need a term that reduces the growth as P gets large. One approach is to add a second multiplicative term, $(1 - P)$:

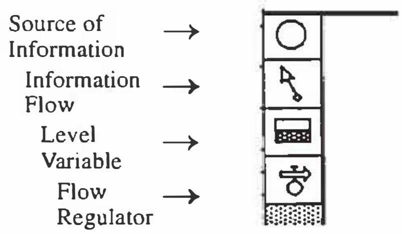
$$P_n = c\,P_{n-1}\,(1 - P_{n-1})$$

This is the same equation that we have been discussing in the preceding sections.

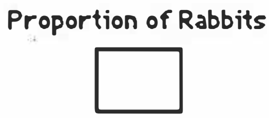
We have approached the question of simple population dynamics from an algebraic perspective. We might just as easily have approached it from a geometric perspective. What we are looking for is a graph that increases for a while and then decreases. One such curve is a parabola that opens downward. Once again, we can end up with the same equation.

Let's look at a third approach using a simulation modeling package called STELLA II. STELLA is an acronym for Systems Thinking, Experiential Learning Laboratory, with Animation. It takes a particular approach to modeling, known as system dynamics. The original Club of Rome report (Meadows et al. 1972), one of the first documents to warn us of the dangers of unlimited industrial expansion, used this approach. The current version of STELLA uses a small set of icons that can be placed anywhere on the screen and joined to other icons to create a flow model.

Two of the four icons represent movement of information; the other two represent movement of some conceptual quantity.



First, we create a level variable (like a bathtub) that contains the level of proportion of rabbits. We simply select the Level Variable icon, drag it onto the screen and type a name to give the icon an identifying label.

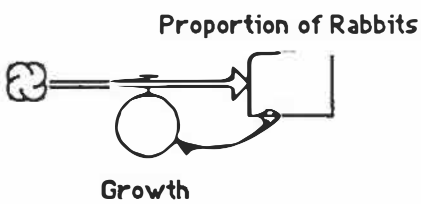


It is simply a box that will show how full it is over time. Opening the box (by double clicking on it) brings up a dialogue box where the user specifies the initial value for this level. I will start with it half full:

INITIAL(Proportion_of_Rabbits) = ...

.5

Next, we need to specify the rate of increase in rabbits. This is accomplished by means of a flow indicator:

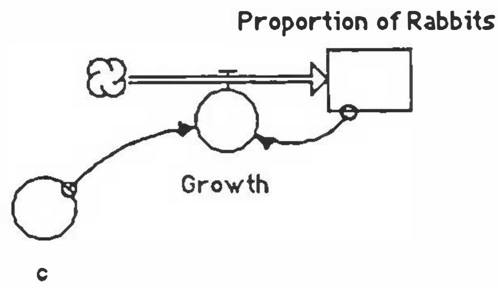


The icons are intended to represent a tap (regulator) that controls the rate of growth of the rabbits. The "cloud" symbol on the left simply means that there is a source of rabbits to begin with. The feedback connector arrow joining the level indicator to the tap is the heart of the model. This permits one to use the current level as a variable controling the tap for the next cycle. Opening the tap calls up another dialogue box where the equation can be typed in:

Growth = ...

Proportion_of_Rabbits*(1-Proportion_of_Rabbits)

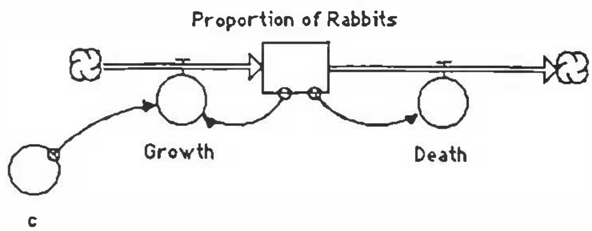
We still need to include the constant c which is intended to take all external factors into a global growth constant:



Opening up the circle icon permits me to specify the initial value of c (as 2).

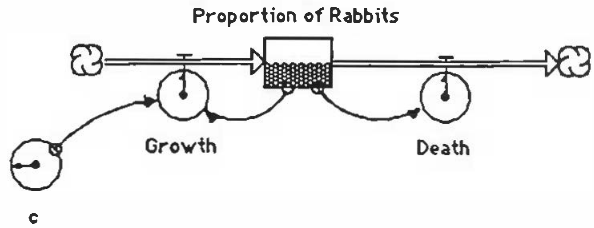
A brief digression on acceptable values for c is appropriate. Clearly, c must be greater than 0, since negative values make no sense in this context (all proportions must remain positive). But if c is too large, the proportions may rise above 1, which also is nonsense. The product of $P(1 - P)$ is a maximum when P is 0.5. This maximum is 0.25. Because c times 0.25 cannot exceed 1, c cannot exceed 4. Thus we may substitute any value of c between 0 and 4 into the model. A discussion of the boundary conditions of any mathematical situation is an important aspect of mathematical understanding.

There is one final step. The model as currently specified represents growth without end. We also need to include a "death" factor. This is also represented by another tap, indicating the outflow.

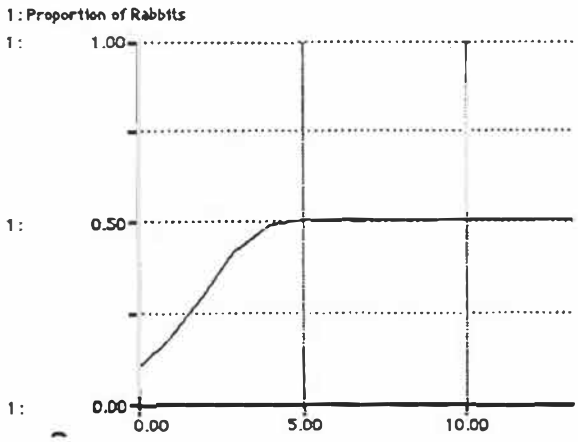


Now that the model is constructed, we need to run the simulation. STELLA maintains a graph of the values as the simulation is run over a specified time (for example, 20 cycles).

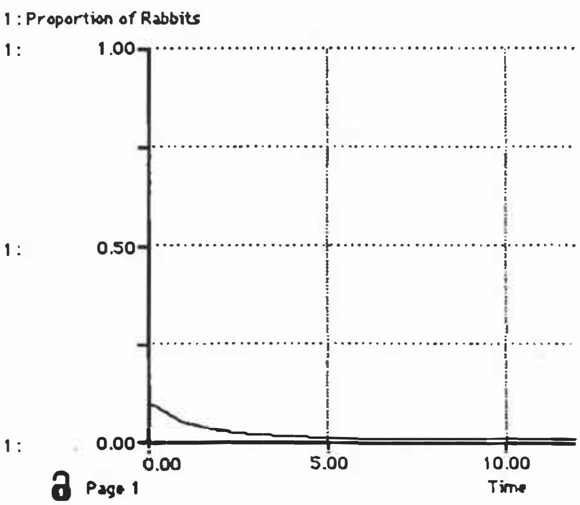
With c set to 2 and the initial value for the Proportion of Rabbits set at 0.1, we obtain the following graph, where the population level quickly approaches a value of 0.5. The final display looks like this:



It is important to realize that the viewer watching the computer screen sees the level rise to the halfway level and then remain constant (it is also possible to ask for a graph that shows how the level changed over time):

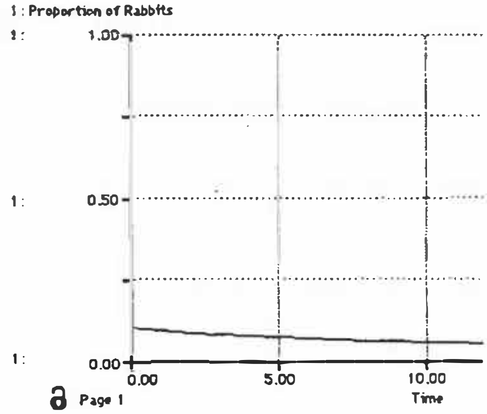


Leaving the structural equations unaltered, one can play with different initial values for the proportion of rabbits, and with the value of c, the constant. Let's see what happens if we play with different values of c. Here is the graph when c is 0.5.

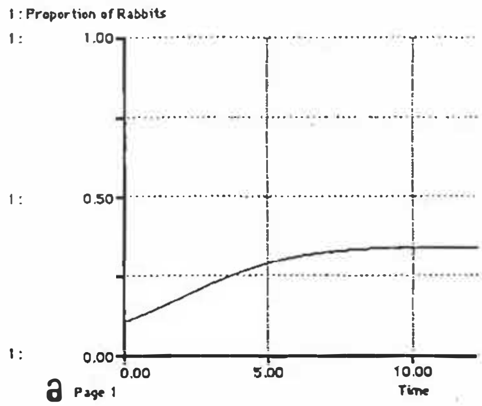


With such a low value of c, even a proportion of 0.1 is not sustainable.

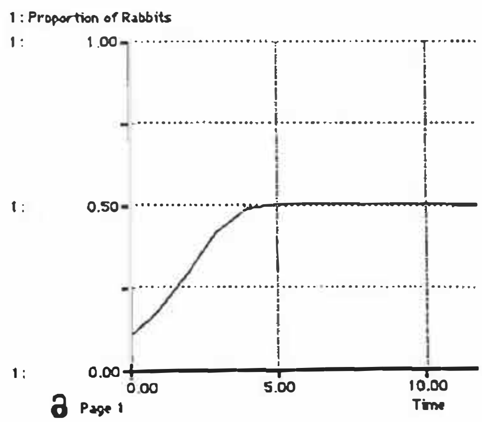
Setting $c = 1$ yields the following graph:



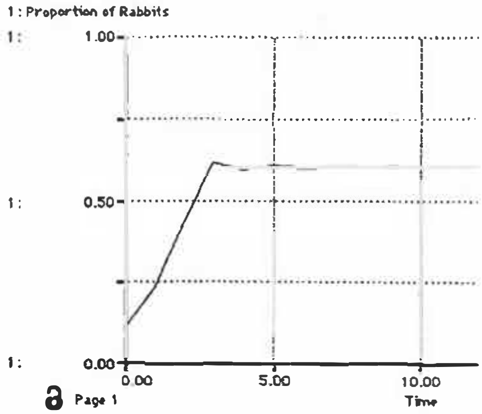
This is a little better, but the proportion of rabbits is still declining. Let's continue up in steps of 0.5. The next case is for $c = 1.5$:



For $c = 1.5$, the proportion of rabbits increases steadily to a value close to 0.3. Recall the graph for $c = 2$:

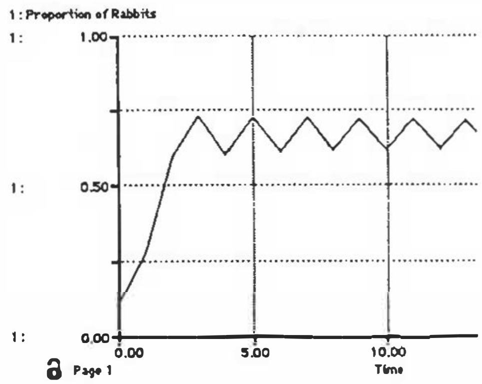


Here the proportion of rabbits increases steadily to a value of 0.5. Now consider the case where $c = 2.5$:



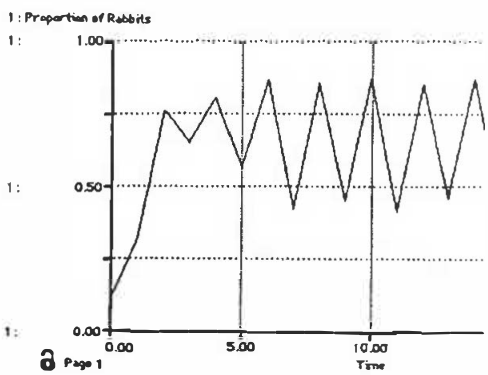
The pattern here is slightly different: there is a mild form of oscillation during the first few cycles before the proportion stabilizes at a value near 0.6.

Here is the graph for $c = 3.0$:

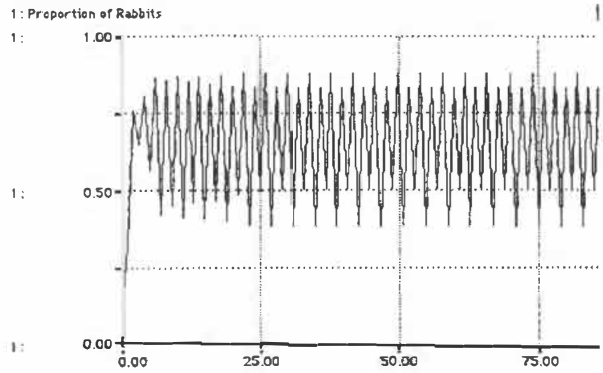


The pattern here is dramatically different. The population of rabbits appears to cycle through a set of values, without converging toward a limiting value.

Here is $c = 3.5$:



Expand the number of cycles from 10 to 100 to get a better picture of what happens in the long run:

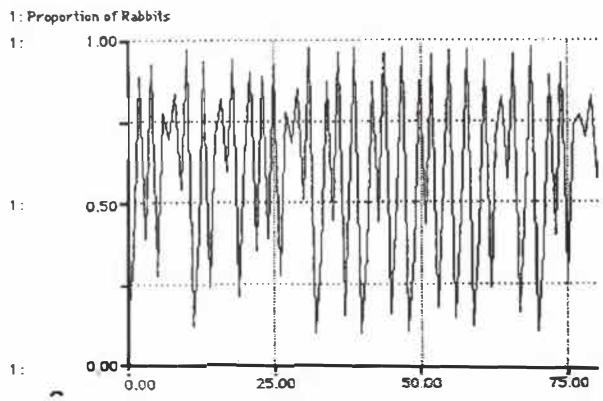


This appears cyclical, but it is difficult to tell by just looking at the graph. The software also provides a corresponding table. Here is a section of the table for the periods from 70 to 78:

Time	Proportion of Rabbits
70.00	0.87
71.00	0.38
72.00	0.83
73.00	0.50
74.00	0.87
75.00	0.38
76.00	0.83
77.00	0.50
78.00	0.87

Examination of the table reveals a cycle of period 4.

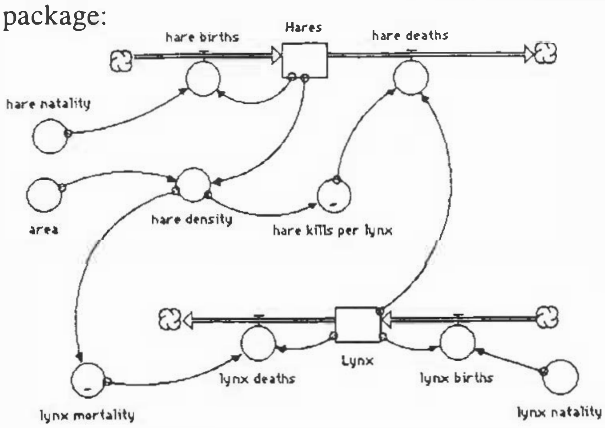
Finally, let's look at the graph and a portion of the table for $c = 3.9$:



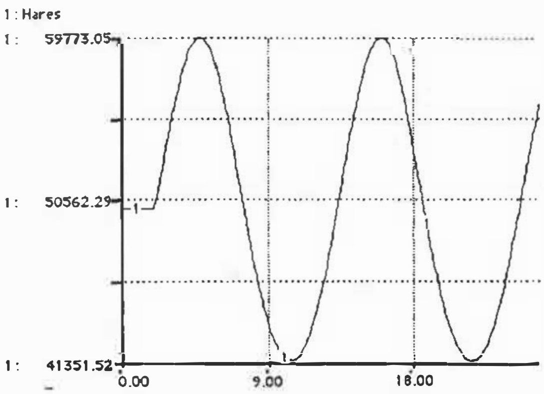
Time	Proportion of Rabbits
70.00	0.10
71.00	0.35
72.00	0.89
73.00	0.39
74.00	0.93
75.00	0.25
76.00	0.73
77.00	0.77
78.00	0.70
79.00	0.82
80.00	0.57
81.00	0.96

There is no discernible pattern at all! The population dynamics do not settle down, as in the other cases. The only parameter that has been altered is the value for *c*. Yet the mathematics of this model leads to chaotic behavior as *c* approaches 4. The chaos is inherent in the mathematics; it is not due to complications in the model caused by other factors.

It is also possible to include other factors, notably the lynx, to make the model more realistic. Here is a sample model that comes with the STELLA software package:



Here is a sample graph run for this model:



Although the model is considerably more complex, the graph is more regular and periodic than in the simpler case of $c\,x\,(1 - x)$ when *c* was set to 3.9.

One must distinguish between the complexity of the model, which is a biological issue, and the complexity of the output, which may be due to the complexity of the model or it may due to the inherent complexity of the mathematics. Because we have little experience with iterating functions, we have yet to acquire a sophisticated intuition of what to expect under different conditions. Certainly, most people, including most mathematicians, did not suspect that iterating a function as simple as $c\,x\,(1 - x)$ could lead to such chaotic results.

I will take one more quick look at the mathematics underlying the simple model where *c* took on different values between 0 and 4. Such an investigation leads into a new topic—the study of chaos.

Section 3 The Unexpected

This section is a brief introduction to the mathematics of chaos. Let's return to the function

$$f(x) = c\,x\,(1 - x)$$

and examine its behavior under iteration. We have two tools at our disposal, Microsoft Excel and Mathematica. Let's begin with a spreadsheet approach and then see if Mathematica can provide some additional insights.

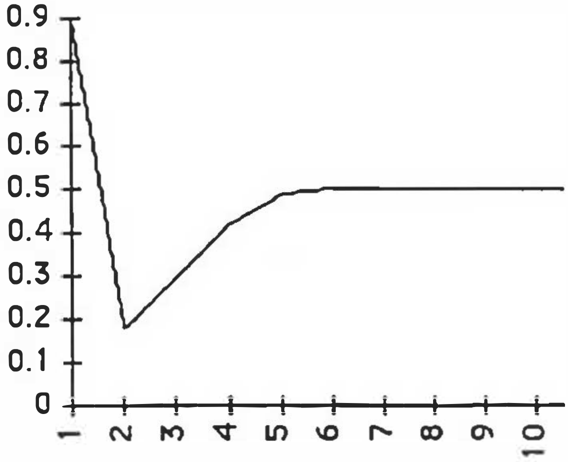
The only parameter we will change will be *c*. In the following examples, we will begin with an initial *x* value of 0.9. We will examine three different functions:

$$f(x) = 2\,x\,(1 - x)$$
$$f(x) = 3.5\,x\,(1 - x)$$
$$f(x) = 3.9\,x\,(1 - x)$$

Although these functions only differ by a small amount, their behavior under repeated iteration is unexpected.

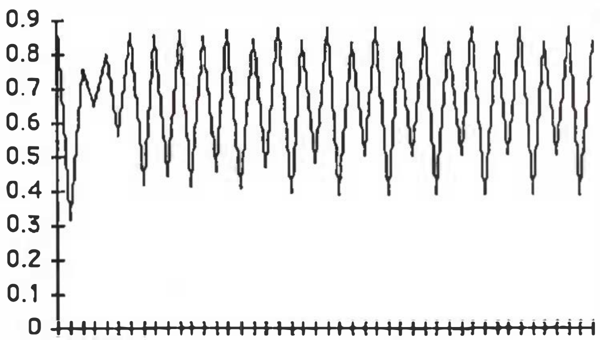
Setting $c = 2$ in Cell A2, setting the formula =$A$2*B3*(1-B3) in cell C3, setting 0.9 in cell B3, setting the formula =C3 in cell B4, and then filling down results in the following table and corresponding graph:

	A	B	C
1	c	x	cx(1-x)
2	2		
3	1	0.9	0.18
4	2	0.18	0.30
5	3	0.30	0.42
6	4	0.42	0.49
7	5	0.49	0.50
8	6	0.50	0.50
9	7	0.50	0.50
10	8	0.50	0.50
11	9	0.50	0.50
12	10	0.50	0.50
13	11	0.50	0.50



First, it is important to note that we are no longer dealing with parabolas. We are looking at the behavior of a function under repeated iteration, where the value of the function at cycle n becomes the argument of the function at cycle $n + 1$. In fact, this is the same graph that we obtained earlier when we were using STELLA. As a reminder, because $c = 2$, we are looking at the function $f(x) = 2x(1 - x)$. The second point to note is that we began the iteration with the initial x value of 0.9. The above graph is called the orbit of the point 0.9 for the function $2x(1 - x)$.

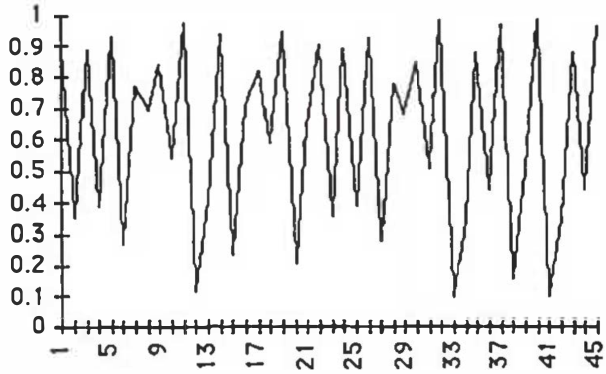
Let's look at two more examples using Excel. First is the case where $c = 3.5$, and the first value of x is 0.9. Here is the orbit of the point 0.9 for the function $3.5x(1 - x)$:



It is dramatically different from the orbit for the function $2x(1 - x)$, yet the only difference between the two functions is the leading coefficient.

A review of the table indicates that the iterations converge toward a cycle that repeats itself every fourth time (that is, 0.87 - 0.38 - 0.83 - 0.50 - and so on).

If we change c to 3.9, we obtain the following chart:

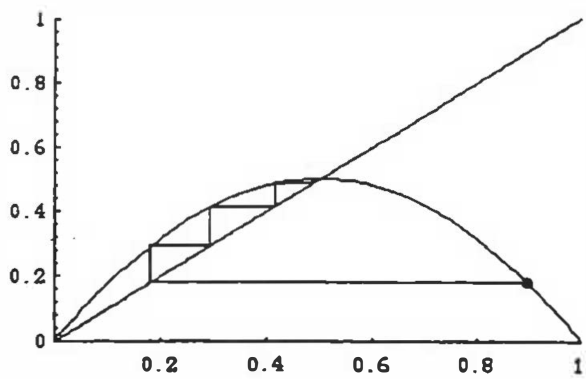


This time, there is no recurring pattern. The graph is said to be chaotic.

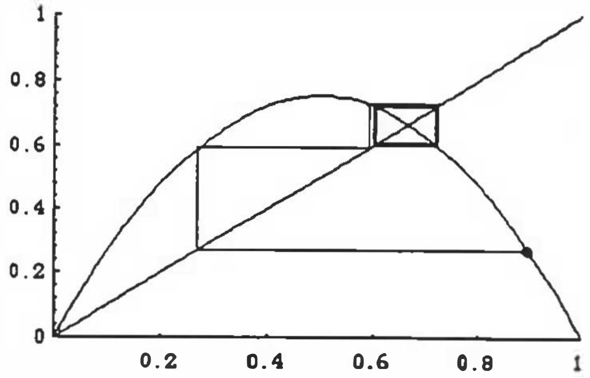
It is possible to explore any function to see its behavior under iteration. Devaney (1990) provides a rich sampling of problems for the beginner, examples easily within the range of high school students.

There is also a graphical procedure for examining the behavior of any function under iteration. The idea is fairly straightforward. Draw both the function of interest, call it $f(x)$, and the function $g(x) = x$ on the same grid. Then select any point x that you wish to begin the iterations with. Locate this point on the line $y = x$. This will be the point (x,x). Draw a vertical line joining this point with the graph of $y = f(x)$. This will be the point $(x, f(x))$. Now draw a horizontal line joining this point to the graph of $g(x)$. This will be the point $(g(x), g(x))$. Now repeat the process. Here is an example:

Consider the function $f(x) = 2x(1 - x)$.
Suppose we begin with $x = 0.9$. Then $f(0.9) = 2 \times 0.9 \times 0.1 = 0.18$. This represents the starting point of the iteration. Here is a graph showing the path of the graphical analysis and produced using Mathematica. Note that the path quickly converges to a value of 0.5. This is another way of viewing the situation that we previously examined using Excel.

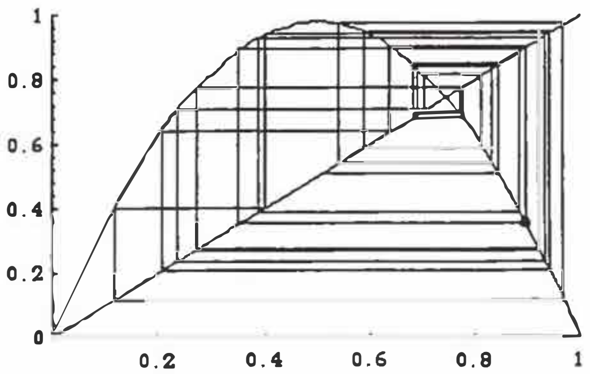


Now let's re-examine the function $f(x) = 3.5x(1 - x)$. Once again, we will begin with $x = 0.9$:



This time the orbit shows a cycle of period 4.

Finally, let's look at $f(x) = 3.9x(1 - x)$, beginning with $x = 0.9$:



The orbit fails to settle down. This is another view of mathematical chaos.

Summary

Pagels (1988) provides a thought-provoking analysis of developments in science over the last decade, using the phrase "sciences of complexity" to capture this new perspective and identifying a number of themes to characterize this new approach to science. His first theme is "the importance of the computer. . . . One of the ways that future science will progress is by a combination of precise observations of actual systems followed by computer modeling of those systems" (p. 43). He goes on to include "the computational viewpoint in mathematics" and then "the rise of computational biology" and "the study of nonlinear dynamics" and concludes with "the study of complex systems" as other themes characteristic of new approaches to science. The book ends with the sentence, "The future, as always, belongs to the dreamers." What are we doing in education to encourage such dreaming?

There is much concern from many interest groups about the present state of our classrooms, and mathematics courses take their fair share of the spotlight. Suggestions for improvement fall into three main categories:

1. How can we improve our present pedagogy?
2. What should we delete, to have more time for an in-depth exploration of what is left?
3. What should we add, because the new topics represent an important part of our evolving knowledge?

This article falls into the first and third categories. While it is true that I have used fairly sophisticated software packages as an integral part of my pedagogy, the main pedagogical thrust has been an open-ended exploratory approach with a conscious effort to understand this function as much as possible. Thus I have attempted to provide a metacognitive perspective to my own investigations. The article might also be viewed as a form of portfolio, a record of my explorations to date. This leads naturally into considerations surrounding portfolio assessment and authentic assessment. Perkins (1992) has written a stimulating book that attempts to address the issue of educational reform. His basic claim is that we need to focus on the curriculum and that our criterion should be deep, meaningful learning. I would like to think that this article represents a tentative beginning toward such an approach.

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Microsoft Excel Version 3.0. Microsoft Corporation, Redmond, Wash.

STELLA II Version 1.00. High Performance Systems, Inc., Hanover, N.H.

Zap-a-Graph Version 4.2. Brain Waves Software, Fitzroy Harbour, Ont.

Mathematically Speaking: Communication in the Classroom

Marie Hauk

One key standard proposed by the National Council of Teachers of Mathematics in *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) is "Mathematics as Communication." *Professional Standards for Teaching Mathematics* (NCTM 1991) offers direction for teachers to address this standard through their teaching practices.

Focusing on communication requires neither a defensive position nor a fabrication of connections between language and mathematics learning. Indeed, the burden of proof should be put on those who wish to dichotomize them. Whether implicitly believed or explicitly stated, connections between language and mathematics are pervasive. Significantly, however, these connections are open to wide interpretation; that is, they depend on how one perceives language, mathematics and learning. This view may be illustrated by three scenarios taken from personal experience.

In the first instance, I was in a store and overheard a conversation between a salesperson and a customer. I came along just as the salesperson was telling the customer that he was a former high school mathematics teacher and what his philosophy of teaching was. "I see mathematics as a language," he said, "and I taught it that way. I taught the grammar of mathematics." The salesperson's perceptions of mathematics and language were evident.

The second scenario involves a student who was taking a course in teaching English as a second language at the same time she was in my curriculum and instruction course in junior high mathematics. On several occasions, she mentioned many similarities between the two courses and that my views of learning were much like those of the other professor. In particular, she noted our mutual emphasis on the need for context and the role of personal experience. Interestingly, when I had a conversation about this with the other professor, he expressed surprise that there were connections between learning a language and learning mathematics. He saw mathematics as comprising rules and facts best learned in a rote fashion. This reminded me that I have encountered teachers who have embraced the principles of whole language learning, yet who still think of mathematics as basically a series of algorithmic procedures such as long division.

The third example arose from an exam question that addressed the importance of using manipulative materials in mathematics. One student gave the following response:

> Most of our language is taught at first with symbolic ideas. "Horse" is given meaning in the presence of the animal or a picture of one. No one tries to teach children what a horse is without some sort of representation. If we make the language of mathematics more tangible, the concepts therein will be more easily learned.

This was a future teacher who was surprised and delighted when he found that even he could benefit from the use of concrete materials.

These incidents show that cognizance and acceptance of the connections between language and mathematics may require re-examination of fundamental beliefs about learning. The role of language in learning mathematics may be clarified by illustrations of how teachers may implement guidelines offered by the NCTM for "Mathematics as Communication."

Use Problems and Materials to Promote Communication

If you want to cultivate communication among your students, you must provide problems or tasks that stimulate discussion. Some problems are more likely to stimulate discussion. This is not due to the mathematical understanding required to solve the problems but rather to other features inherent in them. Ask students to clarify and justify their ideas orally and in writing by relating them to models such as manipulative materials, pictures and diagrams. Where appropriate, the use of concrete materials fosters language and mathematical vocabulary development because children can talk about what they are doing with the materials and see how the concrete action is related to written symbols. While inventive teachers may enjoy making up appropriate assignments, many textbook problems may be adapted to include the particular features discussed in this article.

Problems that require the solvers to supply data necessitate collaboration. Requiring each student in a group to contribute to a problem before it can be solved allows less verbal students to enter into discussion naturally and fosters continued participation. Some typical "involvement" problems are listed below:

- If you each take enough candles for your next birthday cake, how many candles will you need in all for your group? (Model: real birthday candles)
- If all the students in our class bring their sisters and brothers to our class picnic, will the ratio of boys to girls at the picnic be the same as the ratio of boys to girls in our class? (Model: two colors of chips or blocks to represent girls and boys)
- Estimate how many minutes it takes each group member to get to school. Organize and display your data. Find the mean, median and mode. How could this information be useful? (Model: stem and leaf plot)
- Each person selects any two natural numbers whose difference is 2. Find the product of the two numbers. Find the square of the average of the two numbers. Compare the answers of all members in the group. Is there a pattern? Can you show if the pattern would be true for any such pair of numbers? (Model: chart)

Problems that include extraneous conditions and/or data promote communication because students must agree on what information is required to solve the problem.

- Sam had 8 hockey cards and 13 baseball cards. Jenny gave him 5 more hockey cards. How many hockey cards did Sam have then? (Model: hockey cards)
- There are 8 members in a chess club. There are 5 girls and 3 boys. How many different ways can the members pair up to play? (Model: name cards)
- A square garden was enclosed by a fence attached to 2-m high posts 5 m apart. If 20 posts were used to make the fence and 6 posts were left over, what is the area of the garden? (Model: diagram)
- Bill paid $6.10 for his lunch of 2 hamburgers and a soda using only dimes and quarters. He used 9 more quarters than dimes and had 5 dimes left in his pocket. How many quarters and dimes did he use to pay for his lunch? (Model: chart)

Problems that have more than one answer stimulate discussion. When two or more students arrive at different answers to the same problem, a common assumption is that only one of them is correct. "Show us how you got your answer" is a natural instruction that requires each student to justify his or her answer to the other students in the group. Some multiple-answer questions are listed below:

- Tom said, "I have 6 coins worth 30¢ altogether. What coins do I have?" (Model: real coins)
- What is the next number in this pattern?
  [2] [3] [5] [8] [ ]
  (Model: chips to represent each number in the pattern)
- A bag of cookies can be shared fairly among 2, 3, 4, 5 or 6 friends. How many cookies are in the bag? (Model: chips)
- Find seven numbers that have a median of 10, a mode of 5 and a median of 8. (Model: chips and stem and leaf plot)

Problems that may have more than one interpretation provide good catalysts for communication. In mathematics teaching, we tend to avoid ambiguity. If a teacher or student finds that a problem does not seem to be stated clearly, we may disregard it on that basis. The intention is good: we want to be very clear so as not to confuse our students. One reason for this has been the focus on one correct answer. In math, however, as in other aspects of everyday life, when everything is clear and agreeable, little remains to be said. Arguments and opposing opinions tend to generate more talk and thus more language. Examples of this problem type are shown below:

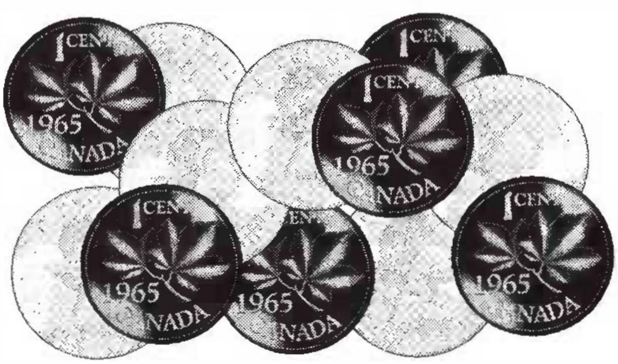
- Jaime ate 3 chocolate raisins and 1 jelly bean. Dale ate 2 gum drops. Who ate the most candy? (Does *most* mean how many candies in a discrete sense, or is the size of the candies important?) (Model: real candies)
- You each have been given a carrot. Who has the biggest carrot? (Students must come to a consensus on how to determine *biggest*. They may use length, mass or diameter as the attribute for comparison.) (Model: real carrots)
- How many different ways can you make double-decker ice cream cones with 8 flavors of ice cream? (Would chocolate on top of vanilla be *different* from vanilla on top of chocolate?) (Model: colored chips)
- There are 99 lockers in a school hallway. Every other locker has a poster on it. Every third locker is unlocked. How many lockers have a poster and are locked? (Does every other locker mean that you start with the first or the second locker?) (Model: diagram)

Problems that require active sharing of decision making or materials for the solution process cannot be solved unless each student in the group understands the strategy being used, such as in the following examples:

- If each person chooses an Attribute block, can you line up them so that each block is different from the block in front of it in exactly one way? (Model: Attribute blocks)
- I will draw four cards, one at a time, from a set of numeral cards from 0 to 9. Work as a team to make the largest 4-digit number possible. You must agree on the place value of each numeral as it is drawn. (Model: sets of numeral cards)
- Each group takes a geoboard. The first person makes the smallest possible square on the geoboard. The area of this square is one. In turn, each person in the group must make a square with a different area and explain how to find the area of that square. (Model: geoboard)
- One person in the group chooses any 2-digit number and any 3-digit number. Using the constant multiplier on the calculator, the others take turns estimating what number to multiply the 2-digit number by to get as close as possible to the 3-digit number. This can be carried out for as many decimal places as possible. (Model: calculator)

Problems that have no correct answer evoke the response, "I can't do it!" The value of such problems lies in the students' justifications for why it can't be done. Four examples of these unsolvable questions are given below:

- Use 14 toothpicks to make a square. Place them end-to-end to make the sides. (Model: toothpicks)
- You have 35¢ in pennies and nickels. If there are 6 coins, what are they? (Model: coins)
- Seven students wrote a math test. The average score was 74 percent. If the marks of six of the students were 72, 61, 88, 93, 54 and 46, what was the mark of the seventh student? (Model: chart)
- A man is three times as old as his son. The sum of their ages is an odd number. How old are they? (Model: chart)



Nurture the Development of Mathematical Terminology

Help students relate their everyday language to mathematical language and symbols. Listening to students when they use their own words to communicate mathematical ideas is as important as talking to them. Encourage them to articulate, through their own language, what they understand implicitly. If we begin with their language, we can help them clarify their understandings and introduce formal mathematical terms and symbols when appropriate. This approach, as illustrated by the following examples, is applicable at all grade levels.

My young son was playing with a pail of plastic magnetic letters, which he dumped on the kitchen floor. I said, "Thomas, please pick them up because we are having supper now." He answered, "But I'm rhyming them." I didn't really have to look to see what he was doing because his language made it fairly obvious. While children are natural sorters, they do not naturally use formal terminology. That is, he did not say, "I am classifying the alphabet."

Picture a Grade 5 student sorting the figures in a set of cardboard polygons. "These shapes are the same," he says, "because they all have four sides and square corners." The teacher responds, "Yes, Steve, you are correct. We call these polygons rectangles."

A junior high school student is asked if she can give answers to the following: 2×7, 4×6, 10×5, $3 \times n$. She responds, "I can answer the first three: $2 \times 7 = 14$, $4 \times 6 = 24$, $10 \times 5 = 50$; but I cannot do $3 \times n$ because I don't know what n is. It can be different numbers." The teacher replies, "That's a good explanation, Cindy. Another way to say that things can be different is to say that they vary, so we call n a variable."

A Math 20 class was using graphing calculators to explore the effects of a on the graph of $f = ax^2$. Describing the changes, one student said, "The graph gets narrower or wider." His teacher agrees, "That's how it looks. In math, we say it compresses or expands."

Difficulties may arise when words have meanings in everyday life different from their meanings in math. The English language has many examples, such as *difference*, *mean*, *root* and *square*. Some words have similar meanings in math and everyday life. *Divide*, *intersect* and *cone* have specific mathematical meanings, but they are related to everyday interpretations. On the other hand, some words such as *decimal*, *polygon* and *integer* are mathematical terms.

Make Discussion a Natural Part of Learning Mathematics

Children should come to realize that discussion is a natural and important part of learning and using mathematics. Discomfort with talking about mathematics leads to mathematics anxiety, a very real problem in our society. Not only is this anxiety prevalent but also it is considered okay to feel that way. In fact, some people even brag about their inability to do mathematics. Mathematics often is portrayed as a rigid, formal system of concepts and skills that are difficult to understand if you are *not mathematically minded.* Once learned, they must be drilled [to be remembered] and are applied through precise algorithmic processes.

Communication in the mathematics class has tended to be a one-way process, from teacher to students. Communication from students to teacher usually has been repeating what the teacher said. This acts as a verification that students understood what the teacher said. Few adults recall being asked for an *opinion* about a mathematical concept or being given a *choice* about which procedure they preferred to use.

Viewing mathematics learning as an individual pursuit means that looking at your neighbor's work is cheating—after all, there is only one correct method and only one correct answer. Working in a small group is less threatening than speaking up in front of the class. Students may be surprised by the different procedures that can be used to solve a problem, and they can clarify their own thinking when justifying their methods to others. Interactive communication in mathematics class does not just happen. Teachers must plan opportunities that foster discussion, model appropriate mathematical language themselves and evaluate their students' progress for the purpose of enhancing communication skills.

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Classroom Activities

Compiled by Betty Morris

The following activities were part of a collection of activities available at the NCTM Canadian Regional Conference held in October 1994. These activities were submitted by teachers or found in *The Arithmetic Teacher*.

The Geochart

This interesting calculator activity uses an unfamiliar version of a familiar item.

On the geochart, make a 2 × 2 square. Look at the sums of diagonal corners. What is the difference in the sums? Look at the products of diagonal corners. What is the difference in the products? Repeat the process for a different 2 × 2 square.

Repeat the process with 3 × 3 squares. Do the same for 4 × 4 squares and 5 × 5 squares. Can you generalize the pattern to $n \times n$ squares?

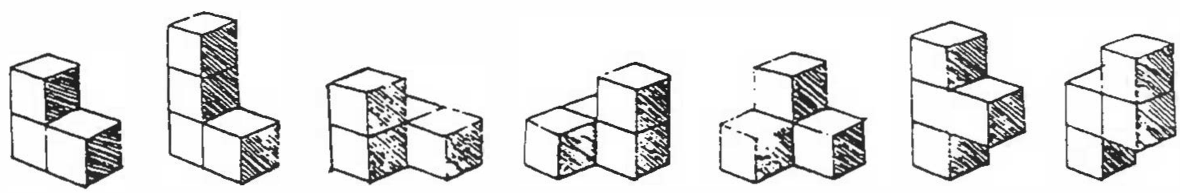
Use the geochart as a model of the first quadrant grid. When looked at in this way, it is easy to see extensions into negative numbers. Can you extend the patterns to include negative numbers? What if some numbers were positive and some were negative?

The Geo Hundreds Chart

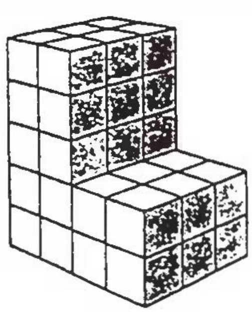
09	19	29	39	49	59	69	79	89	99
08	18	28	38	48	58	68	78	88	98
07	17	27	37	47	57	67	77	87	97
06	16	26	36	46	56	66	76	86	96
05	15	25	35	45	55	65	75	85	95
04	14	24	34	44	54	64	74	84	94
03	13	23	33	43	53	63	73	73	93
02	12	22	32	42	52	62	72	82	92
01	11	21	31	41	51	61	71	81	91
00	10	20	30	40	50	60	70	80	90

Soma Cubes

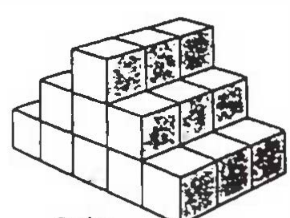
Concept: Geometry
Grade Level: 4–12
Materials: sugar lumps
Directions: Glue sugar lumps together to form the seven Soma pieces.



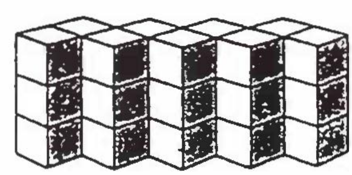
Suggestions: The Soma cube, invented by Danish writer Piet Hein, is probably the most successful three-dimensional version of tangrams. By taking all the irregular shapes that can be formed by combining no more than four cubes, all the same size and joined at their faces, Hein found that these shapes can be put together to form a larger cube. A few of the other shapes that can be made are shown below:



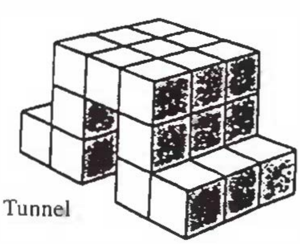
Chair



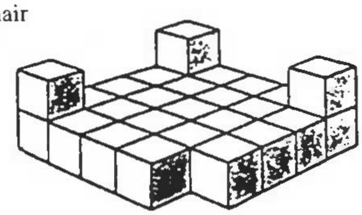
Stairs



Wall



Tunnel



Castle

More information and illustrations of many possible structures that can be made from the seven Soma pieces may be found in *The Second Scientific American Book of Mathematical Puzzles and Diversions*, by M. Gardner (New York: Simon & Schuster, 1965), 65–77. [Reprinted in paperback in 1987 by the University of Chicago Press.]

Centi-Metric

Concept:	Measurement—metric
Grade Level:	3–6
Number of Players:	2–4
Materials:	gameboard ruler deck of 64 cards (see list below) 1 marker per player
Object:	To measure centimetres and reach finish
Rules:	1. Place shuffled cards face down. 2. In turn, players draw a card and follow the directions. Measure lines to the nearest whole centimetre. 3. The winner is the first to reach finish.

Directions: Follow illustration to make gameboard. Around the border of the board, draw segments of the following lengths:

A. 5 cm	B. 12 cm	C. 8 cm	D. 2 cm	E. 19 cm
F. 24 cm	G. 10 cm	H. 15 cm	I. 6 cm	J. 7 cm

Make the indicated number of cards with the following directions:

3 Measure A and move forward that amount
1 Measure A and move backward that amount
3 Measure B and move forward that amount
1 Measure B and move backward that amount
3 Measure C and move forward that amount
1 Measure C and move backward that amount
3 Measure D and move forward that amount
1 Measure D and move backward that amount
3 Measure E and move forward that amount
1 Measure E and move backward that amount
3 Measure F and move forward that amount
1 Measure F and move backward that amount
3 Measure G and move forward that amount
1 Measure G and move backward that amount
3 Measure H and move forward that amount
1 Measure H and move backward that amount
3 Measure I and move forward that amount
1 Measure I and move backward that amount
3 Measure J and move forward that amount
1 Measure J and move backward that amount
5 Take an extra turn
5 Lose one turn
1 Move to square 37
1 Move to square 60
1 Move to square 85
1 Return to start
1 Exchange places with leader
1 Move to leader's square
1 Move to square of person in last place
1 Measure C and move forward twice that amount
1 Measure D and move backward 3 times that amount
1 Pick someone (not you) to get an extra turn

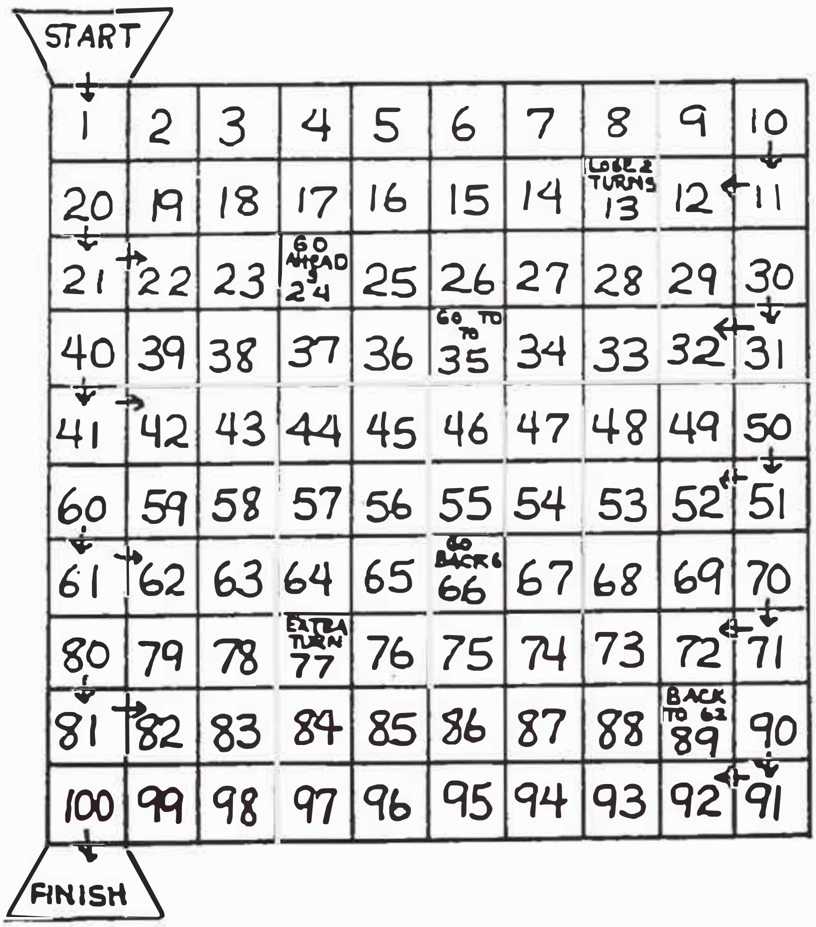
1 Pick someone (not you) to lose one turn
1 Move forward the difference between A and G
1 Move forward the difference between F and E
1 Move backward the difference between J and I

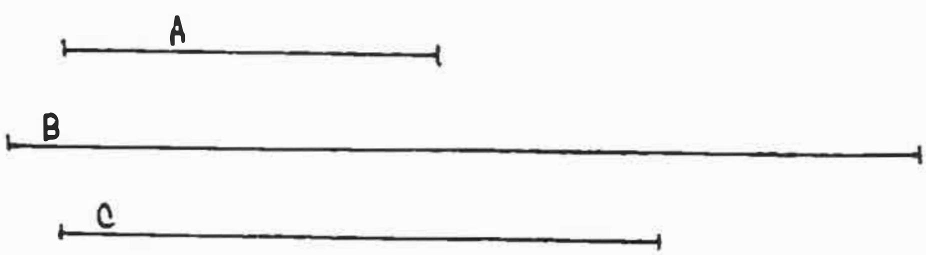
For Teachers

Levels: 2, 3, 4

Objective: To investigate patterns in a geometrical partitioning experiment

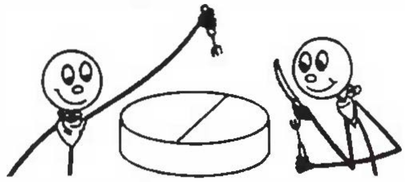
Directions:
1. Duplicate a worksheet for each student.
2. Make sure students understand the directions.
3. When all students have had the chance to investigate this problem, discuss with the entire class the patterns they found.



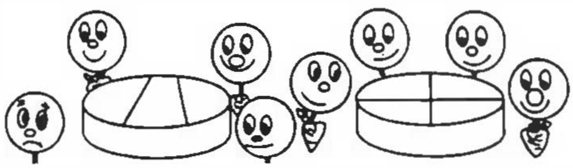


There's More Than One Way to Cut a Cake

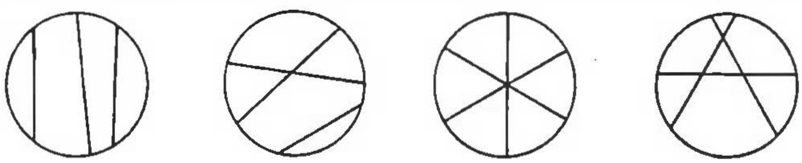
If you cut across a cake with 1 straight cut, you'll have 2 pieces.



If you use 2 straight cuts, you can do it so you'll have either 3 pieces . . . or 4 pieces.

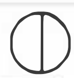
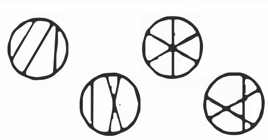


With 3 straight cuts, there are 4 different ways to cut the cake:



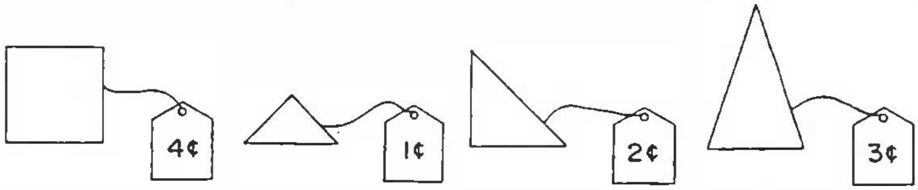
*Remember, every cut must be straight and across the cake.

Investigate this cake cutting pattern. Record what you find in the table below. Continue the table on another paper.

How many cuts?	Drawing	Pieces	How many ways?
1		2	1
2		3 4	2
3		4 5 6 7	4
4			

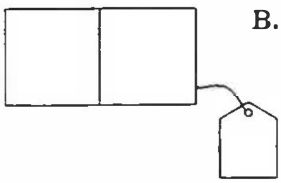
Can you predict what is the greatest number of pieces you could get from 10 cuts? Try it and see!

Congruent Polygons

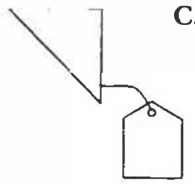


How much are these?

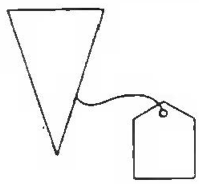
A.



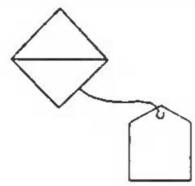
B.



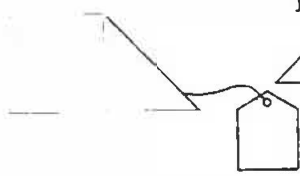
C.



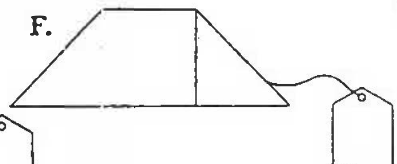
D.



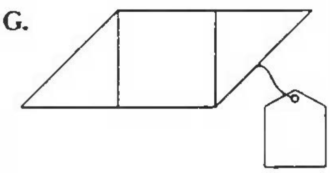
E.



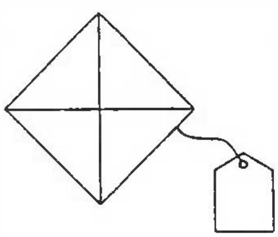
F.



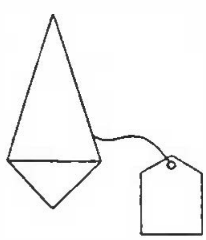
G.



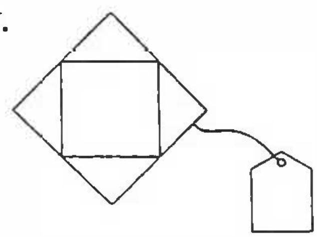
H.



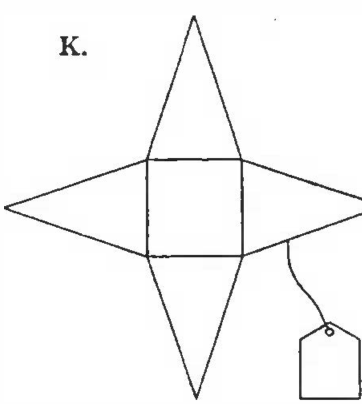
I.



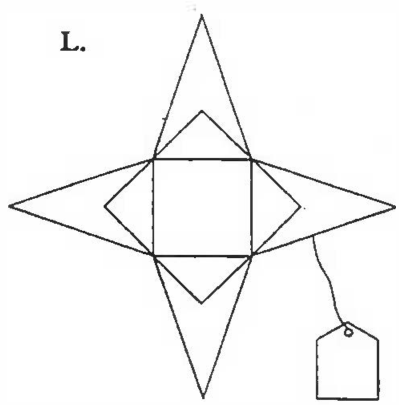
J.



K.



L.



For Teachers

Objective: Experience with identification of congruent polygons

Grade level: 1, 2, 3, 4, 5, 6, 7 or 8

Directions: Remove the student worksheet and reproduce a copy for each student.

For Grades 1 and 2:

1. Discuss the square and triangles shown at the top of the page. Be sure to discuss their cost.
2. Discuss the cost of the first four or five examples and fill in the tags.
3. Let the students do the rest of the examples on their own.

For Grades 3, 4, 5, 6, 7 and 8:

1. Have the students study the polygons at the top of the page and complete the price tags for the other polygons.
2. After they have completed the worksheet, discuss the relation between those polygons that are the same price. The area concept should come out in the discussion.

Comments: Fundamental to the development of many area concepts is the idea of conservation. In this case, the area of the polygon does not change as we move it around or place it with other polygons. The use of price provides a different focus on area and forces students to consider area in a different way. A variety of approaches to the development of a concept broadens the concept for some students and develops understanding for students who didn't see the idea before.

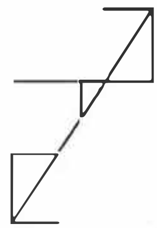
Answers: A. 8 B. 2 C. 3 D. 2 E. 6 F. 8 G. 8 H. 8 I. 4 J. 8 K. 16
Anticipate both 16 and 20 as answers for L.

Corner to Corner

If you have a rectangle like this,

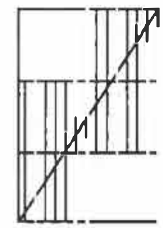
3 by 2

and you draw a diagonal,



3 by 2

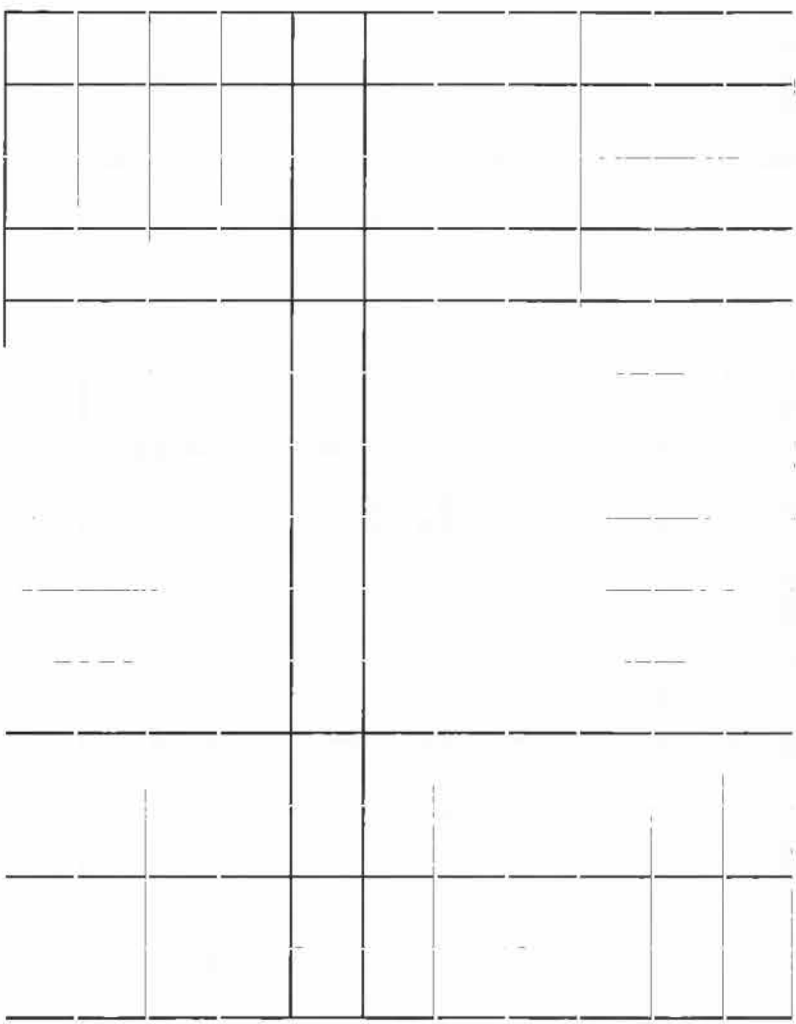
the diagonal goes through 4 squares.



3 by 2

Investigate this diagonal pattern for other rectangles.

Use the squares below to draw rectangles. Use a straightedge.



Dimensions	Squares Cut with Diagonal

For Teachers

Objective: To investigate number patterns from a geometrical experiment

Level: 3, 4, 5, 6

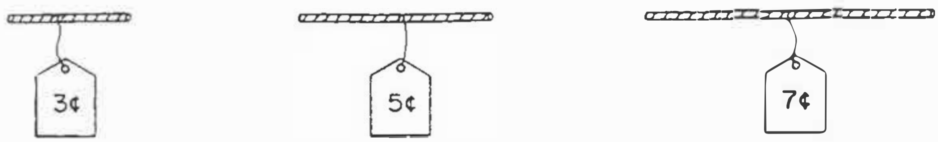
Directions:

1. Duplicate a worksheet for each child.
2. Make sure they understand the directions. It would be helpful to have additional squared paper available.
3. Organizing the data makes it easier to investigate the patterns. One suggestion for doing this is to look at all the rectangles that have one dimension kept the same:

Dimensions	How many?
2×1	2
2×2	2
2×3	4
2×4	4
2×5	6

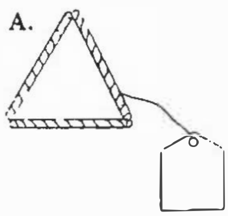
You may want to have different groups of students try the experiment keeping a different dimension constant.

Perimeter and Edges

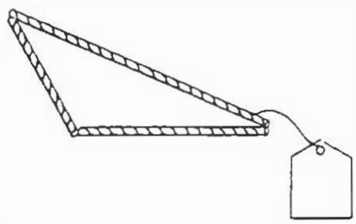


How much are these?

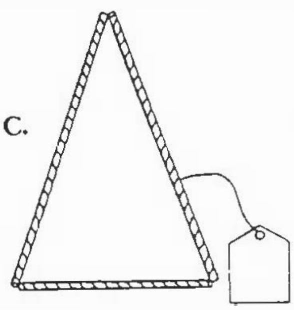
A.



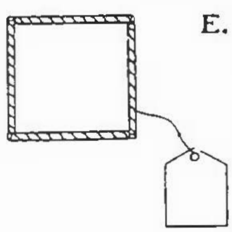
B.



C.



D.



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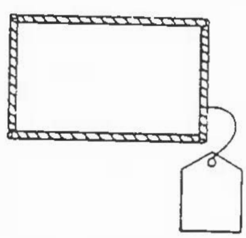
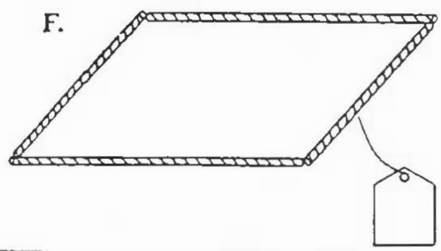
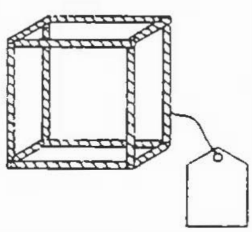


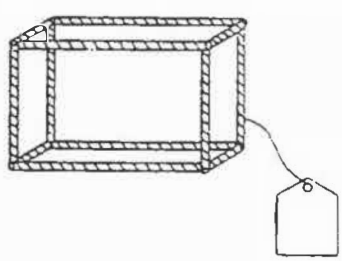
F.



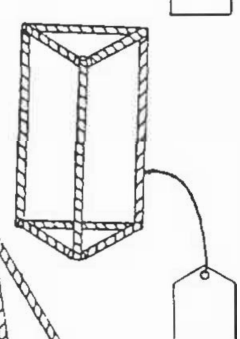
G.



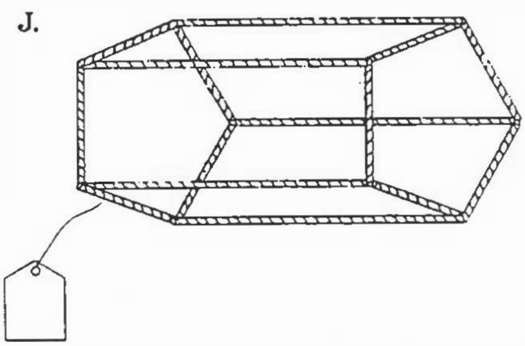
H.



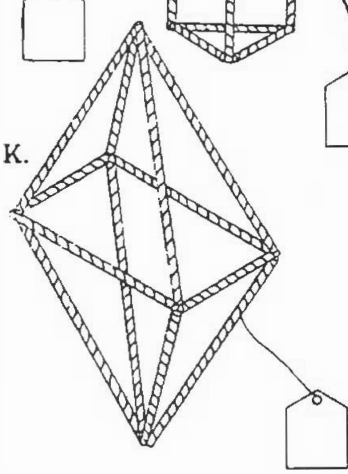
I.



J.



K.



For Teachers

Objective: Experiences with perimeter of polygons and identification of the edges of a solid

Grade Level: 3, 4, 5, 6, 7 or 8

Directions:

1. Remove the student worksheet and reproduce one copy for each student.
2. After handing out the worksheet, ask the students to fill out the price tags on each figure.
3. When the students have completed their answers, discuss the different ways the students arrived at the answers: How did you know which straws made up the sides? Did you need to measure? Which polygons have the largest perimeters? What other polygons can you make from these straws? Can you make a polygon selling for 21¢? For 13¢? For 28¢? What are possible prices for polygons made from these straws?

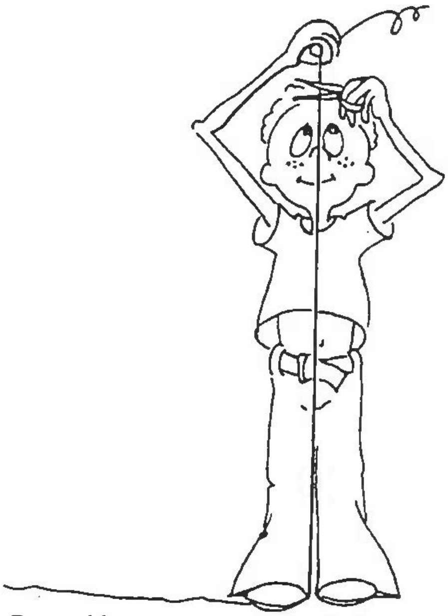
Comments: Students confuse the perimeter and area concepts because they don't have enough experience where the distinction is functional. There are few places in a student's life where he or she uses perimeter and area. An occasional contact in a classroom helps keep the distinction in mind. In many classes, it would be appropriate to discuss the classification of triangles as equilateral, isosceles or scalene. An investigation of pyramids, prisms and other solids might also result.

Grades 3 and 4 students would benefit by building some of the models, using straws and tape.

Answers: A. 9 B. 15 C. 19 D. 12 E. 16 F. 24 G. 36 H. 44 I. 33 J. 65 K. 72

Merry Measuring

Cut a piece of string equal to your height.



Fold it in half and try it on yourself.

What can you find that is ½ your height?

Fold it in thirds. What can you find that is 1/3 your height? 1/4? 1/5? What else?

Record here

½ my height	1/3 my height	¼ my height	1/5 my height	What else?

For Teachers

Objective: Experience with body measures and body ratios

Levels: 3, 4 or 5

Directions:
1. Remove the activity sheet Merry Measuring and reproduce one copy for each student.
2. Discuss it to make sure the directions are understood.
3. Give each child a piece of string. Make sure the string doesn't have any stretch.

Follow-up: Additional problems using the piece of string can be suggested.
1. How many of your widest smile make your height? Guess first, then use your string.
2. Did your mother ever wrap a sock around your fist to see if it was your size? Why would she do a thing like that? Use your string to find out.

Jack's Bean Bag Activities

Introduction: Jack traded the family cow for a handful of "magic" beans. We cannot find magic beans at the store, but you will find many varieties available.

Objective: To provide students with experience in sorting and graphing information

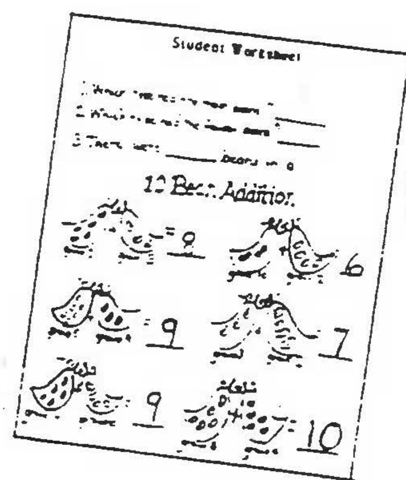
Materials:
- A bag of "Ten Bean Soup" beans
- Student copies of Sorting and Graphing sheets
- 58 g (2 oz.) of the beans in a small brown paper bag
- Pencil, crayons or marker

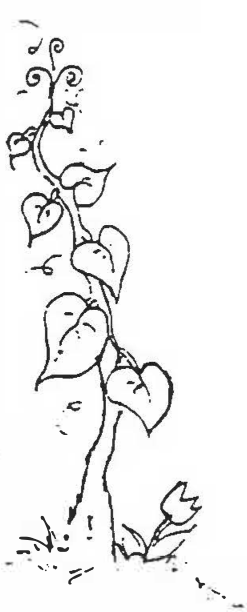
Suggestions: Students may work in pairs or groups of three

Procedure:
1. Give each student group a small bag of beans and the Student Sorting and Graphing Sheets.
2. Using the sorting sheets, have the students sort the beans into groups by "type" of bean.
3. Using the student graphing sheets, the students should graph the "sorted" beans by coloring spaces, drawing beans or gluing beans on the graph.
4. Groups may collect, compare and share data from each sorting and graphing sheet.

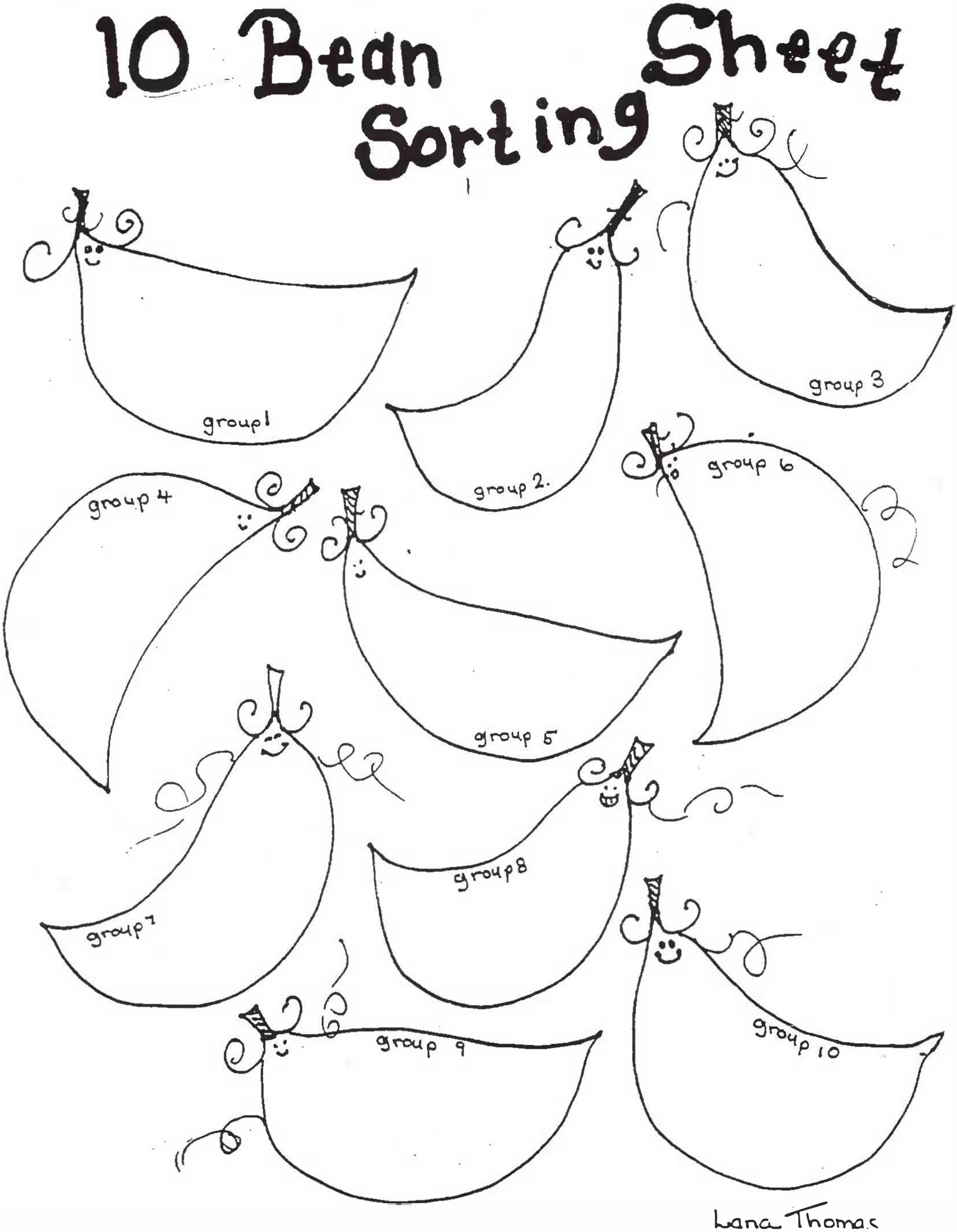
Extension: This extended lesson provides practice in problem solving, using information gained in the sorting and graphing of beans activity, and combining sets to complete addition problems.

Students may use the information from the sorting and graphing activity to complete the student worksheet provided.





Student Sorting Sheet



Student Graphing Sheet

1

2

3

4

5

6

7

8

9

10

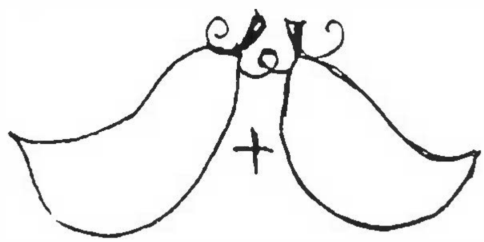
Student Worksheet

1. Which type had the most beans? ____________________

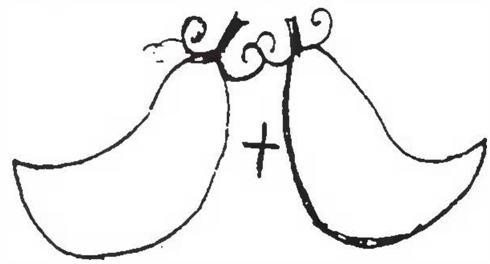
2. Which type had the fewest beans? ____________________

3. There were ____________ beans in all.

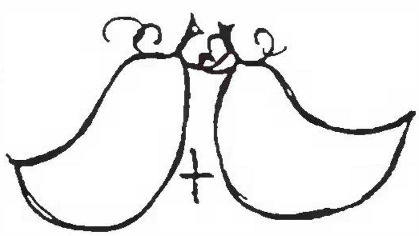
10 Bean Addition



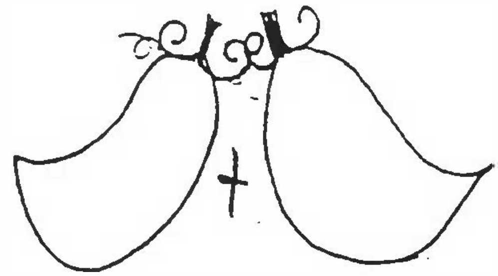
Group 1 + Group 3 = ________



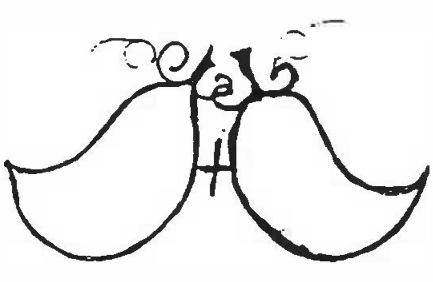
Group 10 + Group 2 = ________



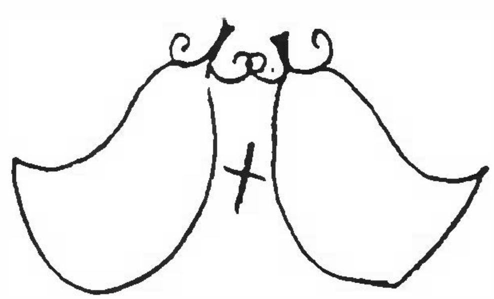
Group 5 + Group 9 = ________



Group 3 + Group 2 = ________



Group 4 + Group 6 = ________



Group 8 + Group 4 = ________

100 or Bust

Objective: To develop relative magnitude for numbers 0–100
To apply place value concepts
To explore concepts of chance
To represent two-digit numbers with models
To count on by tens and/or ones
To develop a winning strategy

Materials: For each group of three—1 scoresheet, 100 grid, place value pieces (ones and tens), calculator. 1 die for the whole class.

Procedure: Place students in groups of three and assign each a role: a calculator, a recorder and a plotter. Have students take turns rolling the die. Each group decides whether the number rolled should be recorded in the tens' column or the ones' column. The recorder writes the decision on the scoresheet. When a number is written in the tens' column, "0" is written next to it in the ones' column, so 4 tens is "40." The plotter records the roll on the 100 grid using place value pieces. The calculator keeps a running total on the calculator. After 7 rolls, the group closest to 100 without going over wins.

Discuss: What was the best total that could have been achieved with those 7 rolls?

What was your group's strategy?

100 or Bust Scoresheet

Game 1	
Tens	Ones
1.	
2.	
3.	
4.	
5.	
6.	
7.	

Game 2	
Tens	Ones
1.	
2.	
3.	
4.	
5.	
6.	
7.	

Game 3	
Tens	Ones
1.	
2.	
3.	
4.	
5.	
6.	
7.	

Game 4	
Tens	Ones
1.	
2.	
3.	
4.	
5.	
6.	
7.	

Toss Up Fractions

Number of Participants:	From one person to the whole class
Materials Needed:	Two-color counters, Toss Up Fractions Record Sheets
Procedure:	Take the thirds record sheet and three two-color counters. Toss up the counters, decide what fractions part shows. (Example: Two of the three counters show yellow. Use the same color during the game.) Write 2/3 in the column of the record sheet or color in one block in that column. Continue tossing the counters and graphing the results. The game is over when one column is full. Try fourths with four two-color counters, fifths with five two-color counters or sixths with six two-color counters.

Toss Up Fractions—thirds 3/3

0/3	**1/3**	**2/3**	**3/3**

Toss Up Fractions—fourths 4/4

0/4	1/4	2/4	3/4	4/4

Toss Up Fractions—fifths 5/5

0/5	1/5	2/5	3/5	4/5	5/5

Toss Up Fractions—sixths 6/6

0/6	1/6	2/6	3/6	4/6	5/6	6/6

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