Developing and Assessing Understanding of Integer Operations

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The study of integers in the middle grades bridges the concretely based elementary school program and the more formal secondary school curricula. While the basic ideas are grounded in real-world situations and can be modeled concretely and semiconcretely, development of the operations with integers also relies on examining the consequences of extending whole number meanings, properties and patterns. As students build on previous understanding, they have many opportunities to make mathematical connections and to use informal mathematical reasoning. The first encounter with integers can be an exciting and rewarding mathematical adventure.

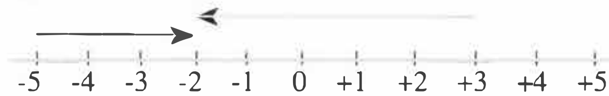
This article describes an introductory unit on integers I taught to a class of 27 Grades 6 and 7 students in an urban public elementary school. During the lessons, the students were encouraged to use materials or extend their previous knowledge to devise solutions to problems and to explain and justify their thinking orally and in writing. Following seven days of instruction, a written test, consisting mostly of multiple-part problems, was administered to assess student understanding of concepts and procedures.

Introducing the Integers

We began by talking about the nature of mathematics and how new mathematics is created. After the invention of the number zero was discussed, the students watched an overhead calculator count down from ten to zero and beyond. I described the use of *negative* numbers in ancient times and by 16th-century mathematicians, who called such numbers "fictitious" or "absurd." The class then generated various real-life situations requiring numbers that indicate direction as well as magnitude. On the second day, the idea that negative numbers are *opposites* of counting (*positive*) numbers was further explored in connection with the number line, and ways of comparing and ordering integers were investigated.

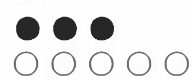
Addition

Everyday life situations were used to introduce addition: you earn $8 and then you spend $10; you take three steps to the left and then five to the right. Addition on the number line was interpreted as moving to the right for positive numbers and to the left for negative numbers. Students found that, although changing the order of the addends represents a different real-world problem and dictates a different number line solution, the final result is the same, as with whole numbers.



-5 + +3 = +3 + -5

The electric charges model using two-color discs (chips) was then introduced—a black disc represented a positive charge of 1 and a red disc represented a negative charge of 1. I demonstrated on the chalkboard using checkers with a magnetic strip glued to one side. Each pair of students was given a container with 10 black and 10 red chips to use in class. In diagrams, the symbols $\oplus$ and $\ominus$ were used to represent the charged chips. I explained that when a negative and a positive charge combine, they cancel each other; therefore, a black and a red chip together represent a value of zero. Students then used their chips to find solutions to various addition questions.

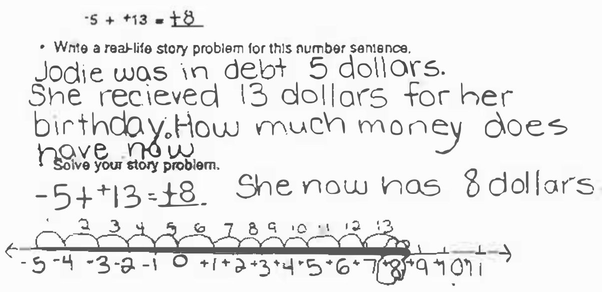


+3 + -5 = -2

Although no rules for addition were given verbally, many students volunteered that adding was easy, and, by the end of the period, most were completing multiple addend examples at the symbolic level.

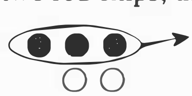
A question on the test asked students to write a story problem to correspond to the addition sentence -5 + +13 = __ and solve their problem (Figure 1). Twenty-six students wrote the correct answer +8 in the blank, and 20 were able to provide appropriate problems and solutions. Sixteen of the 20 acceptable problems involved money. Another 5 students based their problems on temperature but interpreted the second addend as a state (the temperature *is* 13°) rather than as a change or action (the temperature *rose* 13°) and were unable to formulate a meaningful question. A typical problem and solution were "Yesterday's temperature was -5°C. Today's temperature is +13°C. What is the total temperature? The total temperature is +8°C for yesterday and today." It is interesting that the students who used money for the setting did not have the same difficulty (Jean was $5 in debt and found/earned/deposited $13).

Figure 1
Danielle's Responses to the Addition Problem



Subtraction

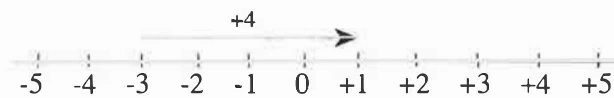
Meaningful development of subtraction of integers builds on students' previous understanding of subtraction of whole numbers, which has several interpretations—take away, comparison, missing addend and difference. The two-color chip model lends itself to the take-away interpretation and was used first. We began with +5 − +2 which means starting with five black chips and removing two. The three remaining black chips represent the expected answer +3. Next we considered -5 − -2: start with 5 red chips and remove 2; the answer then is -3. Students were then challenged to rename +1 − +3 and show their solution with the chips. Using a counting-down strategy, the class decided that the answer should be -2, but they could not immediately see how to remove three black chips when you only have one. Leaving this problem for the moment, I presented the question 32 − 18 in vertical form. We discussed the procedure of renaming 32 as 2 tens and 12 ones so that 8 ones can be removed. Next, we considered 3/4 − 5/8 and recalled that before subtracting, we rename 3/4 as 6/8. Now back to the integer problem—can we represent +1 with chips so that three black chips could be removed? The idea that adding pairs of positive and negative chips would not change the value of the pile was discussed. To solve the problem then, we might first place two more black chips and two red chips with the original black chip. Removing the three black chips leaves two red chips; the answer is -2.



+1 − +3 = -2

Using this strategy, students successfully solved a variety of subtraction questions with the chips. When a student commented that some number sentences did not make much sense, I asked the class how they check subtraction questions. They then used this same procedure to check answers obtained using the chips. For example, -5 − -2 = -3 because -2 + -3 = -5.

During the next class, we investigated subtraction on the number line. We first reviewed subtraction as counting up, using examples such as 7 − 5 and 31 − 28. We then looked at the difference between two numbers in terms of their distance apart on the number line. For example, 2 and 5 are three units apart. If we consider direction as well as distance, the distance from 2 to 5 is +3 and the distance from 5 to 2 is -3. Therefore, to solve 5 − 2 = __ on the number line, we start at 2 and move to 5 (2 + __ = 5); for 2 − 5, we start at 5 and move to 2. Using this interpretation, students completed a variety of subtraction questions, again checking their answers by addition.



+1 − -3 = +4

The students were then asked if they thought that there might be a way of subtracting positive and negative numbers without using colored chips or the number line. Four pairs of number sentences were written on the chalkboard, and the class considered how they were the same and how they were different.

+5 − +2 = +3	+5 + -2 = +3
-1 − +3 = -4	-1 + -3 = -4
+3 − -5 = +8	+3 + +5 = +8
-9 − -4 = -5	-9 + +4 = -5

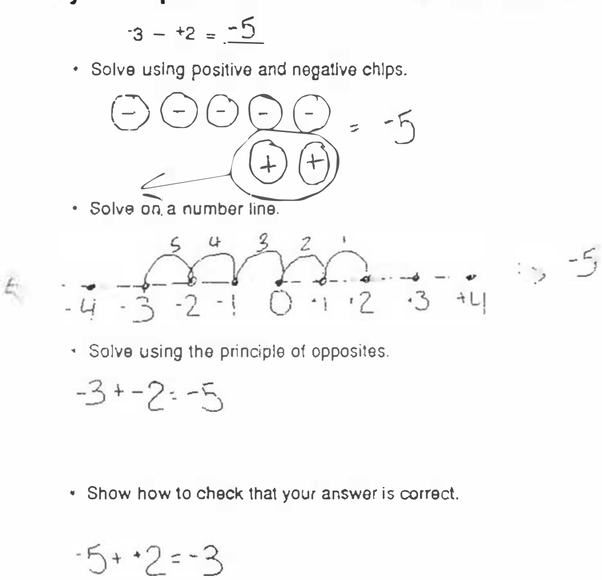
It was noted that the subtraction questions and the corresponding addition questions had the same first number and the same answer and that the second numbers in the addition questions were the *opposites* of the second numbers in the subtraction questions. Students were encouraged to state this relationship in their own words and to verify it using other examples. To further justify the result, the class examined the consequences of continuing patterns such as the following:

+3 − +2 = +1
+3 − +1 = +2
+3 − 0 = +3
+3 − -1 = __

A test question on subtraction asked the students to solve -3 − +2 = __ in three ways and to check the answer (Figure 2). Twenty students wrote the correct answer -5; the other seven wrote -1, which is the correct answer to the addition question -3 + +2 = __.

Of the 20, all showed a correct solution with positive and negative charges. Seven of the people in this group sketched a number line solution indicating the distance from +2 to -3; the other 13 drew the solution for -3 + -2 (start at -3 and move left 2). Only 10 students demonstrated how to solve the subtraction question by adding the opposite of +2, indicating that this rule had not been as well established as the concrete procedures. Furthermore, only 12 students showed how to check their answer by writing +2 + -5 = -3. All seven students who gave -1 as the answer showed correct chip and number line solutions for the addition question, but only two of them tried to check the answer by adding. One acknowledged that the check indicated that something was wrong; the other wrote +2 + -1 = -3.

Figure 2
Kelly's Responses to the Subtraction Problem



In another section of the test, students were asked to write a question to complete the story problem beginning: "At noon, the temperature was +8°C. At midnight, the temperature was -5°C." They were then to write a number sentence and solve the problem (Figure 3). Twenty students wrote meaningful subtraction questions, 12 asking how far the temperature fell and 8 referring to the difference between the two temperatures. Of these 20, 12 correctly solved their problem based on the number sentence 8 − -5 = __. Another 6 also produced this number sentence, but 5 gave the answer +3°C, and 1 student concluded (understandably) that the temperature fell -13°C. Only one of the questions written by the other 7 students made sense: "What was the average temperature?" He correctly found the sum of the two numbers, but, in computing 3 ÷ 2, he divided 3 into 2 (using the long division algorithm).

Figure 3
Laura's Responses to the Temperature Problem

• Write a question to complete the following story problem.

At noon the temperature was 8°C. At midnight the temperature was -5°C. How much did the tempercture fall?

• Write a number sentence for the story problem.

8°C − -5°C = 13°C

• Solve the problem.

The tempputure fell 13°C

Multiplication

I began the lesson on multiplication by reviewing the meaning of 3 × 2 for whole numbers. The general understanding was that it represented three groups of 2 or adding 2 three times. We then modeled +3 × +2 using black chips and the number line. Next, students were challenged to find an answer to +3 × -2 and to be prepared to explain their thinking. The class agreed that the answer should be -6, modeling this with three groups of two red chips and on the number line as three jumps of 2 to the left of zero (-2 + -2 + -2). I then wrote -3 × +2 = __ on the chalkboard. After some initial discussion about whether this had any real meaning, students recalled that 3 × 2 can also be interpreted as two groups of 3 and that 3 × 2 = 2 × 3. The idea was therefore proposed that we could think of -3 × +2 as two groups of -3 (+2 × -3); the answer should be -6. The class then correctly predicted what my next question would be, and the chalkboard looked like this:

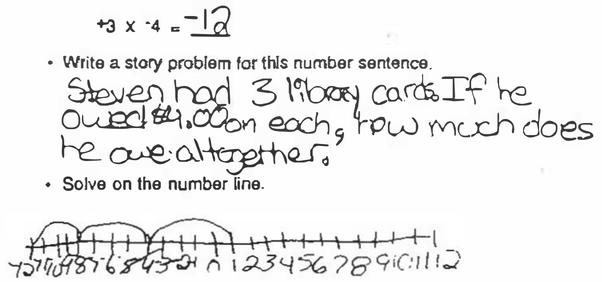
+3 × +2 = +6
+3 × -2 = -6
-3 × +2 = -6
-3 × -2 = __

After students discussed the problem with partners, they shared their ideas with the whole class. One student said that because two odds make an even (he was thinking addition), the answer is positive. Another pair used a calculator. A girl remembered hearing her mother say that two negatives make a positive. To justify the rule, students considered and continued the following pattern:

+3 × -2 = -6
+2 × -2 = -4
+1 × -2 = -2
0 × -2 = 0
-1 × -2 = __

The multiplication test question asked the students to write a story problem for +3 × -4 = __ and show a number line solution (Figure 4). All but 1 of the 27 students wrote -12 as the answer and showed this on the number line. Producing an appropriate story problem was more challenging. Twelve students wrote problems involving the idea of "3 debts of $4," two used the concept "3 times as much as a debt of $3" and four referred to "a debt of 4 times $3." One girl wrote about three groups of four friends who were not her friends. Other students made up amazing stories concerning "three groups of -4 apples/carrots/chips/people."

Figure 4
Pat's Responses to the Multiplication Problem



Division

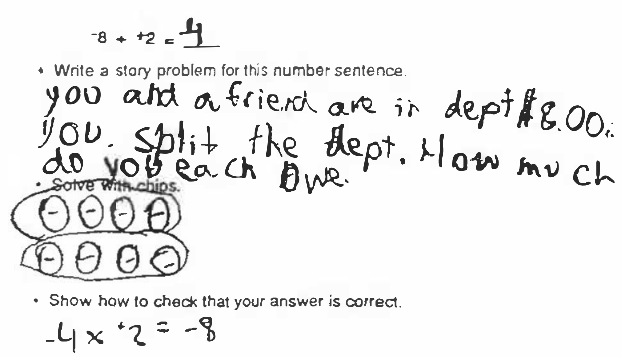
The lesson on division also began by considering whole number meanings. Students recalled that 6 ÷ 2 meant making groups of 2 or sharing six things equally between two people. The procedure for checking a division answer by multiplication was reviewed in connection with a discussion of the relationship between multiplication and division. Similarities to the addition-subtraction connection were also noted. Three new problems were then written on the chalkboard. Students were challenged to use and extend what they knew about division and multiplication to find answers.

+6 ÷ +2 = +3
-6 ÷ +2 = __
+6 ÷ -2 = __
-6 ÷ -2 = __

The students solved the problems using a guess-and-check-by-multiplication strategy. Several remarked that the results for division were the same as for multiplication.

The test question required students to write a story problem for -8 ÷ +2 = __, show a solution with the chips and check the answer (Figure 5). Twenty-five of the 27 students wrote the correct answer, and 23 showed -4 × +2 = -8 as a check. Another student wrote -8 ÷ -4 = +2 as her check. Fifteen of the story problems were based on the idea of sharing something between two people, and each was followed by a sketch showing eight negative chips divided into two groups of four. Six of the eight meaningful problems of this type involved paying half of an $8 debt. The other seven stories referred to sharing such things as bad apples, stale candies, imaginary cookies or -8 donuts. Another five students made sketches of 8 negative chips separated into groups of 2, to represent the solution to their problem. In each case, the numerical answer -4 was given, even though none of the accompanying stories made much sense. For example, "Bill had -8 carrots for lunch. If he ate +2 at a time, how many groups of carrots did he eat?" It appears that most students did not first write a problem to develop a meaningful context that would guide the solution to the number sentence. Rather, after successfully drawing a diagram to show the answer, they then attempted to transfer a familiar whole number setting (8 apples) to one involving negative numbers (-8 apples or 8 bad apples) to produce a related "story."

Figure 5
Michael's Responses to the Division Problem



Computational Skills

In addition to the questions previously described, the test included a section consisting of eight computation items, two for each operation (one in the form a * b = __; the other in the form a * __ = c). The overall success rate was 89 percent for multiplication, 81 percent for division, 69 percent for addition and 54 percent for subtraction. While the results for multiplication, division and subtraction were expected, the outcome for addition was somewhat surprising given the ease with which most students had initially learned this process. On further examination of the data, it was found that, on the item -7 + -9 = __, six of the eight students with errors gave the answer +16, suggesting an incorrect transfer of the multiplication rule to the procedure for adding two negative numbers. The item -4 × -9 = __ was correctly solved by 24 of the 27 students, including all six who had made this particular error in addition. On the other hand, one student who had correctly answered -7 + -9 = -16 then wrote -4 × -9 = -36.

Journal Writing

A reading of students' journal entries during the time they were studying integers provided additional information about their feelings and insights. Midway through the project, students wrote about the mathematics they had learned the previous week. One wrote:

> + and - numbers are ok I supoze. But I don't realy like adding & subtracting them. They are easy doing them with the chips, but harder without. I hope we don't have to × and ÷ them.

In their entries the day after the test, students reflected on what had been easy and hard, interesting and boring:

> What was interesting was why adding and subtracting was so hard in this system and multiplying and dividing was so easy.

> I found the whole concept of two negatives make a positive interesting because it was so weird and confusing.

> The interesting thing was how you would add a negative and a posetive you ended up with a lower number than you started with so it would be like subtracting.

Summary and Conclusion

The instructional approach related computational procedures to the meaning of the operations in the context of real-world applications or to number properties and patterns. In-class practice activities, homework assignments and tests were based on a small number of questions students were to solve in more than one way, using diagrams and written explanations to justify their answers. Students were often asked to relate the numbers and operations to experience and to explain their thinking to their desk partner or a family member at home. Extended practice to promote speed and accuracy was not provided at this time.

Developing an understanding of integer operations and computational procedures is a complex task but can be a rewarding experience for students and teachers. If students are to value sense-making as an important aspect of learning mathematics, a significant portion of instructional time and a major component of evaluation must reflect this emphasis.