Area Graphs from Area Formulas: Connecting Geometry and Algebra

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Two recurring emphases in the NCTM Standards are connections among mathematical ideas and graphical representations. A rich source of examples in which these two emphases can be developed involves area formulas of geometric figures.

We shall consider several familiar area formulas and their corresponding graphical representations.

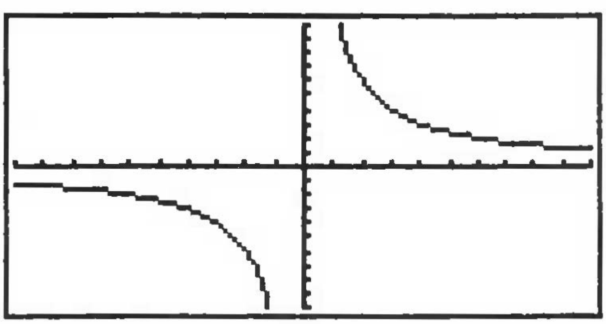
Formula 1

For a rectangle of length x and height y, $A = x \cdot y$. The most familiar use of the formula is to compute areas for the known measures of x and y. A less routine problem is to determine all possible x,y values that would yield a given area. For instance, find all possible dimensions of a rectangle that would yield an area of 12. Obvious answers include integer factors of 12: $x = 1$, $y = 12$; $x = 2$, $y = 6$; $x = 3$, $y = 4$; $x = 4$, $y = 3$; $x = 6$, $y = 2$; $x = 12$, $y = 1$.

The possibilities increase if rational numbers are used, such as $x = ½$, $y = 24$. Further consideration reveals that x could be any positive real number. For a given value of x, y will then be $12/x$.

How can all the x,y length pairs that would yield a rectangle of area 12 be visualized? Use a graphing calculator (such as TI-85) to graph $12 = xy$ or $y = 12/x$ as shown in Figure 1.

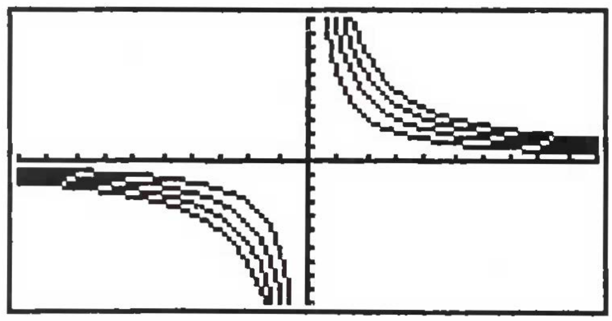
Figure 1



The graph of the equation $xy = 12$ is called a hyperbola. Note that this graph has two distinct branches—one in quadrant 1 and one in quadrant 3. Do both represent solutions to the original problem of finding the dimensions of a rectangle? (Will your students recognize that the branch in quadrant 3 cannot be used because its points have two negative coordinates?) Also note that the first quadrant branch is asymptotic to each axis. The area interpretation is that either of the rectangle's dimensions can become arbitrarily large, so long as the other dimension "shrinks" correspondingly.

The use of 12 as the area of the rectangle was arbitrary. Any positive number could be used. Figure 2 displays the first quadrant branches of the hyperbolas $xy = 6$; $xy = 9$; $xy = 12$; $xy = 15$. Have your students pair each equation with its corresponding area graph.

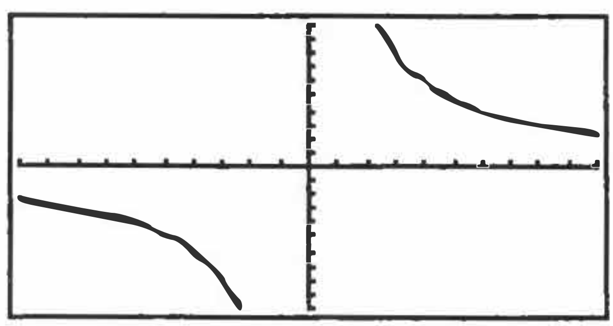
Figure 2



Formula 2

For a triangle of base x and height y, $A = ½xy$. Again, we wish to find all (x,y) values that would yield an area of 12. Expressed symbolically, this yields the equation $½\,xy = 12$ or $xy = 24$ or $y = 24/x$. Figure 3 displays the first quadrant portion of this graph—another hyperbola.

Figure 3



Have your students compare the graphs of Figures 1 and 3. Which is farther from the origin? Why?

Formula 3

For a trapezoid of bases x_1 and x_2 and height y, $A = \frac{1}{2} y(x_1 + x_2)$. If $x_1 + x_2$ is denoted by the single variable x, the formula $A = \frac{1}{2} yx$ or $A = \frac{1}{2} xy$ results. This is identical to the formula for the area of a triangle, so the same graphs would result.

However, if x_1 and x_2 were not combined symbolically, the formula for a given value of A would be $A = \frac{1}{2} y(x_1 + x_2)$. If A is 12, then $12 = \frac{1}{2} y(x_1 + x_2)$ or $y(x_1 + x_2) = 24$. This graph involves three variables, and its graphical representation would be three-dimensional. Use a software package such as Derive on a microcomputer to graph this more complex relationship.

Formula 4

Consider a right circular cylinder of radius x and altitude y. Its total surface area is found in two parts:

- Base area = $2\pi x^2$
- Lateral area = $2\pi xy$

TSA = $2\pi x^2 + 2\pi xy$

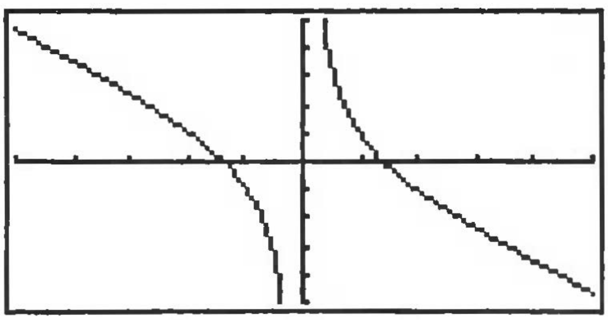
As in previous formulas, let TSA be a constant such as 12. Find the values of x and y that make this possible.

$12 = 2\pi x^2 + 2\pi xy$

$x^2 + xy = 6/\pi$

The graph of this equation is shown in Figure 4; only the first quadrant portion is meaningful in the area situation.

Figure 4



Formula 5

Consider a right circular cone with radius x and altitude y. Again, the total surface area is found in two parts.

- Base area = πx^2
- Lateral area = $\pi x \sqrt{x^2 + y^2}$ (where $\sqrt{x^2 + y^2}$ is the slant height of the cone)

TSA = $\pi x^2 + \pi x \sqrt{x^2 + y^2}$

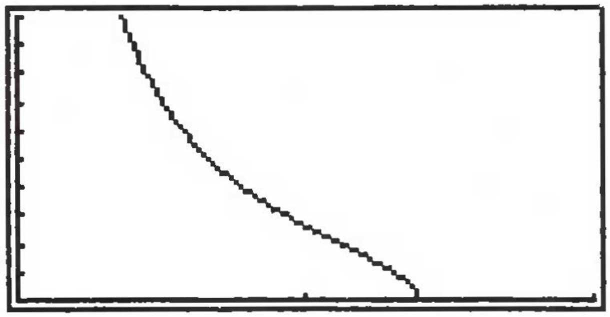
If the TSA = 12, find the values of x and y.

$12 = \pi x^2 + \pi x \sqrt{x^2 + y^2}$

$12/\pi = x^2 + x \sqrt{x^2 + y^2}$

Figure 5 displays the first quadrant portion of the graph of this equation. The scale has been adjusted to show the intervals $0 \le x \le 2$; $0 \le y \le 10$.

Figure 5



The first quadrant portions of all five figures are similar in a major respect—all graphs fall from left to right. In other words, large xs correspond to small ys and vice versa. Can your students explain why this must be true for a constant area?

In another respect, the first quadrant portions of Figures 4 and 5 are distinct from the doubly asymptotic graphs of Figures 1, 2 and 3. In Figures 4 and 5, the graph is asymptotic to the y-axis but intersects the x-axis at $\sqrt{6/\pi} \approx 1.38$.

In other words, the heights of the cylinder and cone can become arbitrarily large (being "balanced" by a shrinking radius), but the radius cannot exceed $\sqrt{6/\pi}$. Can your students explain why this should be so?

The ideas of this article can be extended in several ways:

1. Problems yielding three-dimensional graphs, as mentioned in the discussion of the trapezoid formula, can be pursued. For instance, graph the formula for the surface area of a right square pyramid and a general triangle using Hero's formula.
2. Graph the formulas for the volumes of polyhedrons.
3. The formulas for the area of a circle and the volume and surface area of a sphere involve only one independent variable (radius). Apply the methods of this article to this situation, if possible.
4. Write perimeter formulas for plane figures; graph the dimensions needed to produce a given perimeter in the spirit of this article.
5. Pick's formula is used to compute areas of polygonal figures drawn on dot paper or a geoboard. Investigate this formula and graph it for a given area.