Exploring Products Through Counting Digits: Equal Factors

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The *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, 39) defines number sense as "an intuition about numbers that is drawn from all the varied meanings of number." Five components of number sense are listed, two of which are relevant to this article. Children with good number sense "recognize the relative magnitudes of numbers" and they "know the relative effect of operating on numbers" (NCTM 1989, 38). Furthermore, "students with number sense pay attention to the meaning of numbers and operations and make realistic estimates of the results of computation" (NCTM 1992, 10).

Too often, students will give an unreasonable answer to a computational exercise and not be aware that it is unreasonable. How can we help children "pay attention to the meaning of numbers and operations" so that they can recognize whether an answer is unreasonable or "in the ballpark"? The booklets in the Addenda Series published by the National Council of Teachers of Mathematics contain many helpful suggestions. This article may add more ideas.

Children are sometimes taught to use a variety of estimation strategies (Reys 1986) to judge the reasonableness of an answer or to make an estimate when that is all that is needed. In addition, children could be given experiences that focus on the number of digits they should expect in an answer. The following activities focus on the latter approach. They are, for illustration purposes, restricted to multiplication but could be adapted to other operations. This article is also delimited to the case of equal factors.

Collect the Data

Suggest that students begin with a one-digit number multiplied by a one-digit number (basic facts). What is the smallest product you could get? The largest?

Now do the same for a two-digit number multiplied by another two-digit number. Encourage students to use mental computation to determine the minimum product and the calculator to get the maximum product. Invite students to continue this process for three-digit, four-digit and five-digit multipliers.

Finding the maximum product for two five-digit factors may be problematic, even with a calculator. This calculation will require multiple steps if students have a calculator with an eight-digit display—the most likely case for most middle school students. How children handle this will, in itself, provide some evidence of their level of number sense. One possible strategy follows:

Children with good number sense may recognize that 99 999 × 99 999 can be expressed as (99 999 × 999) + (99 999 × 99 000) and proceed as follows:

Calculator:	99 999 × 999 =	99 899 001	(1)
Calculator:	99 999 × 99 =	9 899 901	(2)
Mental arithmetic:	calculator result (2) × 1000 =	9 899 901 000	(3)
Paper and pencil:	mental arithmetic result (3) + calculator result (1) = 9 999 800 001		

Organize the Data

You might provide students with a blank copy of Table 1 (except for the headings and first column) and ask them to insert the data they obtained above and then complete the remaining columns.

Explore the Data

Students may notice a number of patterns in Table 1. The major purpose in developing the data was to help students reflect on the reasonableness of their product by considering the number of digits it might have. Therefore, this should be the initial focus.

Minimum and Maximum Number of Digits

Ask students to examine the data in columns 1, 3 and 5 in Table 1.

Table 1
Products and the Number of Digits in Products

Factors	Minimum Product	Number of Digits in Minimum Product	Maximum Product	Number of Digits in Maximum Product
1-digit × 1-digit	1	1	81	2
2-digit × 2-digit	100	3	9801	4
3-digit × 3-digit	10 000	5	998 001	6
4-digit × 4-digit	1 000 000	7	99 980 001	8
5-digit × 5-digit	100 000 000	9	9 999 800 001	10

Teacher: Describe the pattern you see in the column containing the minimum number of digits in a product.

Student answers might include the following:

- They go up by 2.
- They are all odd numbers.
- They are consecutive odd numbers, beginning with 1.

Teacher: How does the minimum number of digits in the product relate to the magnitude of the factors?

Allow ample time for students to investigate this because an understanding of this relationship is critical to determining the reasonableness of a product by counting digits. In the end (perhaps after some hints), students should recognize that the minimum number of digits in a product where the multiplier and multiplicand have the same number of digits is one less than the sum of the digits in the two numbers ($2n - 1$, where n is the number of digits in each factor).

Teacher: Describe the pattern you see in the column containing the maximum number of digits in a product.

Student answers might include the following:

- They go up by 2.
- They are all even numbers.
- They are consecutive even numbers, beginning with 2.

Teacher: How does the maximum number of digits in the product relate to the magnitude of the factors?

Again, allow ample time for students to investigate this because an understanding of this relationship is also critical to determining the reasonableness of a product. In the end, students should recognize that the maximum number of digits in a product where the multiplier and multiplicand have the same number of digits is twice the number of digits in each factor ($2n$).

With respect to the main purpose behind this activity, it is important for students to understand that when you have 2-digit × 2-digit, the product has either 3 or 4 digits, 3-digit × 3-digit, the product has either 5 or 6 digits and so on. This will help them to quickly detect unreasonable answers.

Minimum and Maximum Products

Students' attention could now be directed to columns 2 and 4 in Table 1, minimum and maximum products. Encourage students to describe patterns they observe. For example, in the minimum products column, most students will be able to observe that the number of zeros increases by two each time. Others might be a little more mathematical and say that "the minimum product increases by a factor of 100 as you increase the number of digits in the factors by one." Some may relate this to the previous investigations and think of the minimum product as having a 1 followed by $2n - 2$ zeros, where n is the number of digits in each factor.

Teacher: Try to describe the pattern in the maximum products column.

This pattern may be more interesting. It appears that as the number of digits in each factor increases by one, the product has one more 9 annexed at the beginning and another 0 sandwiched between the 8 and the 1.

Students may be challenged in different ways to pursue this pattern. Some junior high students may be encouraged to explain why the pattern holds or to show that it breaks down after a certain point. Others may be simply invited to see whether it will hold for a specific larger case, say an 11-digit number × 11-digit number.

At this point, some children may be challenged to add a generalization row to Table 1:

n-digit × n-digit	1 $(2n - 2)$ 0s	$2n - 1$	$(n - 1)$ 9s 8 $(n - 1)$ 0s 1	$2n$

Maximum Products by Mental Arithmetic

While using a calculator is the most logical and efficient procedure for determining the maximum product in Table 1, some students could be challenged to find a pattern that would enable them to use a mental procedure. With some guidance, they may note that

$9 \times 9 = (10 \times 10) - 19$. *Why?*
$= 100 - 20 + 1$
$= 80 + 1$
$= 81$

$9 \times 9 = (10 - 1)(10 - 1)$
$= 100 - 10 - 10 + 1$
$= 100 - 20 + 1$
$= 81$

Similarly, $99 \times 99 = (100 \times 100) - 199$
$= 10\,000 - 200 + 1$
$= 9\,800 + 1$
$= 9\,801$

The larger products in the maximum column of Table 1 are probably too difficult to do mentally for most middle school students, although a few may enjoy the challenge.

Extension: Switch-Over Points

Returning to the major focus of this article—determining the number of digits to expect in a product—at least one more investigation should be pursued. Is it possible to decide whether the product of, say, two 2-digit numbers has three digits or four digits?

Stimulate the students to think about "switch-over points" by making a statement such as the following:

Teacher: You have discovered that a 1-digit number multiplied by a 1-digit number has either one or two digits in the product. At what point does a product begin having two digits?

Many children will be inclined to begin their investigation with equal factors because that is what they have been using in the earlier explorations. (Because that is the focus of this article, this investigation will be restricted to that case.) Some will think it logical that the switch-over point would be about midway between 1 and 9 and start with 5×5. They may be surprised that this product is considerably greater than the smallest 2-digit number, 10. Using mental arithmetic, they will quickly realize that 3×3 (9) is still a 1-digit product but 4×4 (16) is a 2-digit product.

Teacher: Using your calculator and working with equal factors only, try to determine switch-over points for 2-digit and greater cases.

Using their calculators, students will quickly discover that 31×31 (961) is still a 3-digit product but 32×32 (1024) is a 4-digit product. If students are given a blank table such as Table 2 (except for column headings, first row and first column), the investigation may be facilitated.

Table 2
Switch-Over Points: Equal Factors

Number of Digits in Each Factor	Minimum Number of Digits in Product	Largest Factors for Minimum Digits	Maximum Number of Digits in Product	Smallest Factors for Maximum Digits
1	1	$3 \times 3 = 9$	2	$4 \times 4 = 16$
2	3	$31 \times 31 = 961$	4	$32 \times 32 = 1024$
3	5	$316 \times 316 = 99\,856$	6	$317 \times 317 = 100\,489$
4	7	$3162^2 = 9\,998\,244$	8	$3163^2 = 10\,004\,569$
5	9		10	
6	11		12	

Conclusion and Extension

Students are not often encouraged to consider the number of possible digits a product might have. While this strategy should not replace other estimation strategies, it should be included as one more strategy students can use to help them decide on the reasonableness of an estimate or computed answer. This article focused only on products of two equal factors.

References

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