delta-k

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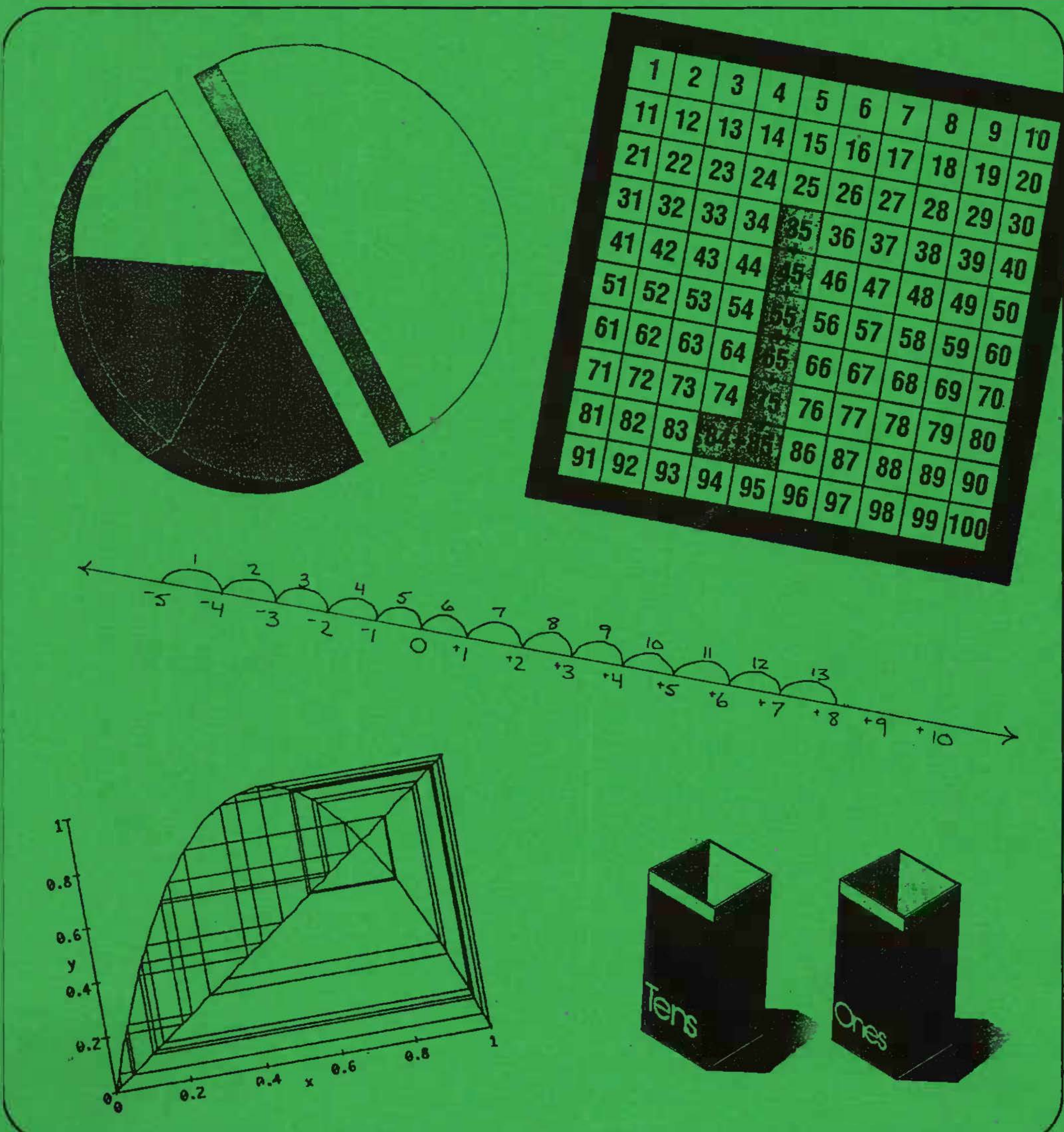

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delta-K is the official journal of the Mathematics Council (MCATA) of The Alberta Teachers' Association (ATA).

The objective of the *Journal* is to assist MCATA to achieve its objective of improving teaching practices in mathematics by publishing articles that increase the professional knowledge and understanding of teachers, administrators and other educators involved in teaching students mathematics. The *Journal* seeks to stimulate thinking, to explore new ideas and to offer various viewpoints. It serves to promote MCATA's convictions about mathematics education.

EDITORIAL

NCTM president Jack Price, in his "President's Message" in *NCTM News Bulletin* (April 1995), issues a real challenge to those of us responsible for teaching students mathematics. We must challenge our detractors. His message, included below, sets the tone for this issue of *delta-K*.

Arthur Jorgensen

NCTM believes that all children can and should learn mathematics. We believe that children construct their own knowledge using reflection and accommodation. We believe that a standards-based classroom is a humane community of learners. And we think of mathematics as problem solving, communication, reasoning and connections.

We can do a better job of letting the public at large know what we profess. If we do not, the public may believe what the detractors tell them, regardless of how twisted, misinformed or incorrect the disparagement is. For example, the NCTM Standards documents are accused of eliminating algorithms and forsaking all drill and practice. That simply is not true. We know that children should be assisted in developing more efficient ways of using basic skills. That is what algorithms are. We encourage mental mathematics and estimation, and we realize that children need to have a good grasp of arithmetic operations before they can be effective problem solvers. Our quarrel is with the way in which these operations may have been taught in the past, not with understanding but with relatively mindless repetition.

This is just one example where our message is not reaching the public. Many other examples exist, such as the argument that "there is no mathematics in the Standards." Those of us who have read the documents know that they call for stronger mathematics earlier in the curriculum. They also call for new, useful topics to be introduced into the curriculum and for other topics—those that can be replaced by technology, for example—to receive less attention.

I could go on, but you know the stories as well as I do. What we need to do is get the word out. Each of us needs to tell what we, as mathematics educators, believe. If parents, for example, believe that their children are receiving a more challenging, more useful mathematics program, they will rally for our movement. Like you, they want the best for their children. Let's pledge to tell at least one nonmathematics person this month how much the Standards can mean for all children. Let's not hide our light any longer.

The Standards documents are a giant step toward better mathematics education for every child. They give us a guide, not a prescription. They tell what we value; they don't mandate a specific scope and sequence. Let's clear up the misunderstandings and provide knowledge where it doesn't exist. But let's also counteract the lies and innuendos. And while we are at it, let's tell the naysayers that the strength of the Standards will keep our efforts moving forward—we are not going away.

The 100s Chart: A Stepping Stone to Mental Mathematics

Carol A. Thornton, Graham A. Jones and Judy L. Neal

"I want a hamburger, cider and apple pie," said Katie. Kirk thought, "That's 25 and 11, that's 35, 36, and 20 for the apple pie makes 46, 56¢." Katie replied, "That's right. I got out a quarter for the hamburger, a dime and a penny for the cider, and two more dimes for the apple pie and counted 25, 35, 36, 46, 56¢" (see Figure 1).

FIGURE 1

Menu used to place orders and calculate cost

MENU

Main dishes		Drinks	
Slice of pizza	35¢	Milk	9¢
Hot dog	25¢	Cider	11¢
Spaghetti	30¢	Soft drink	11¢
Hamburger	25¢	Milk shake	19¢
Vegetables		**Desserts**	
Ear of corn	10¢	Chocolate-chip cookies	20¢
Baked potato	10¢	Apple pie	20¢
French fries	20¢	Ice cream	20¢
Baked beans	10¢	Caramel apples	20¢

Moving on, Mrs. Neal heard another group of her second graders placing their orders. "I want pizza, corn on the cob, a chocolate chip cookie and a milkshake," said Brandon. Brandon used a 100s chart like that shown in Figure 2 to help find his total. Starting on 35 for the pizza, he moved down the columns and counted on by tens for the corn and cookie: "35, 45, 55, 65"; then, for the 19¢ milkshake, he moved down two more rows and went back one space. "That's 84¢ for me." Naoko said, "I worked it out in my head, and I got 84¢ too."

The children were practising for the annual festival time, when almost everyone age seven and up in the small town took turns working in booths. As it would be her students' first time to participate, Mrs. Neal prepared her own menu to let them practise taking orders and calculating costs. On festival day, each child would work with a family helper, but to practise, the children worked in pairs and checked each other's calculations.

Different Solution Strategies

The children performed the calculations in different ways. Kirk and Naoko calculated the cost mentally, Katie used coins and Brandon turned to the 100s chart for help. Except for several students who needed to work with cubes, these solution strategies were typical for these second graders.

Mrs. Neal thought this group was also typical of her former classes, in that children worked at different levels. Over time, she recognized the value of taking a more flexible approach to problem situations, believing children needed to work through learning stages before they spontaneously moved on to mental calculation. She was glad that the school's parent group had sewn "chair packs" to hold each child's mathematics materials (see Figure 3). This arrangement gave children ready access to various manipulatives whenever needed.

Looking Back: Good Beginnings

Why could Kirk and Naoko, as second graders, do mental calculations, whereas some older students could not? People differ, of course, but Mrs. Neal decided that much could also be attributed to a good foundation in mathematics. She recalled an

FIGURE 2

The 100s chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

FIGURE 3 **Mathematics chair pack**

observation that another local teacher had made after being moved from first to fourth grade:

> I'm convinced that you have to start early to nurture number sense and create a mind-set for working mentally with numbers. My first graders last year were more flexible and creative in their thinking for the 'Number of the Day' (Slovin 1992) than my fourth-grade group is now, even after working with them for nearly seven months.

Looking back, Mrs. Neal believed that her second-grade group had had a good foundation in mathematics. She remembered how the first-grade teacher, Mrs. Haas, had also focused on a "Number of the Day" in the spirit of the Slovin (1992) article and had often challenged the children to tell her as much as they could about such problems as 34 + 9 or 125 + 125, but to give no answers.

Mrs. Neal also knew that Mrs. Haas had laid a foundation by incorporating *counting* and *counting on by 10s and 1s* as a regular part of her daily opening. She smiled as she recalled the counting problems Mrs. Haas had created on the *Cloudy with a Chance of Meatballs* theme (Barrett 1978). The children loved this story, in which all kinds of food rained down on the town of Chewandswallow! She remembered stopping by the first-grade class one day when the children were using cubes to solve "meatball" problems:

> After a big meatball storm, Henry's family packed meatballs in bags of 10. There were 7 bags and 3 extras. Then Henry packed another 2 bags from meatballs he had found out back. How many did they find?

In retrospect, Mrs. Neal believed that these experiences had helped Kirk and Naoko to develop successfully the mental skills they had used in calculating festival orders.

At first, Kirk and Naoko had used cube trains to solve such problems. When they learned how quickly they could solve them using a 100s chart, they turned to the chart instead and soon were solving even harder problems. For example:

> On Monday, it rained 33 meatballs in Henry's yard. On Tuesday, it rained 29 more. How many then?

They would start with 33, move down 30, or three rows, to 63, then move back one space to 62.

Developmentally, Kirk and Naoko were ready to capitalize on the power of the 100s chart before many of Mrs. Neal's other students. The strength of the visual imagery furnished by the chart enabled these students to move quickly to carry out most calculations mentally. Kirk said he didn't need the chart because he could " 'see' the moves in my head." Quite naturally, both children sometimes reverted to using the chart, especially when solving problems with larger numbers.

Recognizing that many second graders might be ready to progress as Kirk and Naoko had, Mrs. Neal revisited the 100s chart. She challenged her students to describe and justify the patterns they discovered. Brandon observed, "When you move right, you add 1; when you move left, you subtract 1. When you move down, you add 10, and when you move up, you subtract 10. See [pointing], the numbers in this row are all 10 more than the numbers in the row above it."

To encourage children to apply these patterns, Mrs. Neal used literature-based problems involving addends of 10, 9 and 11:

> On Wednesday morning, Mom found 14 waffles on the porch. She needed 10 more. (How many would that be?) She only found 9 more. (How many did she have for Wednesday's breakfast?)

At first, Mrs. Neal tried to pose pairs of related problems like this one on purpose, so that students could make the connection between 14 + 10 and 14 + 9, or even 14 + 11. In a similar way, she later introduced addends like 40, 39 and 41. Brandon, like Kirk and Naoko at an earlier stage, was beginning to value the power of the 100s chart.

Other children preferred to use cubes grouped in trains of 10 for solving the problems or sometimes checked their 100s chartwork on a calculator. Marita said: "I put out a train of 10 cubes and 4 ones to show how many waffles Mom found on the porch. Then I added an extra 10-train to find how many were needed altogether. I just took one away to see how many Mom actually had for Wednesday's breakfast."

At one time, it would have concerned Mrs. Neal that these children were not progressing more quickly to the 100s chart or mental mathematics. However, she was becoming sensitive to children's thinking and individual learning levels and more reflective in accommodating these differences in her teaching.

Mrs. Neal was delighted that the festival was imminent. It presented a timely context for a learning activity that allowed her to observe and assess children's progress on the concrete-to-mental mathematics continuum.

Reflecting on the Reflection

In this series of reflections, Mrs. Neal recognizes the value of encouraging children to develop various computational processes as well as a range of solution approaches. She believes that such encouragement leads children to be more flexible, confident and independent problem solvers. Note how Mrs. Neal was willing to accept and encourage a range of solution approaches to the waffle problem, including Marita's use of cubes, Brandon's reliance on the 100s chart and other more mature mental approaches.

In addition, she recognizes the significance of posing problems and assessing understanding in a context familiar to the children. In the first example, she relates mathematical tasks to experiences that children were likely to have in the upcoming festival. Later, she uses problems developed from familiar stories from literature.

Consistent with the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, 46), Mrs. Neal believes in delaying written computation to furnish a maximum opportunity for other computational processes to grow and mature. In essence, this approach gives children time to gain competence and confidence in using a variety of nonwritten computational strategies. Her previous experiences in delaying written computation have shown her that children are more inclined to adopt flexible strategies for computing and to make better use of their repertoire of manipulative, mental and written approaches.

Mrs. Neal's thoughts model those advocated in "Standard 6: Analysis of Teaching and Learning" of the *Professional Standards for Teaching Mathematics* (NCTM 1991) and, accordingly, can potentially influence learning in a significant way. In this instructional episode, observing and analyzing the progress her students made toward mental mathematics helped Mrs. Neal recognize the power of the 100s chart as a bridge that enables children to move from solving problems with concrete materials to solving them mentally.

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Exploring Products Through Counting Digits: Equal Factors

W. George Cathcart

The *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, 39) defines number sense as "an intuition about numbers that is drawn from all the varied meanings of number." Five components of number sense are listed, two of which are relevant to this article. Children with good number sense "recognize the relative magnitudes of numbers" and they "know the relative effect of operating on numbers" (NCTM 1989, 38). Furthermore, "students with number sense pay attention to the meaning of numbers and operations and make realistic estimates of the results of computation" (NCTM 1992, 10).

Too often, students will give an unreasonable answer to a computational exercise and not be aware that it is unreasonable. How can we help children "pay attention to the meaning of numbers and operations" so that they can recognize whether an answer is unreasonable or "in the ballpark"? The booklets in the Addenda Series published by the National Council of Teachers of Mathematics contain many helpful suggestions. This article may add more ideas.

Children are sometimes taught to use a variety of estimation strategies (Reys 1986) to judge the reasonableness of an answer or to make an estimate when that is all that is needed. In addition, children could be given experiences that focus on the number of digits they should expect in an answer. The following activities focus on the latter approach. They are, for illustration purposes, restricted to multiplication but could be adapted to other operations. This article is also delimited to the case of equal factors.

Collect the Data

Suggest that students begin with a one-digit number multiplied by a one-digit number (basic facts). What is the smallest product you could get? The largest?

Now do the same for a two-digit number multiplied by another two-digit number. Encourage students to use mental computation to determine the minimum product and the calculator to get the maximum product. Invite students to continue this process for three-digit, four-digit and five-digit multipliers.

Finding the maximum product for two five-digit factors may be problematic, even with a calculator. This calculation will require multiple steps if students have a calculator with an eight-digit display—the most likely case for most middle school students. How children handle this will, in itself, provide some evidence of their level of number sense. One possible strategy follows:

Children with good number sense may recognize that 99 999 × 99 999 can be expressed as (99 999 × 999) + (99 999 × 99 000) and proceed as follows:

Calculator:	99 999 × 999 =	99 899 001	(1)
Calculator:	99 999 × 99 =	9 899 901	(2)
Mental arithmetic:	calculator result (2) × 1000 =	9 899 901 000	(3)
Paper and pencil:	mental arithmetic result (3) + calculator result (1) = 9 999 800 001		

Organize the Data

You might provide students with a blank copy of Table 1 (except for the headings and first column) and ask them to insert the data they obtained above and then complete the remaining columns.

Explore the Data

Students may notice a number of patterns in Table 1. The major purpose in developing the data was to help students reflect on the reasonableness of their product by considering the number of digits it might have. Therefore, this should be the initial focus.

Minimum and Maximum Number of Digits

Ask students to examine the data in columns 1, 3 and 5 in Table 1.

Table 1
Products and the Number of Digits in Products

Factors	Minimum Product	Number of Digits in Minimum Product	Maximum Product	Number of Digits in Maximum Product
1-digit × 1-digit	1	1	81	2
2-digit × 2-digit	100	3	9801	4
3-digit × 3-digit	10 000	5	998 001	6
4-digit × 4-digit	1 000 000	7	99 980 001	8
5-digit × 5-digit	100 000 000	9	9 999 800 001	10

Teacher: Describe the pattern you see in the column containing the minimum number of digits in a product.

Student answers might include the following:

- They go up by 2.
- They are all odd numbers.
- They are consecutive odd numbers, beginning with 1.

Teacher: How does the minimum number of digits in the product relate to the magnitude of the factors?

Allow ample time for students to investigate this because an understanding of this relationship is critical to determining the reasonableness of a product by counting digits. In the end (perhaps after some hints), students should recognize that the minimum number of digits in a product where the multiplier and multiplicand have the same number of digits is one less than the sum of the digits in the two numbers ($2n - 1$, where n is the number of digits in each factor).

Teacher: Describe the pattern you see in the column containing the maximum number of digits in a product.

Student answers might include the following:

- They go up by 2.
- They are all even numbers.
- They are consecutive even numbers, beginning with 2.

Teacher: How does the maximum number of digits in the product relate to the magnitude of the factors?

Again, allow ample time for students to investigate this because an understanding of this relationship is also critical to determining the reasonableness of a product. In the end, students should recognize that the maximum number of digits in a product where the multiplier and multiplicand have the same number of digits is twice the number of digits in each factor ($2n$).

With respect to the main purpose behind this activity, it is important for students to understand that when you have 2-digit × 2-digit, the product has either 3 or 4 digits, 3-digit × 3-digit, the product has either 5 or 6 digits and so on. This will help them to quickly detect unreasonable answers.

Minimum and Maximum Products

Students' attention could now be directed to columns 2 and 4 in Table 1, minimum and maximum products. Encourage students to describe patterns they observe. For example, in the minimum products column, most students will be able to observe that the number of zeros increases by two each time. Others might be a little more mathematical and say that "the minimum product increases by a factor of 100 as you increase the number of digits in the factors by one." Some may relate this to the previous investigations and think of the minimum product as having a 1 followed by $2n - 2$ zeros, where n is the number of digits in each factor.

Teacher: Try to describe the pattern in the maximum products column.

This pattern may be more interesting. It appears that as the number of digits in each factor increases by one, the product has one more 9 annexed at the beginning and another 0 sandwiched between the 8 and the 1.

Students may be challenged in different ways to pursue this pattern. Some junior high students may be encouraged to explain why the pattern holds or to show that it breaks down after a certain point. Others may be simply invited to see whether it will hold for a specific larger case, say an 11-digit number × 11-digit number.

At this point, some children may be challenged to add a generalization row to Table 1:

n-digit × n-digit	1 $(2n - 2)$ 0s	$2n - 1$	$(n - 1)$ 9s 8 $(n - 1)$ 0s 1	$2n$

Maximum Products by Mental Arithmetic

While using a calculator is the most logical and efficient procedure for determining the maximum product in Table 1, some students could be challenged to find a pattern that would enable them to use a mental procedure. With some guidance, they may note that

$9 \times 9 = (10 \times 10) - 19$. *Why?* $9 \times 9 = (10 - 1)(10 - 1)$
$= 100 - 20 + 1$ $\quad = 100 - 10 - 10 + 1$
$= 80 + 1$ $\quad = 100 - 20 + 1$
$= 81$ $\quad = 81$

Similarly, $99 \times 99 = (100 \times 100) - 199$
$= 10\,000 - 200 + 1$
$= 9\,800 + 1$
$= 9\,801$

The larger products in the maximum column of Table 1 are probably too difficult to do mentally for most middle school students, although a few may enjoy the challenge.

Extension: Switch-Over Points

Returning to the major focus of this article—determining the number of digits to expect in a product—at least one more investigation should be pursued. Is it possible to decide whether the product of, say, two 2-digit numbers has three digits or four digits?

Stimulate the students to think about "switch-over points" by making a statement such as the following:

Teacher: You have discovered that a 1-digit number multiplied by a 1-digit number has either one or two digits in the product. At what point does a product begin having two digits?

Many children will be inclined to begin their investigation with equal factors because that is what they have been using in the earlier explorations. (Because that is the focus of this article, this investigation will be restricted to that case.) Some will think it logical that the switch-over point would be about midway between 1 and 9 and start with 5×5. They may be surprised that this product is considerably greater than the smallest 2-digit number, 10. Using mental arithmetic, they will quickly realize that 3×3 (9) is still a 1-digit product but 4×4 (16) is a 2-digit product.

Teacher: Using your calculator and working with equal factors only, try to determine switch-over points for 2-digit and greater cases.

Using their calculators, students will quickly discover that 31×31 (961) is still a 3-digit product but 32×32 (1024) is a 4-digit product. If students are given a blank table such as Table 2 (except for column headings, first row and first column), the investigation may be facilitated.

Table 2
Switch-Over Points: Equal Factors

Number of Digits in Each Factor	Minimum Number of Digits in Product	Largest Factors for Minimum Digits	Maximum Number of Digits in Product	Smallest Factors for Maximum Digits
1	1	$3 \times 3 = 9$	2	$4 \times 4 = 16$
2	3	$31 \times 31 = 961$	4	$32 \times 32 = 1024$
3	5	$316 \times 316 = 99\,856$	6	$317 \times 317 = 100\,489$
4	7	$3162^2 = 9\,998\,244$	8	$3163^2 = 10\,004\,569$
5	9		10	
6	11		12	

Conclusion and Extension

Students are not often encouraged to consider the number of possible digits a product might have. While this strategy should not replace other estimation strategies, it should be included as one more strategy students can use to help them decide on the reasonableness of an estimate or computed answer. This article focused only on products of two equal factors.

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Developing and Assessing Understanding of Integer Operations

Jim Vance

The study of integers in the middle grades bridges the concretely based elementary school program and the more formal secondary school curricula. While the basic ideas are grounded in real-world situations and can be modeled concretely and semiconcretely, development of the operations with integers also relies on examining the consequences of extending whole number meanings, properties and patterns. As students build on previous understanding, they have many opportunities to make mathematical connections and to use informal mathematical reasoning. The first encounter with integers can be an exciting and rewarding mathematical adventure.

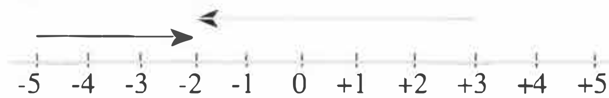
This article describes an introductory unit on integers I taught to a class of 27 Grades 6 and 7 students in an urban public elementary school. During the lessons, the students were encouraged to use materials or extend their previous knowledge to devise solutions to problems and to explain and justify their thinking orally and in writing. Following seven days of instruction, a written test, consisting mostly of multiple-part problems, was administered to assess student understanding of concepts and procedures.

Introducing the Integers

We began by talking about the nature of mathematics and how new mathematics is created. After the invention of the number zero was discussed, the students watched an overhead calculator count down from ten to zero and beyond. I described the use of *negative* numbers in ancient times and by 16th-century mathematicians, who called such numbers "fictitious" or "absurd." The class then generated various real-life situations requiring numbers that indicate direction as well as magnitude. On the second day, the idea that negative numbers are *opposites* of counting (*positive*) numbers was further explored in connection with the number line, and ways of comparing and ordering integers were investigated.

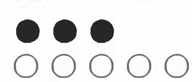
Addition

Everyday life situations were used to introduce addition: you earn \$8 and then you spend \$10; you take three steps to the left and then five to the right. Addition on the number line was interpreted as moving to the right for positive numbers and to the left for negative numbers. Students found that, although changing the order of the addends represents a different real-world problem and dictates a different number line solution, the final result is the same, as with whole numbers.



-5 + +3 = +3 + -5

The electric charges model using two-color discs (chips) was then introduced—a black disc represented a positive charge of 1 and a red disc represented a negative charge of 1. I demonstrated on the chalkboard using checkers with a magnetic strip glued to one side. Each pair of students was given a container with 10 black and 10 red chips to use in class. In diagrams, the symbols $\oplus$ and $\ominus$ were used to represent the charged chips. I explained that when a negative and a positive charge combine, they cancel each other; therefore, a black and a red chip together represent a value of zero. Students then used their chips to find solutions to various addition questions.

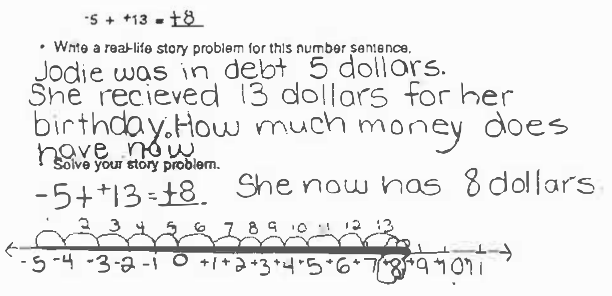


+3 + -5 = -2

Although no rules for addition were given verbally, many students volunteered that adding was easy, and, by the end of the period, most were completing multiple addend examples at the symbolic level.

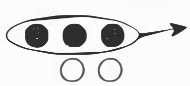
A question on the test asked students to write a story problem to correspond to the addition sentence -5 + +13 = __ and solve their problem (Figure 1). Twenty-six students wrote the correct answer +8 in the blank, and 20 were able to provide appropriate problems and solutions. Sixteen of the 20 acceptable problems involved money. Another 5 students based their problems on temperature but interpreted the second addend as a state (the temperature *is* 13°) rather than as a change or action (the temperature *rose* 13°) and were unable to formulate a meaningful question. A typical problem and solution were "Yesterday's temperature was -5°C. Today's temperature is +13°C. What is the total temperature? The total temperature is +8°C for yesterday and today." It is interesting that the students who used money for the setting did not have the same difficulty (Jean was $5 in debt and found/earned/deposited $13).

Figure 1
Danielle's Responses to the Addition Problem



Subtraction

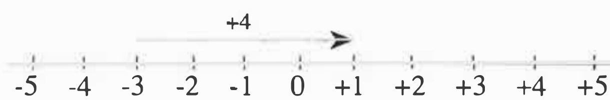
Meaningful development of subtraction of integers builds on students' previous understanding of subtraction of whole numbers, which has several interpretations—take away, comparison, missing addend and difference. The two-color chip model lends itself to the take-away interpretation and was used first. We began with +5 − +2 which means starting with five black chips and removing two. The three remaining black chips represent the expected answer +3. Next we considered -5 − -2: start with 5 red chips and remove 2; the answer then is -3. Students were then challenged to rename +1 − +3 and show their solution with the chips. Using a counting-down strategy, the class decided that the answer should be -2, but they could not immediately see how to remove three black chips when you only have one. Leaving this problem for the moment, I presented the question 32 − 18 in vertical form. We discussed the procedure of renaming 32 as 2 tens and 12 ones so that 8 ones can be removed. Next, we considered 3/4 − 5/8 and recalled that before subtracting, we rename 3/4 as 6/8. Now back to the integer problem—can we represent +1 with chips so that three black chips could be removed? The idea that adding pairs of positive and negative chips would not change the value of the pile was discussed. To solve the problem then, we might first place two more black chips and two red chips with the original black chip. Removing the three black chips leaves two red chips; the answer is -2.



+1 − +3 = -2

Using this strategy, students successfully solved a variety of subtraction questions with the chips. When a student commented that some number sentences did not make much sense, I asked the class how they check subtraction questions. They then used this same procedure to check answers obtained using the chips. For example, -5 − -2 = -3 because -2 + -3 = -5.

During the next class, we investigated subtraction on the number line. We first reviewed subtraction as counting up, using examples such as 7 − 5 and 31 − 28. We then looked at the difference between two numbers in terms of their distance apart on the number line. For example, 2 and 5 are three units apart. If we consider direction as well as distance, the distance from 2 to 5 is +3 and the distance from 5 to 2 is -3. Therefore, to solve 5 − 2 = __ on the number line, we start at 2 and move to 5 (2 + __ = 5); for 2 − 5, we start at 5 and move to 2. Using this interpretation, students completed a variety of subtraction questions, again checking their answers by addition.



+1 − -3 = +4

The students were then asked if they thought that there might be a way of subtracting positive and negative numbers without using colored chips or the number line. Four pairs of number sentences were written on the chalkboard, and the class considered how they were the same and how they were different.

+5 − +2 = +3	+5 + -2 = +3
-1 − +3 = -4	-1 + -3 = -4
+3 − -5 = +8	+3 + +5 = +8
-9 − -4 = -5	-9 + +4 = -5

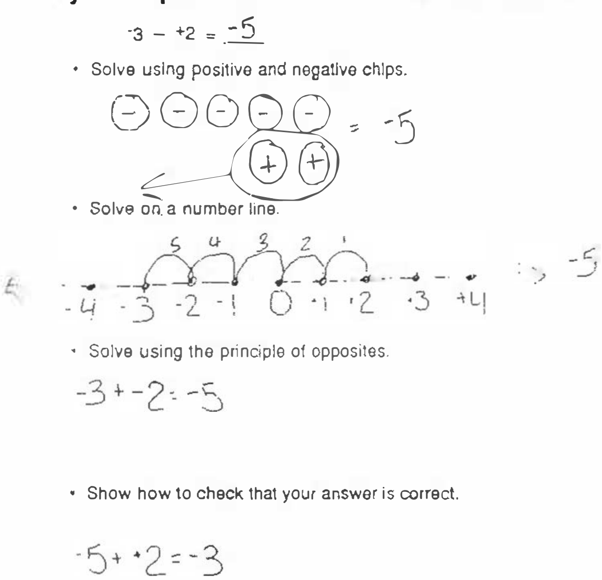
It was noted that the subtraction questions and the corresponding addition questions had the same first number and the same answer and that the second numbers in the addition questions were the *opposites* of the second numbers in the subtraction questions. Students were encouraged to state this relationship in their own words and to verify it using other examples. To further justify the result, the class examined the consequences of continuing patterns such as the following:

+3 − +2 = +1
+3 − +1 = +2
+3 − 0 = +3
+3 − -1 = __

A test question on subtraction asked the students to solve -3 − +2 = __ in three ways and to check the answer (Figure 2). Twenty students wrote the correct answer -5; the other seven wrote -1, which is the correct answer to the addition question -3 + +2 = __.

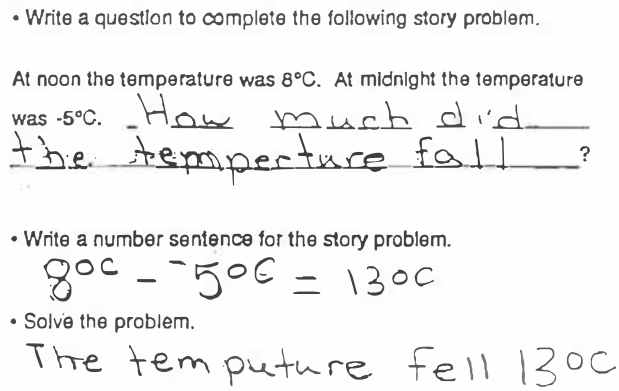
Of the 20, all showed a correct solution with positive and negative charges. Seven of the people in this group sketched a number line solution indicating the distance from +2 to -3; the other 13 drew the solution for -3 + -2 (start at -3 and move left 2). Only 10 students demonstrated how to solve the subtraction question by adding the opposite of +2, indicating that this rule had not been as well established as the concrete procedures. Furthermore, only 12 students showed how to check their answer by writing +2 + -5 = -3. All seven students who gave -1 as the answer showed correct chip and number line solutions for the addition question, but only two of them tried to check the answer by adding. One acknowledged that the check indicated that something was wrong; the other wrote +2 + -1 = -3.

Figure 2
Kelly's Responses to the Subtraction Problem



In another section of the test, students were asked to write a question to complete the story problem beginning: "At noon, the temperature was +8°C. At midnight, the temperature was -5°C." They were then to write a number sentence and solve the problem (Figure 3). Twenty students wrote meaningful subtraction questions, 12 asking how far the temperature fell and 8 referring to the difference between the two temperatures. Of these 20, 12 correctly solved their problem based on the number sentence 8 – -5 = __. Another 6 also produced this number sentence, but 5 gave the answer +3°C, and 1 student concluded (understandably) that the temperature fell -13°C. Only one of the questions written by the other 7 students made sense: "What was the average temperature?" He correctly found the sum of the two numbers, but, in computing 3 ÷ 2, he divided 3 into 2 (using the long division algorithm).

Figure 3
Laura's Responses to the Temperature Problem



Multiplication

I began the lesson on multiplication by reviewing the meaning of 3 × 2 for whole numbers. The general understanding was that it represented three groups of 2 or adding 2 three times. We then modeled +3 × +2 using black chips and the number line. Next, students were challenged to find an answer to +3 × -2 and to be prepared to explain their thinking. The class agreed that the answer should be -6, modeling this with three groups of two red chips and on the number line as three jumps of 2 to the left of zero (-2 + -2 + -2). I then wrote -3 × +2 = __ on the chalkboard. After some initial discussion about whether this had any real meaning, students recalled that 3 × 2 can also be interpreted as two groups of 3 and that 3 × 2 = 2 × 3. The idea was therefore proposed that we could think of -3 × +2 as two groups of -3 (+2 × -3); the answer should be -6. The class then correctly predicted what my next question would be, and the chalkboard looked like this:

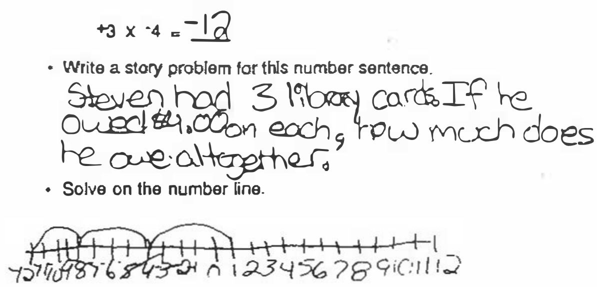
+3 × +2 = +6
+3 × -2 = -6
-3 × +2 = -6
-3 × -2 = __

After students discussed the problem with partners, they shared their ideas with the whole class. One student said that because two odds make an even (he was thinking addition), the answer is positive. Another pair used a calculator. A girl remembered hearing her mother say that two negatives make a positive. To justify the rule, students considered and continued the following pattern:

+3 × -2 = -6
+2 × -2 = -4
+1 × -2 = -2
0 × -2 = 0
-1 × -2 = __

The multiplication test question asked the students to write a story problem for +3 × -4 = __ and show a number line solution (Figure 4). All but 1 of the 27 students wrote -12 as the answer and showed this on the number line. Producing an appropriate story problem was more challenging. Twelve students wrote problems involving the idea of "3 debts of $4," two used the concept "3 times as much as a debt of $3" and four referred to "a debt of 4 times $3." One girl wrote about three groups of four friends who were not her friends. Other students made up amazing stories concerning "three groups of -4 apples/carrots/chips/people."

Figure 4
Pat's Responses to the Multiplication Problem



Division

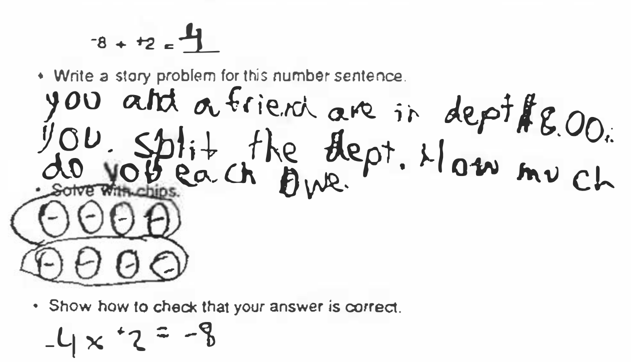
The lesson on division also began by considering whole number meanings. Students recalled that 6 ÷ 2 meant making groups of 2 or sharing six things equally between two people. The procedure for checking a division answer by multiplication was reviewed in connection with a discussion of the relationship between multiplication and division. Similarities to the addition-subtraction connection were also noted. Three new problems were then written on the chalkboard. Students were challenged to use and extend what they knew about division and multiplication to find answers.

+6 ÷ +2 = +3
-6 ÷ +2 = __
+6 ÷ -2 = __
-6 ÷ -2 = __

The students solved the problems using a guess-and-check-by-multiplication strategy. Several remarked that the results for division were the same as for multiplication.

The test question required students to write a story problem for -8 ÷ +2 = __, show a solution with the chips and check the answer (Figure 5). Twenty-five of the 27 students wrote the correct answer, and 23 showed -4 × +2 = -8 as a check. Another student wrote -8 ÷ -4 = +2 as her check. Fifteen of the story problems were based on the idea of sharing something between two people, and each was followed by a sketch showing eight negative chips divided into two groups of four. Six of the eight meaningful problems of this type involved paying half of an $8 debt. The other seven stories referred to sharing such things as bad apples, stale candies, imaginary cookies or -8 donuts. Another five students made sketches of 8 negative chips separated into groups of 2, to represent the solution to their problem. In each case, the numerical answer -4 was given, even though none of the accompanying stories made much sense. For example, "Bill had -8 carrots for lunch. If he ate +2 at a time, how many groups of carrots did he eat?" It appears that most students did not first write a problem to develop a meaningful context that would guide the solution to the number sentence. Rather, after successfully drawing a diagram to show the answer, they then attempted to transfer a familiar whole number setting (8 apples) to one involving negative numbers (-8 apples or 8 bad apples) to produce a related "story."

Figure 5
Michael's Responses to the Division Problem



Computational Skills

In addition to the questions previously described, the test included a section consisting of eight computation items, two for each operation (one in the form a * b = __; the other in the form a * __ = c). The overall success rate was 89 percent for multiplication, 81 percent for division, 69 percent for addition and 54 percent for subtraction. While the results for multiplication, division and subtraction were expected, the outcome for addition was somewhat surprising given the ease with which most students had initially learned this process. On further examination of the data, it was found that, on the item -7 + -9 = __, six of the eight students with errors gave the answer +16, suggesting an incorrect transfer of the multiplication rule to the procedure for adding two negative numbers. The item -4 × -9 = __ was correctly solved by 24 of the 27 students, including all six who had made this particular error in addition. On the other hand, one student who had correctly answered -7 + -9 = -16 then wrote -4 × -9 = -36.

Journal Writing

A reading of students' journal entries during the time they were studying integers provided additional information about their feelings and insights. Midway through the project, students wrote about the mathematics they had learned the previous week. One wrote:

> + and - numbers are ok I supoze. But I don't realy like adding & subtracting them. They are easy doing them with the chips, but harder without. I hope we don't have to × and ÷ them.

In their entries the day after the test, students reflected on what had been easy and hard, interesting and boring:

> What was interesting was why adding and subtracting was so hard in this system and multiplying and dividing was so easy.

> I found the whole concept of two negatives make a positive interesting because it was so weird and confusing.

> The interesting thing was how you would add a negative and a posetive you ended up with a lower number than you started with so it would be like subtracting.

Summary and Conclusion

The instructional approach related computational procedures to the meaning of the operations in the context of real-world applications or to number properties and patterns. In-class practice activities, homework assignments and tests were based on a small number of questions students were to solve in more than one way, using diagrams and written explanations to justify their answers. Students were often asked to relate the numbers and operations to experience and to explain their thinking to their desk partner or a family member at home. Extended practice to promote speed and accuracy was not provided at this time.

Developing an understanding of integer operations and computational procedures is a complex task but can be a rewarding experience for students and teachers. If students are to value sense-making as an important aspect of learning mathematics, a significant portion of instructional time and a major component of evaluation must reflect this emphasis.

Maple: A Computer Algebra System

John Heuver

In the September 1993 issue of *delta-K*, Dale Burnett gave some examples using the computer algebra system Mathematica. This article describes an equivalent system. Since about 1980, mathematicians at the University of Waterloo have developed a computer algebra system called Maple V Release 2 (Maple). Those involved have founded a company that maintains the software package to which mathematicians from around the world contribute. There are 400,000 users worldwide. This company now employs more than 50 people and is associated with three university research groups. More than 20 people are directly involved in design reviews and extensions. Similar computer algebra systems are available, such as REDUCE and MACSYMA. At the moment, Maple is one of the more successful user-friendly packages. Maple performs well on a Macintosh with 4 MB of RAM and a hard drive, making it a resource-efficient software package. The same hardware requirement applies to the IBM-PC and its compatibles. The Ontario Ministry of Education has obtained a provincewide license for its high schools. This was preceded by a study that gave the system a favorable recommendation.

The first computer algebra systems were developed in the '60s, but the programming languages were not portable, and their platforms were mainframe computers. In the '90s, with technological improvements in the capabilities of personal computers, matters have changed. In the '70s, the slide rule vanished from the classroom and was replaced by the handheld calculator. In the '90s, a similar event is taking place in mathematics proper with profound implications.

This time, the change is a little farther removed from everyday life because many applications are geared toward the activities of mathematicians, scientists, engineers and teachers of mathematics and science. It is especially useful in large calculations. Maple is friendly enough that a high school student with proper guidance can learn the system's basic workings. It will solve equations of a single variable up to degree four in terms of radicals or systems of linear equations. It graphs algebraic functions and conic sections and performs statistical plots and so on. Of course, it does the ordinary operations such as factoring, simplifying and multiplying algebraic expressions, which are some regular high school topics. Maple also has a built-in Pascal-like programming language specifically directed toward programming mathematical functions. It will interact with programming languages such as Fortran and C.

Educators will have to start thinking about these systems because some handheld calculators perform similar functions, although they are not quite as powerful. For many students, the computer will be just another fixture in their future working life. In the immediate future, classrooms will not likely have a computer on every desk, but a program such as Maple allows the teacher to demonstrate certain aspects of mathematical routines pertaining to the curriculum and will allow students to experiment with the system.

That the computer can perform complex calculations changes mathematics itself in ways that will affect the curriculum. In the not-so-distant past, logarithmic tables instigated by Napier (1550–1617) and others were a major aid in scientific calculations. Over time, the tables were improved, and certain conventions for easier use were adopted. The same will happen to computer algebra. The introduction of computer algebra at the high school level will eventually force curriculum changes. We have some experience with the use of calculators that perform arithmetical operations, and mental arithmetic is still a necessity of life. Algebraic operations such as factoring and multiplication and many other routines are still necessary, even if they can be done by computer. To illustrate, let me give two anecdotes. I was trying to solve problem 497 from *College Mathematics Journal* (Aboufadel 1994). The problem asked for a limit. Maple produced the answer *1* in 17 seconds on my Macintosh LCIII. It took me several hours to fill in the gaps and show that the answer was correct. However, such searches are not always successful. This means students who use the system should also be required to provide careful documentation of solutions they present. The second example comes from the June 1993 issue of *Crux Mathematicorum*. Editor Bill Sands asks in exasperation for easier arguments after presenting the solution to problem 1680: "The equalities . . . have been verified by helpful colleague Len Bos using MACSYMA. But the editor hasn't the foggiest idea how they were obtained!" (p. 271).

The computer is going to play a permanent role in mathematics. Maple also offers great opportunities for self-study and remedial tasks. Undoubtedly, basic skills in mathematics will always be required. However, we will have to redefine what the specific bounds are. Powerful systems cannot be used without being able to judge the results produced. Students need to be able to document their answers carefully. It is also necessary to be able to discover incorrect results. In the choice of software, you want to be assured that the package is going to be properly maintained, be well documented and that assistance is readily available, which was the case with Maple.

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Appendix 1

```
> P:=proc(n)
local a,b;
a:=1;b:=1;
while a<n do
if gcd(a,b) = 1   then
(a^2 - b^2,2*a*b,a^2 + b^2) fi;
b:=b-2;
if b<= 0 then b:=a ;a:=a+1 fi;
print(a^2 - b^2,2*a*b,a^2 + b^2) ;
od;
end;

P := proc(n)
     local a,b;
         a := 1;
         b := 1;
         while a < n do
                if gcd(a,b) = 1 then a^2-b^2,2*a*b,a^2+b^2 fi;
             b := b-2;
              if b <= 0 then b := a; a := a+1 fi;
                print(a^2-b^2,2*a*b,a^2+b^2)
         od
     end

> P(5);
```

3, 4, 5

5, 12, 13

7, 24, 25

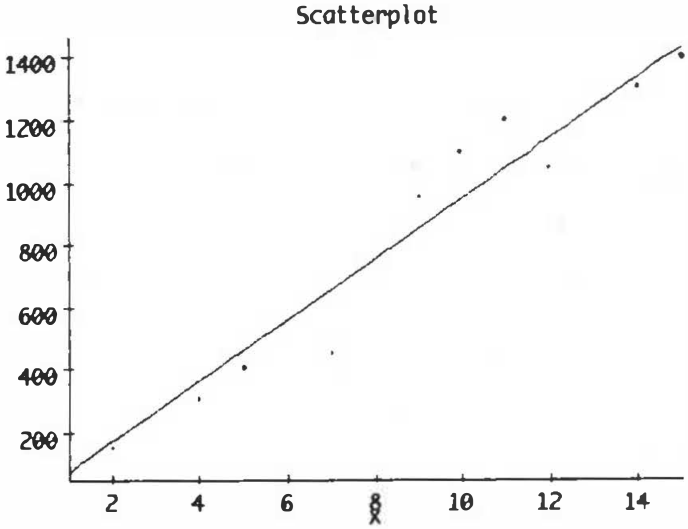
15, 8, 17

9, 40, 41

```
> This program prints the first 5 Pythagorean triples.
```

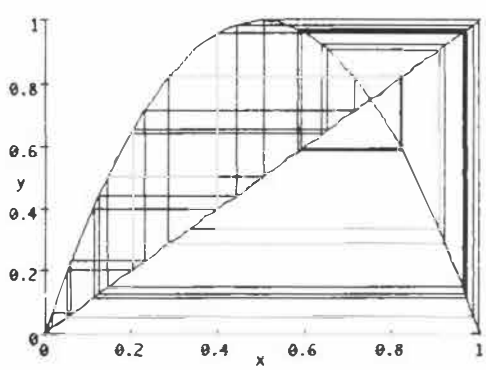
If the program is correct, Maple prints a duplicate version; otherwise, it stops at the error.

Appendix 2



Appendix 3

```
> f:=x->4*x*(1-x):

> plotpoints:=[seq((f@@      (trunc((i+2)/4)))(0.2),i=1..120)]:

> plot2:=plot(x,x=0..1,y=0..1):

> plot3:=plot(4*x*(1-x),x=0..1,y=0..1):

> plot1:=plot(plotpoints,x=0..1,y=0..1,style=LINE,style=PATCH):

> plots[display]({plot1,plot2,plot3});

>
```



This kind of iteration goes back to Fatou and Julia, two French mathematicians from the first half of this century. This was picked up again by Mandelbrot and others.

Compare this with the article "Introducing the Derivative Through Iteration of Linear Functions" in *Mathematics Teacher* (May 1994). To vary the picture, vary the function and the number of iterations.

Appendix 4

```
> with(stats):
```

```
>
dat:=array([[y,x],[200,1],[400,5],[450,7],[150,2],[1100,10],[1300,14],[950,9],[300,4],
[1400,15],[1200,11],[550,6],[1050,12]]):
```

```
> regression(dat,y=A+B*x);
```

$$\{B = 96.95652174, A = -21.48550725\}$$

```
> statplot(dat,y  =subs(",  A  +  B*x),style=point);
```

```
> dat1:=(5,9,4,5,6,7):
```

```
> average(dat1);
```

$$6$$

```
> median(dat1);
```

$$5$$

```
> sdev(dat1);
```

$$\frac{4}{5}\sqrt{5}$$

```
>
```

Area Graphs from Area Formulas: Connecting Geometry and Algebra

Bonnie H. Litwiller and David R. Duncan

Two recurring emphases in the NCTM Standards are connections among mathematical ideas and graphical representations. A rich source of examples in which these two emphases can be developed involves area formulas of geometric figures.

We shall consider several familiar area formulas and their corresponding graphical representations.

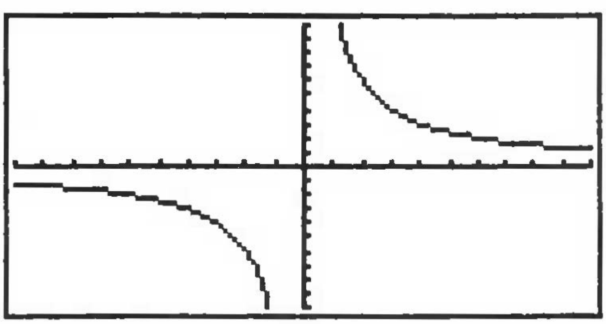
Formula 1

For a rectangle of length x and height y, $A = x \cdot y$. The most familiar use of the formula is to compute areas for the known measures of x and y. A less routine problem is to determine all possible x,y values that would yield a given area. For instance, find all possible dimensions of a rectangle that would yield an area of 12. Obvious answers include integer factors of 12: $x = 1$, $y = 12$; $x = 2$, $y = 6$; $x = 3$, $y = 4$; $x = 4$, $y = 3$; $x = 6$, $y = 2$; $x = 12$, $y = 1$.

The possibilities increase if rational numbers are used, such as $x = ½$, $y = 24$. Further consideration reveals that x could be any positive real number. For a given value of x, y will then be $12/x$.

How can all the x,y length pairs that would yield a rectangle of area 12 be visualized? Use a graphing calculator (such as TI-85) to graph $12 = xy$ or $y = 12/x$ as shown in Figure 1.

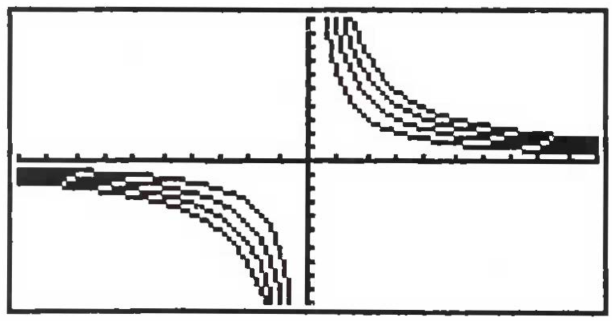
Figure 1



The graph of the equation $xy = 12$ is called a hyperbola. Note that this graph has two distinct branches—one in quadrant 1 and one in quadrant 3. Do both represent solutions to the original problem of finding the dimensions of a rectangle? (Will your students recognize that the branch in quadrant 3 cannot be used because its points have two negative coordinates?) Also note that the first quadrant branch is asymptotic to each axis. The area interpretation is that either of the rectangle's dimensions can become arbitrarily large, so long as the other dimension "shrinks" correspondingly.

The use of 12 as the area of the rectangle was arbitrary. Any positive number could be used. Figure 2 displays the first quadrant branches of the hyperbolas $xy = 6$; $xy = 9$; $xy = 12$; $xy = 15$. Have your students pair each equation with its corresponding area graph.

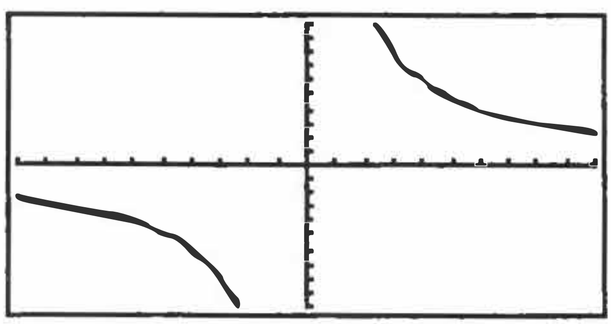
Figure 2



Formula 2

For a triangle of base x and height y, $A = ½xy$. Again, we wish to find all (x,y) values that would yield an area of 12. Expressed symbolically, this yields the equation $½\, xy = 12$ or $xy = 24$ or $y = 24/x$. Figure 3 displays the first quadrant portion of this graph—another hyperbola.

Figure 3



Have your students compare the graphs of Figures 1 and 3. Which is farther from the origin? Why?

Formula 3

For a trapezoid of bases x_1 and x_2 and height y, $A = \frac{1}{2} y(x_1 + x_2)$. If $x_1 + x_2$ is denoted by the single variable x, the formula $A = \frac{1}{2} yx$ or $A = \frac{1}{2} xy$ results. This is identical to the formula for the area of a triangle, so the same graphs would result.

However, if x_1 and x_2 were not combined symbolically, the formula for a given value of A would be $A = \frac{1}{2} y(x_1 + x_2)$. If A is 12, then $12 = \frac{1}{2} y(x_1 + x_2)$ or $y(x_1 + x_2) = 24$. This graph involves three variables, and its graphical representation would be three-dimensional. Use a software package such as Derive on a microcomputer to graph this more complex relationship.

Formula 4

Consider a right circular cylinder of radius x and altitude y. Its total surface area is found in two parts:

- Base area = $2\pi x^2$
- Lateral area = $2\pi xy$

TSA = $2\pi x^2 + 2\pi xy$

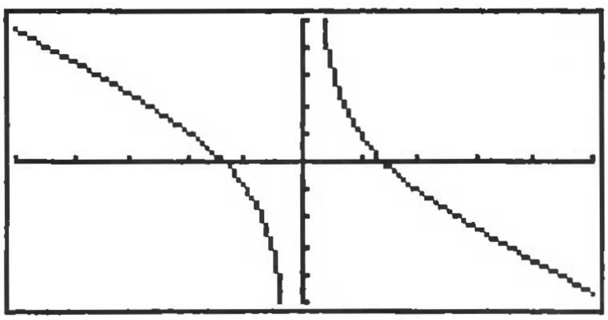
As in previous formulas, let TSA be a constant such as 12. Find the values of x and y that make this possible.

$12 = 2\pi x^2 + 2\pi xy$

$x^2 + xy = 6/\pi$

The graph of this equation is shown in Figure 4; only the first quadrant portion is meaningful in the area situation.

Figure 4



Formula 5

Consider a right circular cone with radius x and altitude y. Again, the total surface area is found in two parts.

- Base area = πx^2
- Lateral area = $\pi x \sqrt{x^2 + y^2}$ (where $\sqrt{x^2 + y^2}$ is the slant height of the cone)

TSA = $\pi x^2 + \pi x \sqrt{x^2 + y^2}$

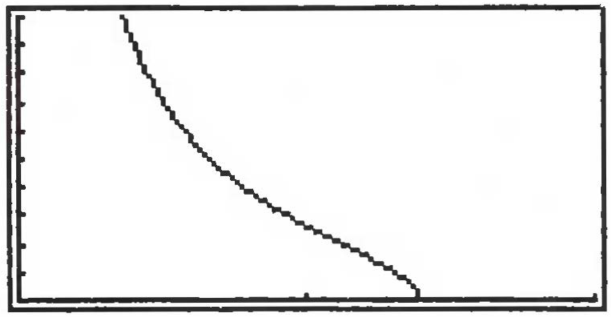
If the TSA = 12, find the values of x and y.

$12 = \pi x^2 + \pi x \sqrt{x^2 + y^2}$

$12/\pi = x^2 + x \sqrt{x^2 + y^2}$

Figure 5 displays the first quadrant portion of the graph of this equation. The scale has been adjusted to show the intervals $0 \le x \le 2$; $0 \le y \le 10$.

Figure 5



The first quadrant portions of all five figures are similar in a major respect—all graphs fall from left to right. In other words, large xs correspond to small ys and vice versa. Can your students explain why this must be true for a constant area?

In another respect, the first quadrant portions of Figures 4 and 5 are distinct from the doubly asymptotic graphs of Figures 1, 2 and 3. In Figures 4 and 5, the graph is asymptotic to the y-axis but intersects the x-axis at $\sqrt{6/\pi} \approx 1.38$.

In other words, the heights of the cylinder and cone can become arbitrarily large (being "balanced" by a shrinking radius), but the radius cannot exceed $\sqrt{6/\pi}$. Can your students explain why this should be so?

The ideas of this article can be extended in several ways:

1. Problems yielding three-dimensional graphs, as mentioned in the discussion of the trapezoid formula, can be pursued. For instance, graph the formula for the surface area of a right square pyramid and a general triangle using Hero's formula.
2. Graph the formulas for the volumes of polyhedrons.
3. The formulas for the area of a circle and the volume and surface area of a sphere involve only one independent variable (radius). Apply the methods of this article to this situation, if possible.
4. Write perimeter formulas for plane figures; graph the dimensions needed to produce a given perimeter in the spirit of this article.
5. Pick's formula is used to compute areas of polygonal figures drawn on dot paper or a geoboard. Investigate this formula and graph it for a given area.

Building a Professional Memory: Articulating Knowledge About Teaching Mathematics

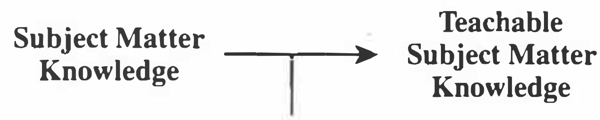
Barry Onslow and Art Geddis

In many ways, teaching is a profession without a memory. Unlike architecture and engineering, few detailed records are kept of what teachers do and how they do it. Architects and engineers leave behind drawings, specifications, models, contracts and buildings. Such artifacts provide a record of the problems faced, solutions tried and products produced. Teachers, however, leave few descriptions that record their experiences introducing a new topic or their struggles with particularly challenging curriculum. Many records that are left do little to capture the complexity of teachers' pedagogy.

An integral part of the education of doctors, lawyers and business people is the study of "cases" that record their profession's history. At present, there is no comparable body of literature to which beginning and practising teachers can turn to discover the wisdom of their predecessors. The development of a case literature of mathematics teaching should be a professional priority.

A useful focus for articulating the complexities of subject matter pedagogy is Shulman's (1987) view that effective teachers transform knowledge of subject matter into forms accessible to their pupils, rather than transmitting knowledge or pouring ideas into someone else's head. Shulman calls this amalgam of subject matter and pedagogy *pedagogical content knowledge*. It arises from deliberations about how to teach particular content to particular pupils in particular contexts and consists (among other things) of misconceptions pupils typically bring to instruction, alternative ways of representing subject matter and effective teaching strategies for changing misconceptions. To a significant degree, the acquisition of relevant pedagogical content knowledge—a way of thinking that helps the teacher understand the learner's difficulty and subsequently transform the content so the learner can understand—distinguishes effective mathematics teachers from those who are less effective.

Figure 1
Transforming Subject Matter Knowledge



Pedagogical Content Knowledge

- Student misconceptions
- Strategies for altering misconceptions
- Alternative representations

When learning primarily involves acquiring information, instruction can proceed in a transmission mode. This typically involves motivating pupils, delivering content, providing opportunities for practice and evaluating learning. In these situations, pupils employ familiar ways of thinking to assimilate new information presented by teachers. Teachers have little need to use pedagogical content knowledge because they can transmit, relatively intact, their knowledge of the subject matter to their pupils. A good deal of mathematics, however, incorporates ways of thinking that are not intuitively obvious. Effective mathematics teaching demands that subject matter be transformed to allow it to be learned meaningfully by novices. Consequently, mathematics teachers find themselves in need of extensive repertoires of pedagogical content knowledge. This knowledge needs to be captured in case studies.

Having discussed mathematics teaching with many prospective teachers over the last few years, we have found that few are able to provide suitable representations for many rudimentary mathematical abstractions. This lack of understanding often stems from their own experiences of school mathematics and the resulting perception of mathematics as a collection of isolated rules to be memorized. We will use an elementary concept, division of fractions, to illustrate the importance of appropriate representations in mathematics and the necessity for teachers to acquire pedagogical content knowledge regardless of the grade they teach.

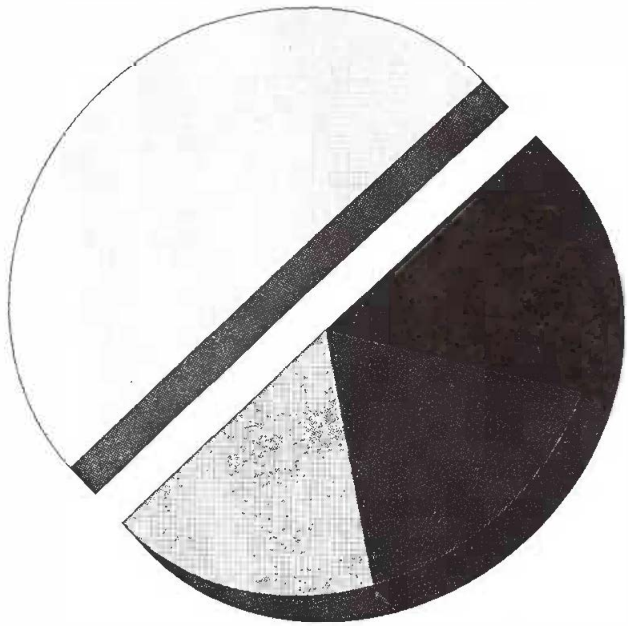
Transforming Knowledge: Division of Fractions

A question that often provides some indication of how a person has learned mathematics is 10 divided by ½. When answering this question, a few prospective teachers contribute the answer 5 (the answer being sensible for the representation they are using—10 shared in half), but most give the correct answer 20. However, when asked to explain why the answer is 20, a majority are unable to provide a representation or model. They resort to rules memorized during their own schooling. As one student teacher wryly explained, "Ours was not to reason why, just invert and multiply."

An understanding of pupils' misconceptions is important pedagogical content knowledge. Without this knowledge, teachers are not in a position to help pupils clarify their understanding. For example, in the situation described above, teachers need to know that many pupils will be comfortable with the smaller answer 5 when they divide 10 by ½ due to their previous experiences with natural numbers. Many pupils have developed the idea that "multiplication makes bigger" and "division makes smaller," which is true for natural numbers. Pupils will also find comfort in the *partition* or sharing model for division and happily share 10 in half, again obtaining 5 for the answer. Unless teachers are aware that such errors are extremely common among 12 and 13 year olds, they are unlikely to spend the necessary time introducing the concept.

Being aware of possible misconceptions, however, is insufficient to enable teachers to advance their pupils' more complete understanding of division. To assist their pupils, teachers need a second model for division, sometimes referred to as the *quotition model*, and have available suitable representations for it. One representation of the quotition model occurs when one asks, "How many people can attend a party if there are 10 pizzas and everyone receives half a pizza?". The correct answer, 20, is usually quickly contributed, but often the connection between the verbal question and its equivalent mathematical expression, 10 divided by ½, is not made. When answering a question of the form 10 divided by ½, pupils have to ask themselves, "How many halves are there in 10?" if they are to make sense of the symbolism. This notion becomes especially important when dividing a fraction by another fraction, for example, ½ divided by 1/6. Many pupils want their answers to be a fraction and, what is more, a rather messy fraction. They are often surprised when they obtain the whole number 3. However, when pupils are provided with a pictorial representation (Figure 2) and are asked, "How many sixths in one half?", most Grade 5 or 6 pupils understand why the answer is 3. Pupils are comfortable that there are six sixths in a whole pie, and therefore there should be three sixths in half the pie.

Figure 2
Pictorial Representation to Illustrate $\frac{1}{2} \div \frac{1}{6}$



Unfortunately, children, and sometimes teachers, are concerned only with the final correct answer to mathematical questions, especially arithmetical questions. This preoccupation often leads to understanding becoming of secondary importance. It is even thought of as inconsequential by many pupils. Justifying the rationale underlying a procedure helps demonstrate its importance. Only when pupils travel comfortably between real entities and symbolic representations can the term *mathematically literate* be appropriately applied to them.

Conclusions

Having suitable representations for oneself and having the knowledge to help pupils develop their own models, stories and analogies for mathematical symbolism comes not only with teaching experience but also with the philosophical outlook that such ideas are important. If mathematics is seen simply as a set of rules and procedures to be reproduced on a test, pedagogical content knowledge is unnecessary. However, if the teacher's role is to assist pupils to understand why their answer is correct or incorrect, the teacher must have more than a knowledge of general pedagogy and mathematics.

To transform subject matter content knowledge into a form accessible to pupils, teachers need to know particulars about the content relevant to its teachability, particulars that probably would not have been revealed until the task of teaching had been assumed. This pedagogical content knowledge is in some sense a result of the interaction of content and pedagogy. It is knowledge about the content derived from consideration of how best to teach it.

Certainly, teachers who are aware of a misconception, are cognizant of its origin and who possess multiple representations to correct it have the pedagogical content knowledge to make the subject matter accessible to pupils. Seldom, however, is this expertise shared systematically among colleagues, and the wisdom of practice is lost to the teaching profession (Shulman 1986). Without records of experience, pedagogical content knowledge has to be continually reinvented by each new generation of teachers—consuming time and energy that should be used constructing new understanding.

Teaching is one of the few professions where most people expect the novice to perform in a similar fashion to the veteran. If teaching is to progress, we have to recognize the intricacies associated with it and understand that teachers are also learners. Teacher educators are gradually recognizing the importance and relevance of using case studies that capture the complexities of teaching. Consequently, there is an increased awareness that the teaching profession needs cases written by practising teachers (Cochran-Smith and Lytle 1990; Shulman and Mesa-Bains 1990). Through such cases, teachers hear their peers' voices. When we have built a shared professional memory that carefully articulates knowledge about teaching mathematics, present and future teachers will have at their command examples of exceptional teaching with which to face teaching's challenges and complexities. Only then will teachers really be able to benefit from peers' wisdom and practice.

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Writing Mathematics

E.G. Bernard

Young people retain stories such as Cinderella accurately after hearing them surprisingly few times. And they remember such stories for their whole lives. Young children, then, possess the ability to learn a system of symbolic knowledge. Can techniques from language experiences assist in the learning and assessment of mathematics?

The Test

To attempt to answer this question, I visited several mathematics classes and told them the following:

> I am going to give you an unusual mathematics assignment today. First, I'll give you a nonmathematical example of the assignment and let you try it. We'll discuss and take up aspects of this assignment. Second, I'll give you the mathematical assignment. Third, I'll return in a few days, take up the assignment, explain to you what I'm trying to accomplish and get your views.
>
> I'm going to put two nonmathematical expressions on the board. When you see the two expressions, I want you to tell me the content (the story) that you associate with these two expressions.
>
> Here are the two expressions: *glass slipper* and *pumpkin*. Now tell me the content (the story) that you associate with these two expressions.

Most students said the expressions referred to the story of Cinderella. When asked to commence telling the story, most students began "Once upon a time" and were able to tell the story. I pointed out that, although they had likely not heard this story for a long time, they had an amazing amount of correct detail in their stories and the content was well sequenced. I continued:

> Now for the mathematics assignment. This time, I will put a title on the board, and you write the story "The Differences Between Adding and Multiplying Fractions."
>
> Remember what you learned in The Writing Process in your English classes? List the appropriate content, that is, what you know about the title; sequence the content; imagine a story into which to put your content; and write out a full account of the story. Give a lot of detail. Use only words.

Results

When I read the stories, I found that many students had employed introductory and concluding sentences. For instance, "There are a few differences between the addition and multiplication of fractions" and "These are the differences between the addition and multiplication of fractions."

In a few cases, students incorporated their content into a traditional type of story. For instance,

> Once upon a time, there was a boy named Bob who didn't know how to do fractions. His friend helped him after school one day. . . . Bob finally understood how to do fractions and passed his fraction test.

In another version, the conclusion stated: "And the boy not only passed his next fraction test, years later he became a math teacher." Another student's story went like this:

> Adding and Multiplying were talking to each other one day. Adding asked Multiplying to tell how he is different than Adding regarding fractions. . . . "Well, although we both deal with numbers, we are different in a few ways."

I had prepared a marking scheme and awarded marks for the title being present, for an introductory sentence (with a bonus for a creative introduction), for a conclusion and for spelling, grammar and writing style. In marking the content of the addition of fractions, I was looking for some discussion of finding a common denominator, of adding the numerators and placing the sum of the numerators over the common denominator. In marking the multiplication of fractions, I was looking for a description of multiplying the numerators and placing this product over the product of the denominators. I also awarded marks for a reference to reducing answers to their lowest terms, of dividing in the multiplication of fractions to reduce the size of the numbers, a reference to the order of operations and a discussion of dealing with mixed numbers.

As many as 40 percent of the students' marks increased on this assignment over their marks on traditional fraction tests—the classes I visited had recently completed their study of fractions, and I had the students' scores on their fraction tests. In many cases, the increases were substantial, for example, 47 to 65, 23 to 45, 63 to 80 and 72 to 100. Some students received marks similar to their marks on the traditional fraction tests. As many as 40 percent of the students' marks decreased, for example, 71 to 60, 60 to 35, 53 to 10 and 63 to 0. But most of the students whose marks decreased were new Canadians who were unable to describe or write about mathematical operations, albeit some of them scored the highest on the traditional types of tests.

Reflections

First, I believe that exercises of this type allow students to exhibit knowledge not requested in traditional tests. It is important, in all subject areas, that we employ many types of questions and various evaluation instruments that suit the varied talents of our students—to allow students to be successful in some area of each course. For instance, in visual arts courses, students are not evaluated in only one medium; various media are explored, such as sketching, watercolors, oil, charcoal and clay. As well, they are evaluated on art history. The greater the variety of tests we use, the more likely we will match an aspect of a course with a talent of a student. We should employ traditional types of testing and assignments of the kind I am suggesting here.

Second, the business world and postsecondary schools want graduates who can communicate clearly and creatively. They want young people who can describe as well as apply what they know. If this is true, a student who can apply theory and articulate it clearly should score higher than someone who can only perform the calculations properly. If writing mathematics is an important aspect in understanding and reporting mathematical concepts, I hope this article points to a way of commencing this activity in our classrooms.

A third factor was emphasized when I returned to the classes to explain what I was doing. When I commented on the fact that a young child can repeat a story such as Cinderella after hearing it surprisingly few times, some students said

> But there was no pressure or expectation to learn Cinderella at our parents' knees. And, at the time, Cinderella was interesting to us. Fractions (or mathematics in general) are not interesting to many of us.

These students pointed out the importance of using this kind of exercise (writing mathematics) at a much earlier stage in the educational process.

To implement "learning and assessing Writing Mathematics," we need to apply principles that pertain to the learning of language and stories:

- In learning language and stories, we give countless examples of how words are used. In mathematics, we need to give countless examples of how numbers are used. Further, we need to give examples of applications in, for instance, science, technology, computer studies, sports, geography, business and so on.
- Mathematics topics need to be presented in a large context rather than as isolated pieces. Language and stories are learned holistically. We did not learn Cinderella by hearing bits of the story now and then, and out of sequence. By learning (or seeing) the whole story, association and sequence are present. Association and sequence are two aspects of what is meant by the material having "meaning," not being "nonsense." There is a lot of research on the amount of time and effort required to learn nonsense syllables as opposed to learning meaningful material (research suggests four times the effort is required).

  Does mathematics come across as nonsense syllables to some children? We need to present the "whole fraction story": what fractions are, examples of what fractions look like, how fractions are used, pictures of pieces of pie, the parts of a fraction named and labeled, the four operations as they apply to fractions (noting the similarities and differences among the operations), examples of the four operations and so on. As well as association and sequence being present as a result of this approach, students would have the whole story of fractions in their notes in one place for study purposes.
- Part of the testing of mathematics knowledge, then, should include the natural questioning used in evaluating language and stories as well as the traditional means of testing.
- Assignments and assessments in mathematics need to involve great variety, not just textbook-like questions. For instance, in introductory geometry, assignments and assessments need to include problem solving, research, writing, conceptual knowledge, esthetics and so on. These aspects can be addressed while children are learning the shapes; learning the parts of each shape; labeling; learning perimeter, area and volume; applying the formulas for perimeter, area and volume; creating mobiles of shapes; researching how the shapes appear in nature; writing the story of, say, Area; using computers in generating shapes; estimating widths of rivers and heights of trees; and so on.
- The common elements of "problem solving" need to be discussed, noting that there is often more than one solution to a problem. The learning style

of each student may mean that each student prefers one way over another.

- One important method of learning in the humanities involves students asking questions prior to commencing a topic and then having their questions answered during their readings. We remember the answers to our questions better than we remember the answers to someone else's (the teacher's) questions. As students accumulate experience asking their own questions, they will get better at asking questions prior to commencing a topic—and will get better at remembering significant details about a topic.

In the case of fractions, some questions students will come up with include: What are fractions? What does the word *fraction* mean? Can you add, subtract, multiply and divide fractions? What are the rules for these operations with fractions? What are some applications of fractions? How are fractions used? What new vocabulary arises from the study of fractions? Is there a relationship between fractions and decimals, and fractions and percent, and so on?

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