Conic Aerobics

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When faced with questions involving the conic sections, many students have the impression that an infinite number of possible solutions exist, but only one acceptable to the teacher. With the gradual inclusion of graphic calculators, the laboriousness and many difficulties of the pencil-and-paper graphing approach have been removed. Other methods can assist students as they try to make sense of the topic for themselves. The activity that follows is a "low-tech" approach that has been used successfully to help Grades 11 and 12 students increase their awareness of how the different components of equations representing the conic sections affect the location and shape of their associated graphs. The methods used here do not resolve all issues, but they are easy to do, interesting, engaging for the students and contain mathematically worthwhile concepts in a memorable and enjoyable series of settings.

Conic aerobics has few requirements—a little space to move is essential and a willingness on the part of the teacher to be a bit of a ham certainly helps. A Richard Simmons wig and jogging suit might be appropriate—an imitation of the Coneheads might be going too far . . . then again . . .

The purpose of conic aerobics is to help more students remember how the various parts of the equations for parabolas, circles, ellipses and hyperbolas affect their appearance on the coordinate plane. The process described below has been used as an introduction and review of the conic sections. The process is based on the beliefs that many approaches to teaching and learning are more desirable than one and that the commitment to find and develop experiences has a strong impact on the students; one powerful enough that the students remember the approaches and can invoke them when they are needed. In this case, the intent is to give the students a feel for what happens as the components of equations representing the conic sections are altered.

The activities that follow have been tried in a classroom. They were introduced over several periods and were used along with calculators, textbooks and other materials as part of a complete mathematics program. The demonstrations and exercises are designed to lead students through each component of the conics and culminate in an activity where they work individually or in small groups to write the "script" for, and perform, a sequence of moves entitled An Invitation to the Conic Dance.

Getting Started: The Standard Position

Figure 1 sets a standing figure positioned against a background composed of a vertical Cartesian Plane with the x-axis in line with the collarbone, the y-axis vertically dividing the body and the origin at the intersection of these. The arms are held upward in a gentle U-shaped curve representing the equation $y = x^2$ and is referred to as the standard position. All activities that follow start and end in this position.

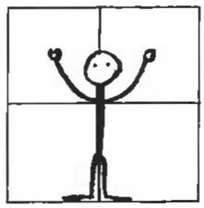

Figure 1. The Standard Position

Opening and Closing the Curve: A Change in the Leading Coefficient Alters the "Tightness" of the Curve

While all students can move their arms up and down, making tighter and looser U-shapes, it is unlikely many of them have made a connection between this movement and the equations for various parabolas. It is likely the teacher will have to demonstrate that a slightly tighter U indicates a change from $y = x^2$ to $y = 2x^2$ as in Figure 2.

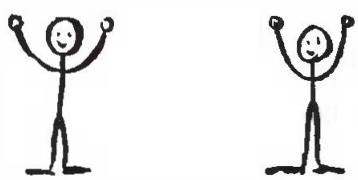

Figure 2. Transforming from $y = x^2$ to $y = 2x^2$

Questions and Tasks

1. Model, draw or demonstrate $y = 4x^2$ and $y = 6x^2$. (The arms should be placed higher and tighter together as the coefficient increases.)
2. Demonstrate how you could show $y = \frac{1}{2}x^2$ or $y = \frac{1}{4}x^2$.
3. Show how a change in the coefficient in the written representation of the equation is reflected in the graph of the same equation. Do this through a series of sketches with equations and captions indicating the transformation.
4. Explore how a change in the leading coefficient moves circles or ellipses on the coordinate plane.

An interesting situation arises as a parabola opens up to become a straight line. In body language, the arms are outstretched while in the language of math textbooks the parabola degenerates to a straight line. In the conic aerobics exercise, this awareness can arise naturally from situations created in the activity. This experience can be used to reinforce earlier work focused on the horizontal quality of $y = 0$ in linear equations.

The next step is to investigate what happens when the coefficient is negative. An exaggerated imitation of the familiar "grunting he-man muscle-builder" posed with arms in an inverted "U" position would likely fix the image in the mind of most students.

Figure 3. "He-Man" Negative Coefficient

Representations so far have included

- the effects of various positive whole number coefficients,
- the effects of proper fraction coefficients between 0 and 1,
- the results when the coefficient is 0, and
- the views when the coefficient is less than 0.

Students have likely been exposed to enough information to allow them to create and label their own examples. One type can involve a series of sketches showing a flower opening ($y = 10x^2$) as it blooms ($y = 5x^2$ . . . $y = 2x^2$ . . . $y = 0x^2$) followed by the gradual transformation to death ($y = -10x^2$).

The equations or the drawings do not need to be exact representations. The goal is to help students develop an awareness or an ability to visualize how a particular change in the coefficient results in a specific change to the corresponding graphical representation of it.

Students should now have the tools to illustrate or construct a sequence of steps such as in Figure 4.

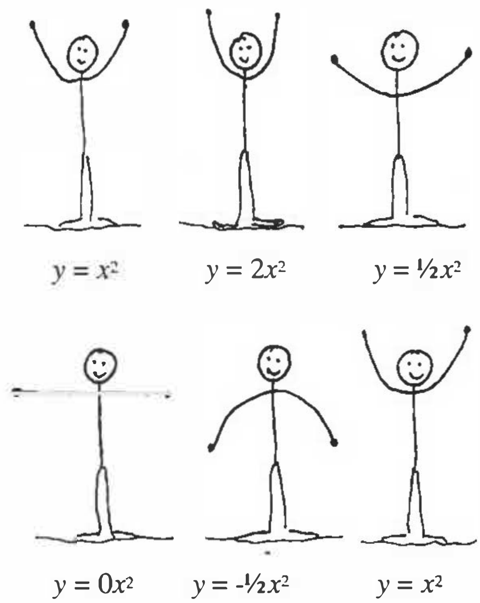

Figure 4. Six Steps Beginning and Ending with $y = x^2$

Vertical Moves

Once students have mastered opening and closing the parabola by attending to the value of the coefficient, new steps can be added to their repertoire. The next series introduces the constant value or the last term usually found in a quadratic equation of this type.

Starting from the standard position, the teacher models or has a student demonstrate $y = x^2$, but standing with heels raised slightly off the floor. This represents $y = x^2 + 1$. The next position is created by returning to the standard position with both feet flat on the ground and then *slightly* bending the knees, lowering the torso and arms to a position that can be identified with $y = x^2 - 1$. With little imagination and practice, students could demonstrate (probably with exaggerated movements) $y = x^2 + 5$ or $y = x^2 - 10$.

Sequenced steps can now include focusing on only the up and down movements, but students may want to include "opening" and "closing" in addition to the "up" and "down." Now, it may be preferable to allow them to explore and write their own scripts. One restriction that may help to avoid mild chaos is to introduce a rule that restricts each move so that it only differs from the previous one by one characteristic. For example, if one step shows $y = 2x^2 + 3$, the next step could change the vertical orientation by replacing the 3 with 2 or 5. An alternative to this would be to change only the leading coefficient from 2 to 4 or -2.

In other words, it is not as important to restrict the extent to which each variable changes as it is to only change one at a time. More students seem able to work within these limits than are able to generate random examples. Working on a restricted sequence also focuses their attention on one detail at a time, probably helps them discriminate between the types of moves permitted and reinforces the mental pictures generated.

A possible sequence of moves is demonstrated in Figure 5.

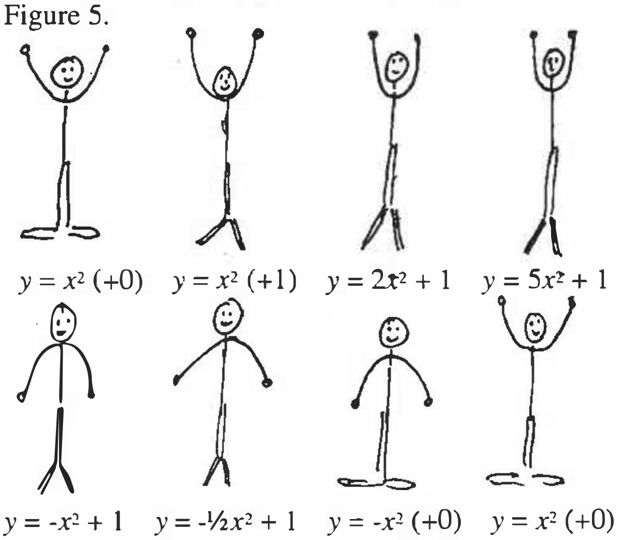

Figure 5. Leading Coefficient and Constant Sequence

Horizontal Transformations

There is no one best order for introducing the components of equations for the conic sections. It is important, however, for students to realize that there are restrictions to the types of representations. The graphic representation can show students parabolas in a variety of opened or closed positions, upright (in standard form) or inverted, and as the special case when it becomes flat (or degenerates to a straight line). The parabola in any of its states can also be placed anywhere on the y-axis. In terms of keeping the same vertical orientation on a coordinate plane, there is only one more condition for consideration—movement to the left or right.

The process of introducing this component motion can be similar to that of the previous two transformations. Students generally do not have difficulty understanding that the movement to the left or right will be reflected in the equation, but they may seem uncomfortable with the idea that $y = (x + 1)^2$ indicates a shift to the "negative" side of the y-axis. One method of working through this dilemma is for the teacher to show an equation, demonstrate the move and then ask the students to make their own generalization, rule or mnemonic device to help them remember the direction they must move.

Once the three components are in place, increasingly complex settings can be created. Students may enjoy creating moves that are sequential and have a sense of rhythm. As they become knowledgeable and comfortable with the different variations, they also take more risks and move further away from the standard position and the origin of the coordinate plane.

At this stage, it may be ideal to introduce the group activity called An Invitation to the Dance. The template in Figure 6 can be used for small groups to design their own "dance" which then can be performed for the class. For one type of presentation, the completed template is placed on an overhead like the libretto at the opera. This allows the entire class to follow along. Designating one student as choreographer and announcer can help to keep the group moving together and serves as a reminder of the moves associated with the equations.

The "dance" commences at the standard position with each "step" having only one attribute different than its immediate predecessor. The moves continue in a similar way until it is resolved in step 12 where the dance ends in the same position it started.

Conclusion

The conic aerobics activity is a simple idea that allows students to explore the variations and limits of allowable "moves" in graphing the conic sections. It is one more way that conics can come to be understood by students. Restricting the types of moves and requiring them to be presented in a sequence requires students to examine each step, discriminate between moves and assess which options are open for successive moves. This may be especially advantageous for long-term effect. Many textbook exercise sequences move quickly from simple to complex types of questions with little discernible rationale for the choice of each particular question in a set. In the sequence of tasks suggested as part of this article, students must consider each step in relation to the previous one. In a sense they are creating their own exercise and task analysis on this specific topic.

The suggested approach lends itself to the qualities of mathematics instruction promoted by the National Council of Teachers of Mathematics in the Curriculum and Evaluation Standards. Extensions to the activity are easily introduced. For example, arching horizontally suggests the graph of $x = y^2$. The parabola itself can be replaced with a circle created with the arms gracefully curved with fingers just touching. Many students are likely to suggest their own variations. On more than one occasion, the author encountered students presenting overacted representation of $y = x^3$ by imitating Steve Martin's gesticulating arms in his well-known King Tut sketch.

Activities such as these have other benefits. Students have opportunities to collaborate with each other in a lighthearted yet serious way. They can learn to appreciate the contributions of others. They can also become aware that serious mathematics can be enjoyable, countering the view that portrays mathematics as little more than a series of exercises to be completed in sequence following exact procedures in a solitary setting. Students are also allowed to ask questions that go beyond the current work. For example, what do the equations look like for parabolas symmetrical about the diagonal line represented by the equation $y = x$? How can physical rotations about an axis be accounted for in an equation? And isn't that what many of us want from our students—to become aware, to understand, to question, and to challenge themselves . . . and us?

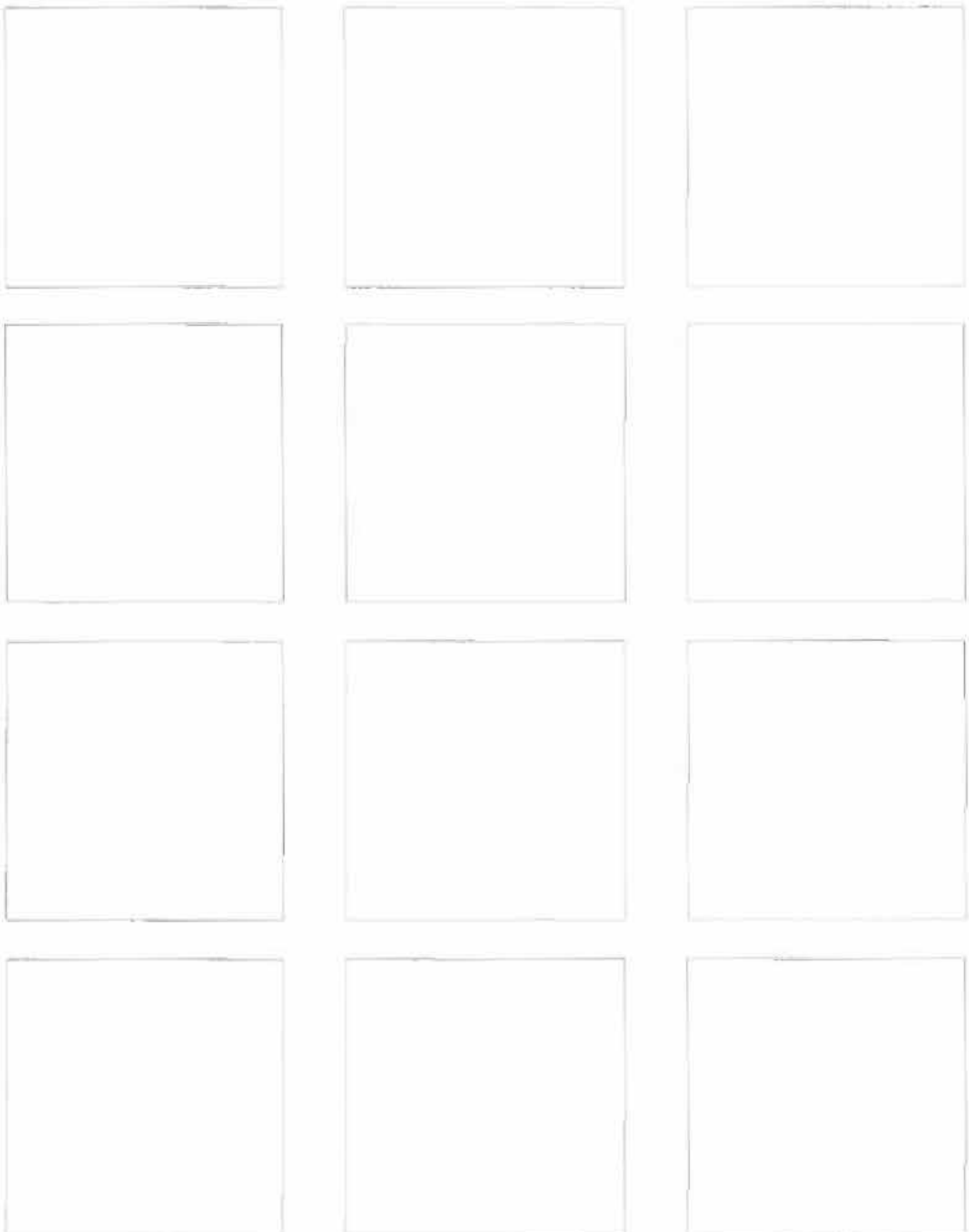

Figure 6. An Invitation to the Conic Dance