The Seeds of Tomorrow: Iterating Functions

J. Dale Burnett

It's *déjà vu* all over again.

—*Yogi Berra*

The idea of something repeating itself is one of the strongest themes in mathematics. The idea lies at the heart of mathematics: the definition of the natural numbers 1, 2, 3. . . .

It also lies at the heart of computing science. Two of the most powerful ideas in the study of computing algorithms are the loop and recursion.

What happens when you combine mathematics and computing science? Bateson (1979) suggests that an alternative title for his book *Mind and Nature* could have been "The Pattern That Connects." It would also make an appropriate title for this article. The connections among the examples to be discussed may be envisioned at many levels. I am reminded of two books: *The Cerebral Symphony: Seashore Reflections on the Structure of Consciousness* (Calvin 1990) and *Descarte's Error* (Damasio 1994), which discuss our current thoughts about thinking. Hofstadter's (1979) *Gödel, Escher, Bach: An Eternal Golden Braid* and Pickover's *Computers, Pattern, Chaos and Beauty* (1990), *Computers and the Imagination* (1991) and *Chaos in Wonderland* (1994) represent additional readings that capture the spirit of this new era. A recent addition to my library is *Frontiers of Complexity* (Coveney and Highfield 1995). The foreword for this latter book is written by Baruch Blumberg, a nobel laureate in medicine, who says,

> My experience is that, in medicine, where observational science is crucial, the complexities of a phenomenon can be understood, at least in part, by repeated observations of a whole organism or a population of organisms under a wide variety of circumstances (p. xi).

That is the theme of this article.

One of my favorite books is Alfred North Whitehead's (1929) *The Aims of Education*. I find it still timely today. Quoting from the first page:

> In training a child to activity of thought, above all things we must beware of what I will call "inert ideas"—that is to say, ideas that are merely received into the mind without being utilised, or tested, or thrown into fresh combinations.

Extending this theme, my aim in this article is to provoke one into embarking on a personal voyage of exploration. I will mention two vehicles useful for the journey and even suggest a few interesting sites, but, if this is just an article to be read, then I will not have achieved my goal.

Computing has evolved to provide a variety of human-machine interfaces. It is no longer necessary to learn how to program to make use of the computational power of computer technology. Spreadsheets represent an important alternative for harnessing the computational power of the computer. Both approaches have appeal for one interested in exploring certain mathematical topics.

This article will explore a few fairly common mathematical functions using spreadsheets and a programming approach in part to compare the results, in part to compare the effort involved and in part to see what is noticeable from the different output displays. Although this article provides a comparison between two types of tool, the focus is on gaining a deeper understanding of the behavior of certain mathematical functions under a process of iteration.

Because a tool rightfully occupies a supporting role, the article will be organized around the exploration of different mathematical functions under the process of iteration. However, it is first necessary to get a feel for the tools, if only so they can recede into the background and leave us to focus on the product.

But even this latter statement misses an important feature. The process of using the tool, of thoughtfully engaging in the activity of creating a new personal understanding, is at least as enjoyable as examining the result. Csikszentmihalyi (1993) has captured this essence in his use of the term "flow": a metaphor that conveys the feeling of "being carried away by a current, everything moving smoothly without effort." He then points out, "Contrary to expectation, 'flow' usually happens not during relaxing moments of leisure and entertainment, but rather when we are actively involved in a difficult enterprise, in a task that stretches our physical or mental abilities" (p. xiii). This is an article about "flow."

Two Vehicles for Mathematical Journeys

Programming

Let's begin with an arbitrary linear function f(x) = 2x – 6 and calculate the first three values under repeated iteration. Suppose we begin with x = 1. Then

f(1) = 2 – 3 = -1
f(-1) = -2 – 3 = -5
f(-5) = -10 – 3 = -13

We could continue the process by hand. We could also continue it using a basic handheld calculator. We could also write a computer program, using a variety of computing languages. Here is one such program, written in LogoWriter, that computes the first 20 values:

```
To ITERATE1 :n :x
   repeat 20[make "n :n+1
             make "x 2*:x-3
             print se :n :x]
End
```

In English: this is a short procedure called ITERATE1 that takes two input values (*n*: this is a counter that keeps track of the number of iterations, and *x*: the initial value of the function as we begin the iterations). The procedure thus consists of three statements that are repeated 20 times. The first statement increments the counter by 1. The second statement calculates a new value for *x*, equal to 2 times the old value for *x*, minus 3. The third statement prints the values for *n* and *x*.

Here is the resulting output when one types ITERATE1 0 1

```
1 -1
2 -5
3 -13
4 -29
5 -61
6 -125
7 -253
8 -509
9 -1021
10 -2045
11 -4093
12 -8189
13 -16381
14 -32765
15 -65533
16 -131069
17 -262141
18 -524285
19 -1048573
20 -2097149
```

One can also write a LogoWriter procedure that uses the idea of recursion (that is, a procedure that calls itself).

```
To RECURSIVE1 :n :x
        if :n=21 [stop]
        print se :n :x
        RECURSIVE1 :n+1 2*:x-3
End
```

The logic of this algorithm is fundamentally different than the iterative approach described a moment ago. The first statement within the procedure is a stop criteria: if *n* is equal to 21 then stop the process. The second line is the familiar print statement. The third line is the essence: the procedure calls itself but alters the value of *x* to the new value 2*x* – 3.

Here is the output from typing RECURSIVE 0 1

[figure 4 goes here]

```
1 -1
2 -5
3 -13
4 -29
5 -61
6 -125
7 -253
8 -509
9 -1021
10 -2045
11 -4093
12 -8189
13 -16381
14 -32765
15 -65533
16 -131069
17 -262141
18 -524285
19 -1048573
20 -2097149
```

It is reassuring to see the results are identical!

One can use these two simple procedures to investigate almost any function. One need only alter one line in the procedure, the line that specifies the function.

Spreadsheets

A variety of spreadsheet programs are available for all computers. I will use Excel, because it is one I use regularly. Here are the first two rows of the spreadsheet:

	A	B	C
1	n	x	2x - 3
2	1	1	-1
3			

The first row is just for typing in a heading for each column. The second row contains the initial values for each column. Thus n (the number of iterations) is 1. The second column contains the value of x. The third column is computed (using the formula =2*B2 – 3, which means 2 times the value in cell B2 minus 3).

The power of the spreadsheet begins to become apparent in the next row. Cell A3 is computed, using the formula =A2 + 1. That is, take the value in the cell above and add one. Cell B2 is also computed: the formula is =C2. Thus it takes the value that was computed in the previous row. Finally cell C3 is computed using the same formula as before: =2*B3 – 3, adjusted for row 3.

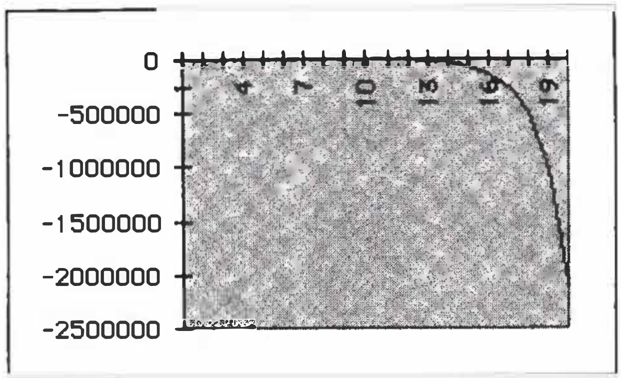
Here are the results:

	A	B	C
1	n	x	2x - 3
2	1	1	-1
3	2	-1	-5
4			

Now comes the action. One can use a command called Fill-Down and select the next 18 rows. The results are immediately displayed as follows:

	A	B	C
1	n	x	2x - 3
2	1	1	-1
3	2	-1	-5
4	3	-5	-13
5	4	-13	-29
6	5	-29	-61
7	6	-61	-125
8	7	-125	-253
9	8	-253	-509
10	9	-509	-1021
11	10	-1021	-2045
12	11	-2045	-4093
13	12	-4093	-8189
14	13	-8189	-16381
15	14	-16381	-32765
16	15	-32765	-65533
17	16	-65533	-131069
18	17	-131069	-262141
19	18	-262141	-524285
20	19	-524285	-1048573
21	20	-1048573	-2097149
22			

Once again, it is nice to see the results agree with the two LogoWriter procedures! The spreadsheet approach has another nice feature: graphing. It is a relatively easy matter to obtain the following graph:



This graphing feature may be useful later.

Interesting Sites for Mathematical Journeys

Investigating a function under repeated iteration requires a few preliminary ideas. First, the iteration must begin with a specified initial value. This initial value is usually called the *seed.* Second, the successive values of the iterated function, for a particular seed, are called the *orbit* of that point. Thus each initial value of the function has its own orbit. The investigation of all the orbits of a given function is called *orbit analysis*.

Let's begin with a simple example, that of the linear function from the preceding section: $f(x) = 2x - 3$.

A Linear Function: $f(x) = 2x - 3$

The previous section showed, using two approaches, that the orbit of the point $x = 1$ approached negative infinity. What about the orbits of other points? Do they also approach negative infinity?

A first venture into this question might proceed by simply trying a few disparate values of x for the seed, and seeing what happens. Let's try a few prototypical values: -100, -1, 0, 0.5, 10 and 100. Also let's look at $x = 3/2$, the root of the equation. Thus the first goal in our exploration is to develop a quick informal feel for the situation.

Here is the result of the first 20 iterations using the LogoWriter program ITERATE1 for the point $x = -100$:

```
1 -203
2 -409
3 -821
4 -1645
5 -3293
6 -6589
7 -13181
8 -26365
9 -52733
```

10	-105469
11	-210941
12	-421885
13	-843773
14	-1687549
15	-3375101
16	-6750205
17	-13500413
18	-27000829
19	-54001661
20	-108003325

Each value is less than the preceding value. The orbit is approaching negative infinity.

Let's look at the orbit for $x = +100$:

1	197
2	391
3	779
4	1555
5	3107
6	6211
7	12419
8	24835
9	49667
10	99331
11	198659
12	397315
13	794627
14	1589251
15	3178499
16	6356995
17	12713987
18	25427971
19	50855939
20	101711875

This is different. The orbit for this value of x tends to plus infinity. Are there other values that orbits for this function approach? Also, if not, where is the dividing point, such that for values of x less than this value, all orbits tend to minus infinity, and for all values greater than this value, all orbits tend to plus infinity. Recall that the previous example showed the orbit for $x = 1$ was minus infinity.

Let's try the seed $x = 5$.

ITERATE1 shows that this orbit appears to tend to positive infinity. Let's try the root $x = 3/2$. The LogoWriter procedure shows this value tends to negative infinity.

As a result of trying a number of values, we can see that if the seed is less than 3 then the orbit moves to minus infinity, and if it is greater than 3, then the orbit moves to plus infinity. If the seed is 3, then the orbit consists of the single point 3.

This type of orbital analysis could be extended to any linear function. Is it always the case that orbits move to plus or minus infinity depending on the value of the seed?

A Quadratic Function

Let's examine a slightly more complicated function: $f(x) = x^2 - 3$. What are the orbits for various starting points? This time, let's use a spreadsheet for a preliminary venture into the behavior of this function.

Let's try a seed of zero:

	A	B	C
1	n	x	x^2 - 3
2	1	0	-3
3	2	-3	6
4	3	6	33
5	4	33	1086
6	5	1086	1179393
7	6	1179393	1.391E+12
8	7	1.391E+12	1.9348E+24
9	8	1.9348E+24	3.7434E+48
10	9	3.7434E+48	1.4013E+97
11	10	1.4013E+97	1.964E+194

The orbit very quickly approaches infinity!

Let's use a spreadsheet to see what happens when the seed is -2, -1, 1 and 3. For $x = -2$:

	A	B	C
1	n	x	x^2 - 3
2	1	-2	1
3	2	1	-2
4	3	-2	1
5	4	1	-2
6	5	-2	1
7	6	1	-2
8	7	-2	1
9	8	1	-2
10	9	-2	1
11	10	1	-2
12	11	-2	1
13	12	1	-2
14	13	-2	1
15			

This is surprising! Let's try the others. For $x = -1$:

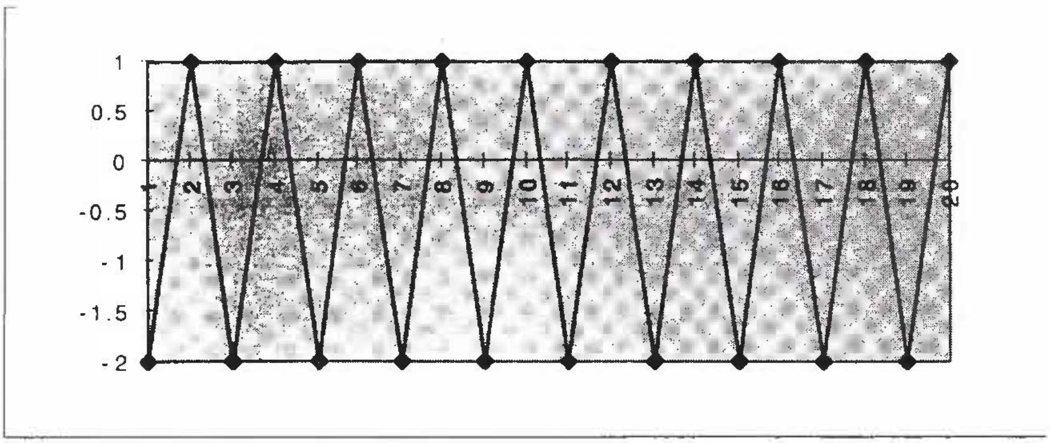
	A	B	C
1	n	x	x^2 - 3
2	1	-1	-2
3	2	-2	1
4	3	1	-2
5	4	-2	1
6	5	1	-2
7	6	-2	1
8	7	1	-2

9	8	-2	1
10	9	1	-2
11	10	-2	1
12	11	1	-2
13	12	-2	1
14	13	1	-2
15			

For $x = 1$:

	A	B	C
1	n	x	x^2 - 3
2	1	1	-2
3	2	-2	1
4	3	1	-2
5	4	-2	1
6	5	1	-2
7	6	-2	1
8	7	1	-2
9	8	-2	1
10	9	1	-2
11	10	-2	1
12	11	1	-2
13	12	-2	1
14	13	1	-2
15			

Here is the graph for these three cases:



For $x = 3$:

	A	B	C
1	n	x	x^2 - 3
2	1	3	6
3	2	6	33
4	3	33	1086
5	4	1086	1179393
6	5	1179393	1.391E+12
7	6	1.391E+12	1.9348E+24
8	7	1.9348E+24	3.7434E+48
9	8	3.7434E+48	1.4013E+97
10	9	1.4013E+97	1.964E+194

These orbits for $x = -2$, -1 and 1 were a bit different—they all oscillate between just two values: -2 and 1.

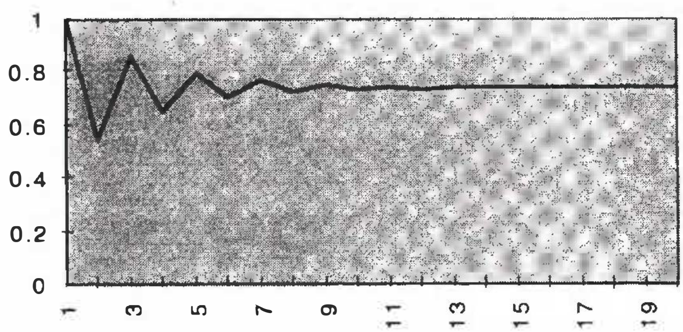
Although most orbits for this function tend to plus infinity, a few orbits have period 2, oscillating between just two values. The analysis has not been exhaustive. Are there other points that have interesting orbits? We will return to the case of quadratic functions again in a moment, but first let's have a quick look at a trigonometric function.

The Trigonometric Function: cos x

We will begin with a seed of 1.

	A	B	C
1	n	x	cos x
2	1	1.0000	0.5403
3	2	0.5403	0.8576
4	3	0.8576	0.6543
5	4	0.6543	0.7935
6	5	0.7935	0.7014
7	6	0.7014	0.7640
8	7	0.7640	0.7221
9	8	0.7221	0.7504
10	9	0.7504	0.7314
11	10	0.7314	0.7442
12	11	0.7442	0.7356
13	12	0.7356	0.7414
14	13	0.7414	0.7375
15	14	0.7375	0.7401
16	15	0.7401	0.7384
17	16	0.7384	0.7396
18	17	0.7396	0.7388
19	18	0.7388	0.7393
20	19	0.7393	0.7389
21	20	0.7389	0.7392
22	21	0.7392	0.7390
23	22	0.7390	0.7391
24	23	0.7391	0.7391

In this case, the orbit appears to converge to a value of 0.7391.



From the examples we have looked at so far we have discovered orbits that

1. fly off to infinity,
2. exhibit periodic behavior, and
3. converge to a particular value.

Another Quadratic Function

Earlier we considered the quadratic function $f(x) = x^2 - 3$

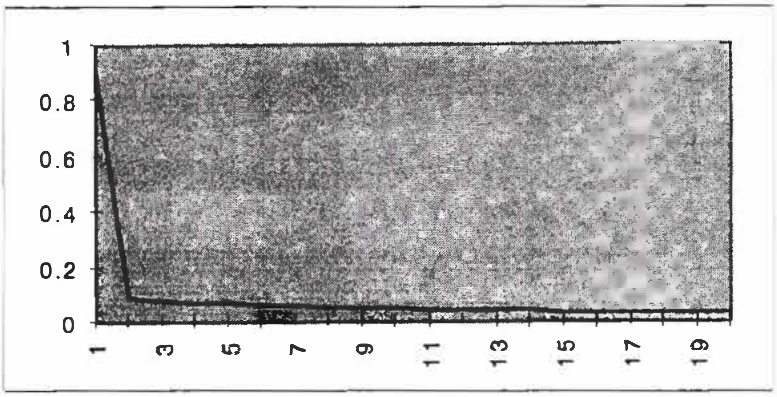
Here is another quadratic equation: $f(x) = cx(1 - x)$. This equation has a rich history, particularly in the life sciences, where it is used to model population dynamics. Because the value for c is not specified, this equation represents a family of equations.

This time I want to hold the seed constant at $x = 0.9$ and see what happens to its orbit as I vary the value of c.

Case 1: $c = 1$

	A	B	C
1	n	x	x(1 - x)
2	1	0.90	0.09
3	2	0.09	0.08
4	3	0.08	0.08
5	4	0.08	0.07
6	5	0.07	0.06
7	6	0.06	0.06
8	7	0.06	0.06
9	8	0.06	0.05
10	9	0.05	0.05
11	10	0.05	0.05
12	11	0.05	0.05
13	12	0.05	0.04
186	185	0.01	0.01
187	186	0.01	0.01
188	187	0.01	0.00
189	188	0.00	0.00
190	189	0.00	0.00
191	190	0.00	0.00

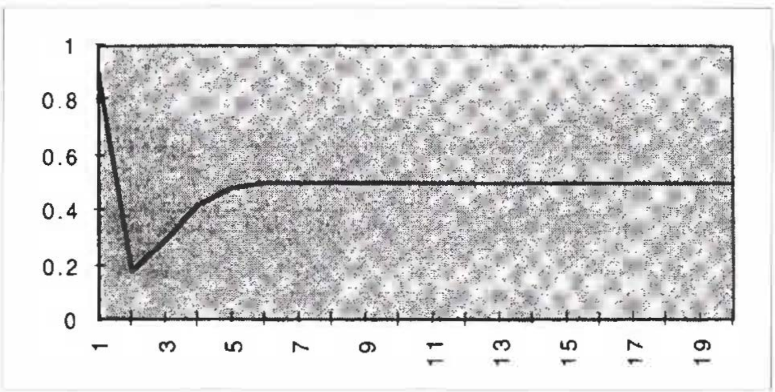
The orbit converges to zero.



Case 2: $c = 2$

	A	B	C
1	n	x	2x(1 - x)
2	1	0.90	0.18
3	2	0.18	0.30
4	3	0.30	0.42
5	4	0.42	0.49
6	5	0.49	0.50
7	6	0.50	0.50
8	7	0.50	0.50
9	8	0.50	0.50
10	9	0.50	0.50
11	10	0.50	0.50
12	11	0.50	0.50
13	12	0.50	0.50
14	13	0.50	0.50

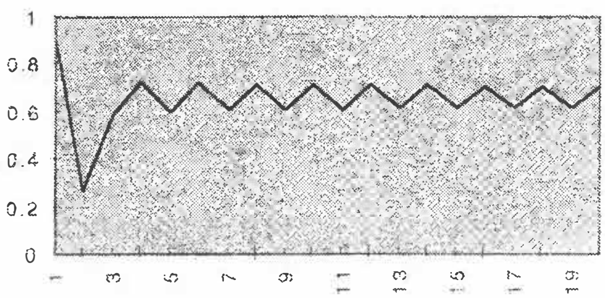
After a momentary drop, the orbit appears to converge to 0.5.



Case 3: $c = 3$

	A	B	C
1	n	x	3x(1 - x)
2	1	0.90	0.27
3	2	0.27	0.59
4	3	0.59	0.72
5	4	0.72	0.60
6	5	0.60	0.72
7	6	0.72	0.60
8	7	0.60	0.72
9	8	0.72	0.61
10	9	0.61	0.72
11	10	0.72	0.61
12	11	0.61	0.71
13	12	0.71	0.61
14	13	0.61	0.71

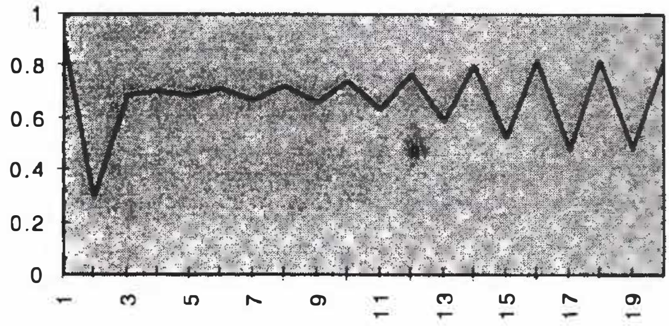
The orbit oscillates but appears to converge to two values.



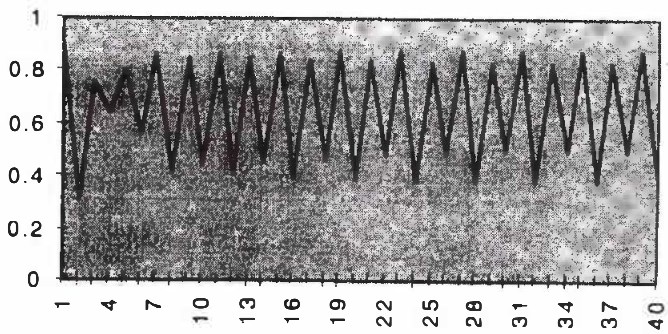
Case 4: $c = 3.3$

	A	B	C
1	n	x	3.3x(1 - x)
2	1	0.90	0.30
3	2	0.30	0.69
4	3	0.69	0.71
5	4	0.71	0.68
6	5	0.68	0.71
7	6	0.71	0.67
8	7	0.67	0.73
9	8	0.73	0.66
10	9	0.66	0.74
11	10	0.74	0.63
12	11	0.63	0.77
13	12	0.77	0.59
14	13	0.59	0.80
15	14	0.80	0.53
16	15	0.53	0.82
17	16	0.82	0.48
18	17	0.48	0.82
19	18	0.82	0.48
20	19	0.48	0.82

Once again, the orbit converges to two values.

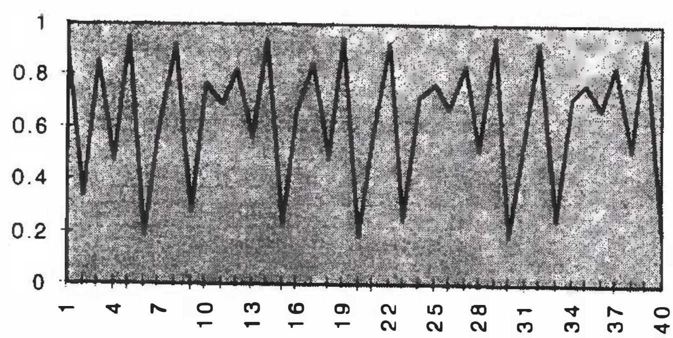


Extending the number of cycles gives:



Case 5: $c = 3.8$

296	295	0.63	0.89
297	296	0.89	0.37
298	297	0.37	0.89
299	298	0.89	0.38
300	299	0.38	0.89
301	300	0.89	0.36
302	301	0.36	0.87
303	302	0.87	0.42
304	303	0.42	0.93
305	304	0.93	0.26
306	305	0.26	0.69



There is no noticeable pattern! This orbit is called chaotic.

A review of this last example has illustrated how a seemingly simple idea, repeated iteration of a relatively simple equation, can give rise to genuine unpredictability. It also introduces a new branch of mathematics, chaos theory.

The above examples illustrate how existing software (programming languages and spreadsheets) can be used to explore new topics in mathematics, topics that are only accessible with computer support.

As we moved from the pencil to the calculator, there was spirited debate about the potential implications. Much of this discussion was predicated on assumptions about how people learn and, in particular, about how they learn mathematical ideas. It is sobering to realize that after 50 years of research, we still have only a vague sense of how people learn or of the conditions that support such learning.

It is also sobering to read about the many crosscultural studies of students' mathematical performance and of the relative standing of most North American students in these studies. Such findings do not support a position of complacency. There is a growing realization that we need to emphasize a deeper understanding of the ideas and concepts that constitute mathematics. However, agreeing on the destination and getting there are different things.

In conclusion, I would like to emphasize three points. First, this article has suggested that we should consider some new destinations for mathematics education. I would like to see new topics inserted into the curriculum that reflect the enthusiasm of practising mathematicians. Second, in considering pedagogical issues, the mathematics curriculum of the near future needs to shift from its dominant behaviorist position with its emphasis on practice, to a more cognitively oriented position that compares, interprets and discusses different mathematical situations. Finally, we must learn to incorporate new tools, such as computers, into the curriculum—tools that permit us to journey further into mathematical domains.

The self is a repeatedly reconstructed biological state. . . .

—*Antonio R. Damasio* (1994, 227)

Bibliography

Bateson, G. *Mind and Nature*. New York: Bantam, 1979.

Burnett, J.D. "Playing with $cx(1 - x)$." *delta-K* 32, no. 2 (1995): 14–26.

Calvin, W.H. *The Cerebral Symphony: Seashore Reflections on the Structure of Consciousness*. New York: Bantam Books, 1990.

Cohen, J., and I. Stewart. *The Collapse of Chaos: Discovering Simplicity in a Complex World*. New York: Viking, 1994.

Coveney, P., and R. Highfield. *Frontiers of Complexity*. New York: Fawcett Columbine, 1995.

Csikszentmihalyi, M. *The Evolving Self*. New York: HarperCollins, 1993.

Damasio, A.R. *Descarte's Error*. New York: Grosset/Putnam, 1994.

Devaney, R.L. *Chaos, Fractals, and Dynamics*. Reading, Mass.: Addison-Wesley, 1990.

Hall, N., ed. *Exploring Chaos*. New York: Norton, 1991.

Hofstadter, D.R. *Gödel, Escher, Bach: An Eternal Golden Braid*. New York: Basic Books, 1979.

Jones, J. *Fractals for the Macintosh*. Corte Madera, Calif.: Waite Group, 1993.

Lauwerier, H. *Fractals*. Princeton, N.J.: Princeton University Press, 1991.

Mandelbrot, B.B. *The Fractal Geometry of Nature*. New York: Freeman, 1977.

Pagels, H.R. *The Dreams of Reason: The Computer and the Rise of the Sciences of Complexity*. New York: Bantam Books, 1988.

Peak, D., and M. Frame. *Chaos Under Control: The Art and Science of Complexity*. New York: Freeman, 1994.

Peitgen, H.-O., and P.H. Richter. *The Beauty of Fractals*. Berlin: Springer-Verlag, 1986.

Peitgen, H.-O., and D. Saupe, eds. *The Science of Fractal Images*. New York: Springer-Verlag, 1988.

Peitgen, H.-O., H. Jürgens and D. Saupe. *Chaos and Fractals*. New York: Springer-Verlag, 1992.

Pickover, C. *Computers, Pattern, Chaos and Beauty*. New York: St. Martin's, 1990.

—. *Computers and the Imagination*. New York: St. Martin's, 1991.

—. *Chaos in Wonderland*. New York: St. Martin's, 1994.

Poundstone, W. *The Recursive Universe*. Chicago: Contemporary Books, 1985.

Schroeder, M. *Fractals, Chaos, Power Laws*. New York: Freeman, 1991.

Waldrop, M.M. *Complexity*. New York: Touchstone, 1992.

Whitehead, A.N. *The Aims of Education*. New York: Free Press, 1929.