An Intuitive Meaning for the Number *e*

Murray L. Lauber

Two irrational numbers that crop up continually in mathematics are π and e. It is easy to attach an intuitive meaning to π: the ratio of the circumference of a circle to its diameter. But in the minds of the majority of students, the number e has no such meaning. And yet, as is the case for many mathematical concepts, attaching an intuitive meaning to e might enrich students' conceptual frameworks and enable them to make conjectures about how expressions involving e should behave.

The Number *e* in Terms of Continuously Compounding Interest

One such meaning for e grows out of continuously compounding interest. It is easiest to make the connection with e if one begins with the problem of calculating the amount after 1 year of $1 invested at 100 percent per year compounded annually. One can then consider how the amount changes when the interest is compounded quarterly, monthly, weekly, daily and hourly. The logical extension is the case of compounding continuously. Using the formula $A=P(1+i)^n$ where P represents the principle, n the number of compounding periods, i the interest rate per compounding period and A the amount, one may compute the amounts as in the following chart. In this problem, $P=1$. The values of n and i depend on how often the interest is compounded each year. For compounding annually, $n=1$ and $i=1$ (100 percent); for compounding quarterly, $n=4$ and $i=\frac{1}{4}=0.25$ (corresponding to 25 percent per quarter for 4 quarters); for compounding monthly, $n=12$ and $i=1/12$; and so on. A calculator can be used to calculate the decimal approximations in the last column, although the limits of the calculator become apparent as one proceeds to shorter compounding periods. For example, the value of A in the last row and column is suspect.

Compounding Period	Number of Periods, n	Interest per Period, i	Amount A after 1 year	Decimal Approximation of A (Using Calculator)
Yearly	$n=1$	$i=1$	$(1+1)^1$	2.0000000
Quarterly	$n=4$	$i=\frac{1}{4}$	$(1+\frac{1}{4})^4$	2.4414063
Monthly	$n=12$	$i=1/12$	$(1+1/12)^{12}$	2.6130353
Weekly	$n=52$	$i=1/52$	$(1+1/52)^{52}$	2.6925969
Daily	$n=365$	$i=1/365$	$(1+1/365)^{365}$	2.7145677
Hourly	$n=8760$	$i=1/8760$	$(1+1/8760)^{8760}$	2.7181209
By the Minute	$n=525600$	$i=1/525600$	$(1+1/525600)^{525600}$	2.7180100[1]

So, in general, the amount after 1 year of $1 at 100 percent per year compounded n times in the year is $\left(1+\frac{1}{n}\right)^n$. Further, the amount after 1 year of $1 at 100 percent per year compounded continuously should be the limit of this expression as n becomes arbitrarily large, that is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. One of the definitions for e is precisely this limit. Thus we have an intuitive meaning for e in terms of continuously compounding interest: the amount after 1 year of 1 unit of currency compounded continuously at 100 percent per year.

Using this Concept of *e* to Make Conjectures About Limits

One can extend this reasoning to express the amounts in more general compound interest problems in terms of e. Changing the amount of the principle P while keeping other factors constant changes the amount A by the same proportion as P is changed. For a \$1 principle, let's consider the amount after several years.

Example 1

Consider the case where we wish to determine the amount after 2 years of \$1 invested at 100 percent per year compounded hourly, and then compounded continuously.

Solution

We first use an iterative process to determine the amount with hourly compounding. The number of compounding periods or hours per year is n=8760 and the interest rate per hour is i=1/8760. The amount after 1 year is $\left(1+\frac{1}{8760}\right)^{8760}$. Using this as the principle for the second year of investment and substituting in the formula $A=P(1+i)^n$ with n=8760 (the number of hours in the second year) and $i=\frac{1}{8760}$, we obtain

$$A=\left(1+\frac{1}{8760}\right)^{8760}\left(1+\frac{1}{8760}\right)^{8760}=\left(1+\frac{1}{8760}\right)^{17520}\approx 7.3881769.$$

Alternatively, we could use the interest rate of $i=\frac{1}{8760}$ per hour and the fact that there are 17,520 hours in 2 years to obtain $A=\left(1+\frac{1}{8760}\right)^{17520}$ directly.

Using an iterative process to determine the amount with continuous compounding, we first note that the amount after 1 year is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Using this as the principle for the second year of investment, we obtain

$$A=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2=$$

$$\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^2=e^2\approx 7.3890561$$

Alternatively, we could use an interest rate of $i=\frac{1}{n}$ where there are n compounding periods per year, or $2n$ compounding periods altogether to obtain $A=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2$ directly.

Consideration of problems with differing yearly interest rates can prompt conjectures about related limits. This is illustrated in the next example. Here a conjecture is made employing the notion of continuously compounding interest and then proved using l'Hospital's Rule.

Example 2

Consider the case where we wish to determine the amount after 1 year of \$1 invested at 200 percent per year compounded hourly, and then compounded continuously.

Solution

First consider the case of compounding hourly. The number of compounding periods (hours) in the year is n=8760 and the interest rate per compounding period is $i=\frac{2}{8760}$. Thus we obtain

$$A=\left(1+\frac{2}{8760}\right)^{8760} \approx 7.3873531$$

Note that the decimal approximation here is close to but not the same as the decimal value in Example 1 for the amount after 2 years of \$1 invested at 100 percent/year compounded hourly.

The amount for continuous compounding is then given by $A=\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^n$. Intuitively, compounding continuously at 200 percent per year for 1 year should be the same as compounding continuously at 100 percent per year for 2 years (since the interest rate for the total period is 200 percent in both cases). Thus the amount for continuous compounding should be the same in examples 1 and 2. This leads to the conjecture that

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2 \text{ or } e^2.$$

This can be proven using l'Hospital's Rule as follows.

Let y $= \lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

$= e^{\ln\left[\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right]}$

$= e^{\lim\limits_{n\to\infty}\left[\ln\left(1+\frac{2}{n}\right)^n\right]}$

Let x $= \lim\limits_{n\to\infty}\left[\ln\left(\frac{n+2}{n}\right)^n\right]$

Then y $= e^x$

x $= \lim\limits_{n\to\infty}\left[\ln\left(\frac{n+2}{n}\right)^n\right]$

$= \lim\limits_{n\to\infty}\left[n\ln\left(\frac{n+2}{n}\right)\right]$

$= \lim\limits_{n\to\infty}\frac{\ln\left((n+2)/n\right)}{1/n}$

Because both numerator and denominator of this limit approach 0, we may apply l'Hospital's Rule. However, it is convenient to modify the numerator before differentiating the numerator and denominator.

x $= \lim\limits_{n\to\infty}\frac{\ln(n+2)-\ln n}{1/n}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+2} - \frac{1}{n}}{-1/n^2} \qquad \text{l'Hospital's Rule}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n(n+2)} = 2$$

Then y $= e^x = e^2$. Q.E.D.

One can make conjectures about limits similar to those in Examples 1 and 2 by thinking of them in terms of continuously compounding interest. The following examples lead to conjectures that can be proved using l'Hospital's Rule by a process analogous to that used in Example 2.

Example 3

Find $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n$

This is the amount of $1 after 1 year compounded continuously at 300 percent per year. It should equal $\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^3$, the amount after 3 years of $1 compounded continuously at 100 percent per year. This leads to the conjecture that $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^3$.

Example 4

Find $\lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{2n}$

This is the amount of $1 compounded continuously at 500 percent per year for 2 years. It should equal $\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{10}$, the amount after 10 years of $1 compounded continuously at 100 percent per year. This leads to the conjecture that $\lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{2n} = e^{10}$.

Example 5

Find $\lim_{n\to\infty} \left(\frac{2n+5}{2n+1}\right)^{4n}$

It may not be immediately apparent how to make a direct association between this limit and continuously compounding interest, but by changing its form and making a substitution we may do so. First we change its form:

$$\lim_{n\to\infty} \left(\frac{2n+5}{2n+1}\right)^{4n} = \lim_{n\to\infty} \left(\left(1+\frac{4}{2n+1}\right)^{2n}\right)^2.$$

Now let $m = 2n + 1$ or $2n = m - 1$.

Then as $n \to \infty$, $2n + 1 \to \infty$ and $2n \to \infty$, that is $m \to \infty$ and $m - 1 \to \infty$. Thus, the above limit should be equivalent to

$$\lim_{m \to \infty} \left(\left(1 + \frac{4}{m} \right)^{m-1} \right)^2$$

$$= \lim_{m \to \infty} \left(\left(1 + \frac{4}{m} \right)^{m} \right)^2$$

This should equal $(e^4)^2 = e^8$, which leads to the conjecture that

$$\lim_{n \to \infty} \left(\frac{2n+5}{2n+1} \right)^{4n} = e^8.$$

Proofs of the conjectures in Examples 3, 4 and 5 are left to the reader. Conjectures about similar limits can often be made easily by a quick inspection and thus serve as an efficient check for finding their limits by the more laborious methods that employ l'Hospital's Rule.

Example 6 (A More General Limit)

Make a conjecture about the value of $\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx}$ and prove it using l'Hospital's Rule.

Solution

First, the formation of the conjecture $\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx} = \lim_{x \to \infty} \left(\left(1 + \frac{a}{x} \right)^{x} \right)^b$

This should equal $\lim_{x \to \infty} \left(\left[\left(1 + \frac{1}{x} \right)^{x} \right]^a \right)^b$

$= (e^a)^b$ or e^{ab}

Proof by l'Hospital's Rule:

Let $y = \lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx}$

Then $y = e^{\ln \left[\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx} \right]}$

$= e^{\lim_{x \to \infty} \left[\ln \left(1 + \frac{a}{x} \right)^{bx} \right]}$

Let $u = \lim_{x \to \infty} \left[\ln \left(1 + \frac{a}{x} \right)^{bx} \right] = \lim_{x \to \infty} \left[\ln \left(\frac{x+a}{x} \right)^{bx} \right]$

Then $y = e^u$. We proceed first to find u, using the methods employed in Example 2.

$$u = \lim_{x \to \infty} \left[bx \left(\ln \frac{x+9}{x} \right) \right]$$

$$= b \lim_{x \to \infty} \left[\frac{\ln \left(\frac{x+a}{xx} \right)}{1/x} \right]$$

The numerator and denominator of this limit both approach 0, so we may apply l'Hospital's Rule. As in Example 2, we change the form of the numerator before differentiating the numerator and denominator.

$$u = b \lim_{x \to \infty} \left[\frac{\ln(x+a) - \ln x}{1/x} \right]$$

$$= b \lim_{x \to \infty} \left[\frac{\frac{1}{x+a} - \frac{1}{x}}{-\frac{1}{x^2}} \right]$$

$$= b \lim_{x \to a} \frac{a\ x^2}{(x+a)(x)}$$

$= ba$

Thus $y = e^u = e^{ba}$. Q.E.D.

Continuously Compounding Interest and Exponential Growth

Now let us return to a more general problem in continuously compounding interest. Suppose that $\$P$ is invested at a rate of k (or $100k$ percent) per year. With continuous compounding, what should be the amount after t years?

We can find the amount here with the aid of the formula $A = P(1 + i)^n$ and the results of Example 6. Assuming n compounding periods per year, or nt compounding periods altogether, and noting that the interest rate per compounding period is then $\frac{k}{n}$, the amount after t years is given by

$$A = P \left(1 + \frac{k}{n} \right)^{nt}$$

The amount after t years with continuous compounding is given by

$$A = \lim_{n \to \infty} P \left(1 + \frac{k}{n} \right)^{nt}$$

$$= P \lim_{n \to \infty} \left(1 + \frac{k}{n} \right)^{nt}$$

From the results of Example 6, this equals Pe^{kt}.

Thus we have $A = Pe^{kt}$. This is the formula for exponential growth encountered by students first at the high school level. The results should not be surprising because continuous compounding and exponential growth are two descriptions for the same phenomenon.

Conclusion

Perhaps the attachment of an intuitive meaning to *e* is illustrative of what can happen with many mathematical concepts. Relating *e* to continuously compounding interest can, on the one hand, serve as an efficient means for making conjectures about limits—conjectures which may then be subjected to more rigorous inspection using tools such as l'Hospital's Rule. Or, conversely, the conceptual framework for making conjectures provided by this meaning for *e* can be used as an efficient check for results found by methods that, despite the advantages of their rigor, may be fraught with opportunities for errors.

As with *e*, so with many mathematical concepts. Although it may not always be possible or desirable to attach intuitive meanings to mathematical concepts, often attaching an intuitive meaning, or several intuitive meanings, to a mathematical concept, enriches one's understanding of the concept. It enables one, on one hand, to make conjectures that one would not be able to make otherwise and, on the other hand, to detect errors in results arrived at by more formal arguments.

Note

1. There is a significant rounding error in this figure due to the limitations of the calculator in storing the number 1 + 1/525600.

Bibliography

Hughes-Hallet, D., et al. *Calculus.* New York: Wiley, 1994.

Bittinger, M.L., and B.B. Morel. *Applied Calculus.* 2d. ed. New York: Addison-Wesley, 1988.

Ferve, J., and C. Steinhon. *Calculus with Discrete Mathematics.* New York: Harcourt Brace Jovanovich, 1991.