Mathematics as Problem Solving— A Japanese Way

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We know from many international studies that Japanese students (and Asian students in general) do quite well in mathematics on knowledge-level questions and on deeper conceptual and problem-solving items (Stevenson and Stigler 1992; Stedman 1994 and many others). A question arises: How are mathematical concepts and problem solving approached in Japanese elementary schools?

My response to this question was largely formed during a research visit to the University of Michigan in Ann Arbor at the invitation of Harold W. Stevenson during spring/summer 1993.[1] His project team was in the midst of analyzing voluminous data gathered from classrooms in Taipei, Beijing, Sendai and Chicago. Dr. Stevenson kindly made it possible for me to immerse myself in a set of classroom observations of 160 lessons taught by 40 teachers of Grades 1 and 5 in Sendai, Japan.

To share with you what I found, I do not want to provide a "once over lightly" treatment of a few general characteristics of problem solving in Japanese elementary schools. Rather I have selected a typical Grade 5 lesson taught by a typical Grade 5 teacher with typically 40 students in her classroom. This teacher does not have many striking characteristics that would mark her as exceptional among the set of 40 teachers included in the data set. I would describe her as competent, careful and caring. That description covers most of the other teachers as well. I present the lesson in some detail to provide readers with a feel for the problem. I then identify and comment on some aspects that were particularly striking to me. I close the article with a question and a partial answer to the main question. [Note: Parenthetical material is enclosed in square brackets.]

Lesson Description (Grade 5)

The teacher, Ms. Sato (a pseudonym), introduces the lesson with a discussion of "crowdedness" (a notion quite meaningful for people of Japan). After pointing out that one cannot say that a place is crowded simply because it has a lot of people, she focuses discussion on a particular comparison: "Which is more crowded, Hokkaido or Okinawa?" She reminds the class of problems they have already worked on in previous classes and raises one of them again: "Which tatami room is more crowded, the one that measures 10 mats and has 1 person, or the room that measures 10 mats and has 10 persons?" [Note: The area of a tatami room is measured in terms of the number of tatami mats it takes to cover the floor. The floor space of a tatami room is always designed as a particular tessellation of tatami mats. A tatami mat itself is 1 m by 2 m.] A student volunteers a response: "The second room." Ms. Sato agrees and explains that this problem is easy because the rooms are the same size: "But in our case, Hokkaido and Okinawa have different areas." She then asks if the two areas can be compared. A student suggests figuring out how many times bigger Hokkaido is than Okinawa. The teacher explains that this method is similar to the one used in the "quantity of salt experiment" they worked on earlier, and that this too can be a good method.

Another student suggests that one can "turn over a certain amount of area from Hokkaido to Okinawa to make them the same size." Other students object, suggesting that this method is too cumbersome to use here. Another student volunteers, "Like what we did with the tatami room problem, we should try to figure out crowdedness for the same amount of area first" (instead of messing around with exchanging area, as in the previous suggestion). Ms. Sato takes up this suggestion asking, "What then shall be done with the tatami room case?" A student responds, "Compare number of persons per mat." The teacher writes this suggestion on the board while repeating it orally.

Ms. Sato recalls the figures from the tatami mat problem from a previous lesson (Room A—9 persons in a 6-mat room; Room B—15 persons in a 15-mat room) and calculates the number of persons per mat for each tatami room. She then asks, "Which room is more crowded?" She confirms the answer by drawing two diagrams on the board showing one and one-half bodies per mat for Room A and one body per mat for Room B. She summarizes saying that we calculated "quantity per unit."

Turning to the Hokkaido-Okinawa problem, she asks what unit to use. A student responds with "one square kilometre." Accepting this, the teacher points out again that they can base the calculation on how many persons per square kilometre. She then summarizes with "population divided by area = population per square kilometre" and writes this on the board. Students copy this statement into their notebooks. The teacher explains further that by dividing population by area, one figures out how many live in the same amount of space.

In preparation for the calculations, the teacher reminds students about the number of significant digits to keep when rounding. A student responds, "Round off to significant digits when doing calculations" and explains that the numbers given in the problem are already rounded off to two significant digits, and the teacher confirms. She tells students to solve the problem starting with Hokkaido and helps by having students identify the rounded numbers they should use and writes them on the board. Students work on the calculations at their desks. Ms. Sato circulates. She then asks for an answer and gets "70.8" from a student. She tells him to round it off and then states that Hokkaido has about 71 people per square kilometre.

Students then calculate using the numbers for Okinawa, and the teacher again reminds them to use two significant digits when calculating and not to round off until after they find the answer. A student volunteers "478," and Ms. Sato asks, "Rounding this number to two significant digits we have?" A student responds with "480." The teacher then asks which island is more crowded, receives the answer that Okinawa is more crowded and confirms this by reiterating that Hokkaido has 71 people per square kilometre whereas Okinawa has 480 per square kilometre.

Ms. Sato directs students to look at page 79 in the textbook and asks, "What do we call this "crowdedness" that we are trying to figure out?" In a choral manner, students respond, "Population density." The teacher confirms, saying, "We have just learned to figure out population density today."

At this point, 35 minutes have elapsed. During the remaining 7 minutes, students do two more problems involving population density. [Two days later, students were still working on density problems but with some variation: the opening problem involved iron and silver—250 cc of iron weighing 1,975 g; 350 cc of silver weighing 3,675 g. This problem took 25 minutes to solve and discuss. The next day, the concept of average was developed as a particular case of density. The problem consisted of comparing the output per factory of two kinds of production where the procedures for handling density generated a quantity which could be called the "average production" (per factory).]

Looking Back

On reflection, these points seem worthy of notice:

- A deliberate effort was made to connect the present problem to *previous problems* and solutions.
- *Simpler problems* were used.
- Teacher functioned as a *guide with a definite agenda*.
- Crowdedness provided a meaningful context for *embedding the concept* of population density.
- In turn, the concept of density provided a context for embedding the concept of average.
- More than 35 minutes were spent working on *one problem*.
- *Student contributions* were used to determine the content of the lesson as well as its flow.
- The notion of rounding and *significant digits* was used consistently when doing the computations.
- The problem was rich in *social studies* content.
- *Formulaic statements* did not enter the lesson until they could be used as summary confirmations.
- The lesson was not followed up by assigning several *application exercises*. Instead, two related problems were discussed and solved, each widening the interpretation of the idea of density.

Commentary

The comments to follow also reflect my study of the 159 other lessons taught by Sendai teachers in the data set.

Embedding

Problem solving occurs *pervasively* in this lesson. Problem-solving techniques are used not only in solving the problem about density but also in the approach to the lesson structure itself. I have used the term *embedded* to describe this more pervasive use of problem solving in teaching mathematics in which problem solving is the *medium* for mathematics teaching and learning. In the lesson described, there is as much emphasis on the medium (problem solving process) as there is on the message (the particular mathematical content to be learned). This emphasis on the medium in many Grade 5 lessons in Sendai often takes precedence over the message so that instructional decisions (how to respond to mathematical "errors," for example) are made with the health of the medium as much in mind as the need to assess the correctness of the message. (For example, a so-called "error" is often discussed at length and thereby contributes to the diversity of the "solution space" and therefore to the health of the medium, and thus is valued even if not particularly "correct" as a message.) Indeed, such contributions, because they do enrich the solution space are actively sought by teachers through such simple leads as, "Any other methods?" None of the responses to such questions are rejected at the outset, and this decision not to reject indicates that the health of the medium is at least as important as the correctness of the message. This priority on the medium occurred in the lesson in many particular as well as general ways.

From a mathematical viewpoint,

- crowdedness was used as a medium for (to embed) density;
- density was used as a medium for rate/ratio; and
- in its turn, rate/ratio was used as a medium for average.

From a pedagogical viewpoint,

- the general problem situation was a medium for the structure of the lesson;
- the particular problem was a medium for the mathematical concept; and
- the multiple solutions generated by students were a medium for the "correct" solution.

From a learning viewpoint,

- the process of solving the problem was a medium for learning of mathematics;
- right or wrong, student answers de facto were the medium for exploring the solution space; and
- generating and discussing solutions was the medium for arriving at and critiquing answers.

Spending 35 Minutes or More on One Problem

Spending nearly the whole period on one problem might seem an inefficient and drawn-out way to teach a mathematical concept. On the other hand, if only four or five minutes were spent on a nonroutine problem, I would have grave doubts that a problem-solving approach was being used at all. For a rich problem-solving process to occur, there needs to be time for investigating the nature of the problem, generating and proposing multiple solutions, discussing and critiquing various types of solutions, assessing and comparing the relative merits of each solution within the set of solutions, reflecting on what one has learned and what else could be learned. This sounds like a lot of rhetoric, but it happens regularly in most Grade 5 classes in Sendai (statistically speaking, 86 percent of Sendai Grade 5 teachers use problem solving to embed mathematical concepts, while the comparable figure for Chicago elementary schools is 14 percent), and it means that the whole period will often be spent on only one problem.

Acting Out the Problem

Although the students in the lesson described did not act out the solution process, in other Sendai classrooms this happened quite often. For example, the teacher would bring cushions to simulate tatami mats, children would select which one(s) they would prefer to sit on and a problem of density would be enacted. After this, rooms with so many mats would be drawn on the board with students selecting which room they preferred, and the density problem thus enacted would be ready to be solved.

Multiple Solutions

Multiple solutions are the "rice and fish" of the problem-solving approach used in Sendai. If only one solution exists, it would have to be the correct one so nothing would function as, nor need to function as, a medium for anything else. The correct solution would be demonstrated or illustrated for all to learn. Multiple solutions, and establishing the conditions for generating, articulating, understanding, comparing and critiquing multiple solutions are required components to the way problem solving happens in Sendai classrooms.

Interesting Problems to Solve

While the problem of the relative population density of Hokkaido and Okinawa may be of interest to Japanese students, as a generalization, most problems are rather mundane. Many of them remind me of typical word problems occurring by the hundreds in Canadian texts. For the classrooms in Sendai, the problems per se are not different; the way these problems become the medium of instruction and of learning is contrastive.

Use of Manipulative Aids

The NCTM publication *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan* (Stevenson et al. 1990) reports a surprising finding: When manipulatives are introduced into American lessons, the amount of talk decreases, while in Japanese elementary classrooms it increases. This finding may have been surprising, but, when problem solving is the pervasive mode of teaching and learning, introducing manipulative aids contributes immensely to the variety of multiple solutions to be generated, assessed, compared and so on, thus providing many more opportunities for talk.

Interpretation Rather Than Application

Currently in North America, we talk about developing problem-solving strategies and skills and then applying them. In this sense, problem solving is split into two rather distinct parts:

1. The learning of the concepts and skills (perhaps through problem solving)
2. The use or application of these concepts and skills in similar situations

This second part is often taken to be the full extent of problem solving. It is a matter of applying concepts learned, not for learning concepts and skills. Our textbooks are organized this way: concepts are taught (often by demonstration or explanation, as well as problem solving), and then students are given a collection of similar "problems" to do. Because of this practice, lessons in North America are likely to end with students working at their desks. The contrast in Sendai is striking. Classes often end with discussion. And when children are working at their desks at the end of the class, they are not only applying the concepts just learned but also *interpreting* problem situations that extend the ideas beyond the initial circumstances. In the lesson presented, the problem-solving approach is not two parts but just one. Problem solving becomes two parts when the concepts learned (the messages) become so important that they need to be separated and dealt with differently (as applications). On the other hand, if we keep the problem-solving process intact and pervasive, the messages learned will never dominate the medium that created them.

Conclusion

Several other aspects about problem solving in Sendai deserve mention, such as the deliberate making of errors (by the teacher) or the occurrence of memorization, but I think the important points have been made. I close this article with a question and a partial answer. Why has the problem-solving approach as medium taken hold so pervasively in Japanese elementary schools but remains largely problematic in North American schools despite a decade in which nothing in mathematics education received more attention than problem solving? While not wishing to oversimplify, I should like to suggest that, in Japan, accepting ways of doing things as being as important as the things themselves is a familiar and comfortable stance. We see it in martial arts, flower arranging, the tea ceremony, pottery making and in teaching, too. Problem solving as a medium for teaching mathematics is another of these important ways.

Note

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