# delta-k

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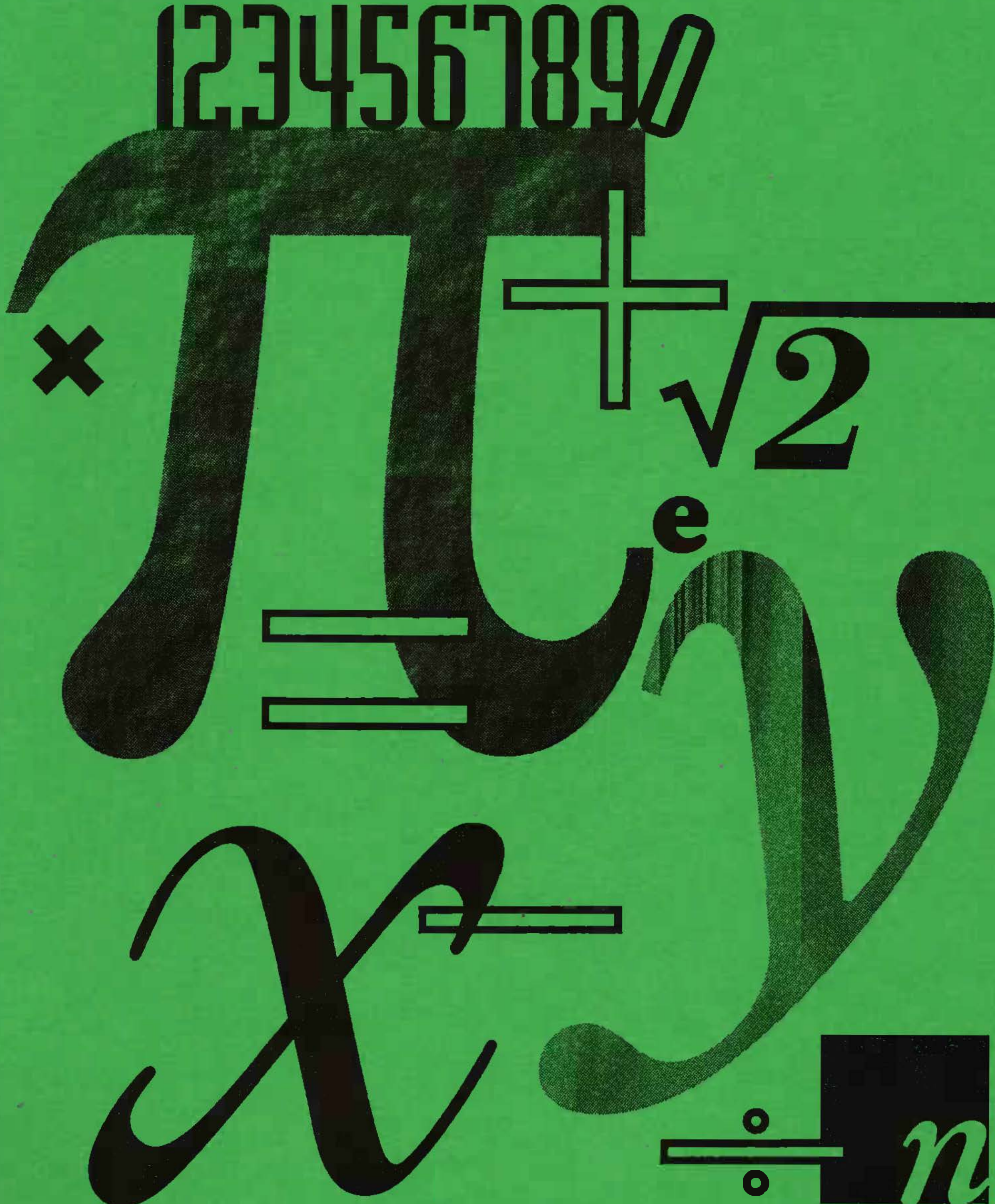

## MCATA Mission Statement

*Providing leadership to encourage the continuing enhancement of the teaching, learning and understanding of mathematics.*

delta-K

Volume 33, Number 2 June 1996

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# COMMENTS ON CONTRIBUTORS

*J. Dale Burnett* is a professor in the Faculty of Education at the University of Lethbridge.

*Alison Dickie* is a regular contributor to the *Home & School* magazine.

*Ken Harper* is a professor in the Faculty of Education at the University of Victoria.

*Louise M. Lataille* is a professor of mathematics at Springfield College in Springfield, Massachusetts.

*Murray L. Lauber* is a professor in the Division of Mathematical Sciences at Augustana College in Camrose.

*Bonnie H. Litwiller and David R. Duncan* are professors of mathematics at the University of Northern Iowa, Cedar Falls, Iowa.

*A. Craig Loewen* is a professor at the University of Lethbridge.

*Marian Oberg* teaches at Parkland Composite High School in Edson.

*Daiyo Sawada* is a professor in the Faculty of Education at the University of Alberta.

*delta-K* is the official journal of the Mathematics Council (MCATA) of The Alberta Teachers' Association (ATA).

The journal seeks to stimulate thinking, to explore new ideas and to offer various viewpoints. It serves to promote MCATA's convictions about mathematics education.

# EDITORIAL

The Western Canada Protocol, under the leadership of Alberta, is developing a common core mathematics program for British Columbia, Alberta, Saskatchewan, Manitoba, the Yukon and the Northwest Territories. The Grades K–9 curriculum is already established, and the Grades 10–12 program will be in place before the end of this school year. This should allow easier transfer of students throughout western Canada. Publishers should be able to prepare materials for a larger market.

As I see it, the biggest challenge to the success of the new curriculum will be a shortage of qualified teachers. Significantly more emphasis is placed on such areas as statistics and probability, estimation, mental mathematics, the use of technology, and manipulatives. Currently, many mathematics teachers have little or no training beyond high school in either area of content or methods.

If the new curriculum is to be effectively implemented, teachers are going to have to be provided with extensive inservice sessions in mathematics. Postsecondary institutions, Alberta Education, school boards and the ATA are going to have to share responsibility for this training. Teachers, as individuals, must not only pressure for these inservice programs but also attend them. Costs will have to be a shared responsibility.

The new mathematics program should be exciting to teach and should develop in students a genuine enthusiasm for mathematics.

*Arthur Jorgensen*

### What Do You Think?

Stop "experts" from spoiling our schools.

Those who sabotage our state schools always refuse to believe the obvious—that there can be no teaching without peace and order; that mistakes must be corrected; that spelling, sums and neatness are the building bricks of a good education.

The symbol of their willful stupidity is the use of calculators in class. Any fool can use a calculator. But if that fool should press the wrong button, how will he be able to tell the answer is wrong?

The "experts" who brought the calculator into British schools are the same people who flatly refuse to teach young children their tables.

The countries that actually make the calculators, and sell them to us in millions, know better. Their youngsters still sit in orderly rows and learn things by heart, and their economies will overtake ours if we do not mend our ways.

From the "Opinion" section of the *International Express*, November 15–21, 1995, p. 8.

# From the President's Pen

In November '95, the MCATA executive met in Red Deer at the annual "Thinkers Conference." The main focus on Friday evening and all day Saturday was the "Changing Role of the Teacher: What Are the Implications for Specialist Organizations?" To help guide our thinking, we considered the effect on the following: mathematics standards; philosophy of public education; Alberta Education's new directions (accountability, governance, Western Canada Protocol, the quality teaching document); education as a learning community; identification of constituents in the learning community and their involvement; what we should maintain as "critical" to mathematics; and mathematics education and financial realities on teachers.

Following a productive discussion, two groups were established, under the direction of the vice presidents, to create strategies for working within these realities. One group related its work to communications, public relations and education. The second group related its discussion to conferences, regional activities and membership. Sunday was spent bringing the discussions and recommendations together into set action plans. Following this, an executive meeting was held.

The results of our discussions and the actions that follow will provide a number of benefits to members. Reaching all areas of our regionals and satisfying our responsibilities is not as easy as we would like. We desperately need input from all members. If you have any issues, suggestions, questions or comments, please contact a member of the executive.

*George Ditto*

# From the Membership Director

I would like to remind you to renew your membership as soon as possible if it is coming up for renewal. I would still like to hear about activities that members are performing to promote math education in their classrooms, schools or communities. Let me know if any fellow teachers have any innovative ideas to share. Drop me a line to keep me abreast of what is happening with math education in your school: 1826 51 Street NW, Edmonton T6L 1K1; fax 469-0414.

*Daryl Chichak*

# President's Report 1994–95

*The following was presented at the annual general meeting in Lethbridge on September 29, 1995.*

As I look back on my two years as MCATA president, I have many fond memories. Our conferences have been a source of much enjoyment. Meeting with, learning from and socializing with other math teachers has been fun! Working with the dedicated group of professionals that comprises our council has been a wonderful experience. During my term, I have traveled to other parts of Canada and have met and worked with mathematics educators from across the continent. My appointment to curriculum advisory committees with the ATA and Alberta Education, and my involvement with the Western Canada Protocol for a common math curriculum, result directly from my position as MCATA president. I am honored to have had some input in curriculum decision making.

As we look at MCATA's 1994–95 activities, I regret to announce that four executive members have resigned: Dennis Burton, Bob Hart, Myra Hood and Bob Michie. Their service is greatly appreciated, and we wish them well in their future endeavors. Welcome to new executive members Klaus Puhlmann, coeditor of *delta-K*; Linda Brost, public relations director; Sandra Unrau, regional activities director; and Kay Melville, Department of Education representative.

The executive had one teleconference and met four times last year—twice in Red Deer, once at the annual conference and once at our annual "Thinkers' Conference" in Edmonton. We continued to work on the four focus areas of service to our members: publications, conferences, membership and issues. In view of the recent enormous changes in education, we spent considerable time reexamining MCATA's structure and purpose.

Conferences are the Council's ongoing source of pride. The Edmonton Regional of NCTM held in October '94 was a great success, attracting 1,500 participants. Congratulations to Florence Glanfield and her committee members. Mary Jo Maas and her committees have spent countless hours over the last two years organizing "Math Trek—The Next Generation" in Lethbridge. We thank them for their efforts and enthusiasm. Thanks to Myra Hood and Florence Glanfield for organizing a highly successful parent conference in Edmonton and a mini-conference in Calgary.

One strength of MCATA continues to be our excellent publications. Art Jorgensen, publications coeditor, has kept members well informed via five issues of the newsletter and three issues of *delta-K*. I would also like to commend Art and Joan Worth for their fantastic job in gathering materials for *Thirty-Four Years and Counting*. I know that Art would like to publicly commend the publications staff at Barnett House for the wonderful job they did in putting it together. Thank you, all.

Executive members attended a number of different conferences during the year. MCATA was represented at the NCTM annual conference and delegate assembly in Boston in April. Executive members serving on NCTM committees were Florence Glanfield on the Regional Services Committee and George Ditto on the Conventions and Conference Committee. I attended a Canadian Mathematics Society Forum in Education in Mathematics in Quebec City in May. MCATA sent Richard Kopan and Betty Morris to the NCTM Leadership Conference in Regina in July. Thanks to Past President Bob Hart for representing MCATA as an observer to ARA in Edmonton in May. Donna Chanasyk and Graham Keogh attended the ATA Summer Conference in Banff in August. Thanks to all executive members who gave up holiday time to attend these conferences.

The Council financially supported three math contests (Junior High Contests in Edmonton and Calgary, and the Alberta High School Math Contest), as well as a teacher inservice project.

In closing, I would like to thank all executive members for their work this past year. They can be proud of all that MCATA has accomplished. Special thanks and recognition again go to Past President Bob Hart and ATA Staff Advisor Dave Jeary for all their help while I was president.

*Wendy Richards*

# The Right Angle

## Evaluation Branch

### Achievement Assessment

Grades 3, 6 and 9 achievement test information bulletins are now in schools. Please note that page 14 of the Grade 9 mathematics 1995–96 information bulletin contains an error. The instructions and examples should inform students to enter the first digit to the answer in the left-hand box and leave any unused boxes blank.

### Diploma Exams

Mathematics 30 and 33 diploma exam information bulletins are now in schools.

## Curriculum Branch

### Western Canada Protocol

Development work on the Grades 10–12 common curriculum framework is ongoing. For more information, contact Hugh Sanders at 422-3220.

### Program of Studies

The Grades K–9 outcomes in the common curriculum framework are being reformatted to be included in the Program of Studies. Implementation dates follow:

- Mathematics Grades 7 and 9 September 1996
- Mathematics Grades K–6 and 8 September 1997

A resource review occurred in March, and an announcement is planned before June 1, 1996.

## Distance Learning

### Mathematics 30

Distance-learning materials for the 1994 revisions to Mathematics 30 are now available from the LRDC. The materials contain student modules and assignment booklets, and a learning facilitator's manual. The course is self-contained; textbooks are not required. Contact the Alberta Distance Learning Centre (ADLC) at 674-5333.

### Mathematics 31

Distance-learning materials for the new Mathematics 31 course are now available from the LRDC. The materials contain student modules and assignment booklets, and a learning facilitator's manual. This is a self-contained course; textbooks are not required. Contact the ADLC at 674-5333.

### Mathematics 7

Distance-learning materials for Mathematics 7 are now available from the LRDC. The materials contain student module and assignment booklets, a learning facilitator's manual, and a student support guide. This course is based on the common curriculum framework. This is a self-contained course; textbooks are not required. Contact the ADLC at 674-5333.

### Zap-A-Graph Software

Zap-A-Graph 4.2.1 is now available from the ADLC. Zap-A-Graph is an easy-to-use mathematics graphing application tool for Macintosh. The software allows students to interact directly with on-screen data, manipulate graphic functions and discover mathematics relationships. To obtain this program, send a floppy disk to Alberta Distance Learning Centre, Instructional Technology and Media Unit, PO Box 4000, Barrhead T7N 1P4.

### Question Bank (ADLC) CD-ROM

This CD-ROM contains question banks in LXR test format for the Macintosh and for Windows.

The Macintosh LXR version 5 question banks include Mathematics 10, 20, 30, 31, 13, 23 and 33; Social Studies 7, 8, 10, 20, 30, 13, 23 and 33; Physics 20 and 30; Biology 20; Chemistry 20 and 30; and Science 7, 8, 9 and 10. French Mathematics 20 is included for the French version 5.0 of LXR.

The Windows LXR version 5 question banks include Mathematics 10, 20, 30, 31, 13, 23 and 33; Social Studies 8, 10, 20, 13, 23 and 33; Physics 20 and 30; Biology 20; Chemistry 20; Science 7, 8, 9 and 10.

This CD-ROM is available at the LRDC. The product order number is 309072. If you require more information or if you are an educator outside Alberta, contact the Instructional Technology and Media Unit of the Alberta Distance Learning Centre at 674-5333.

*Kay Melville*

# Western Canada Protocol Mathematics Team

The Western Canada Protocol Mathematics Project began in 1994 as part of the Western Canada Protocol for Collaboration in Basic Education, which was signed by ministers of education for British Columbia, Alberta, Saskatchewan, Manitoba, the Yukon and the Northwest Territories. The team leader for the project is Hugh Sanders from Alberta Education. Through the collaborative efforts of teachers and ministry personnel, the first part of the common curriculum framework has been completed. Copies were sent to all schools in Alberta before September '95. Implementation in Alberta begins in September '96 with Grades 7 and 9.

In August and November '95, the team worked on common curriculum outcomes for Grades 10–12 mathematics programs. The team comprised over 30 people, about 75 percent of whom were actively teaching in various high school programs. The purpose was to develop and exemplify curriculum outcomes for high school students who have successfully completed the outcomes documented in the K–9 program. Outcomes were developed under the working headings "pure" and "applied"; many of these outcomes were later chosen as "common" because all high school students, no matter what their program, should be achieving these. The various jurisdictions will group the outcomes into courses that fit their own high school framework. A high school course sequence may be structured from a combination of common, pure and applied outcomes, depending on the needs of the students for whom it is being developed. Each province or territory will continue to determine how much of this mathematics program is required for graduation.

Participants struggled with the problem of how technology should be used with the program. After trying out the T192 calculators, provided by Bob Hart, we became even more aware of the effect such technology will have on mathematics education. Many were concerned about who will pay for this technology so that students will have it available in classrooms.

Alberta teachers involved in this work were Lea Beeken, Bob Hart*, Elaine Manzer*, Kathy McCabe, Bruce Peers and Marian Oberg*. (* Representatives chosen by the ATA.)

*Marian Oberg*

*Western Canada Protocol Mathematics Team*

# Would You Believe?

On February 5, 1992, results of the International Assessment of Educational Progress II were released. Alberta's science students scored an average of 76 percent on questions involving the integration of science knowledge, higher than any other country participating in the tests. On questions on the nature of science, Alberta's students scored an 84 percent average, again the highest average among all provinces and countries.

*—Alberta Education News Release,*
*February 5, 1992*

"Public schools have so many people to look after that they couldn't possibly meet everyone's needs."

*—Parent promoter, Suzuki Elementary School,*
*Edmonton*

"People look for simple solutions, but it has not been shown that there's some magic in the private sector—and you can't say it hasn't been tried."

*—David Sarasohn,*
*"Forever Profit Schools Are Not the Answer,"*
The Oregonian, *September 9, 1995*

Reductions in spending per capita for all provinces between 1992–93 and 1994–95 follow:

- *Health care.* Only three provinces have reduced spending—Ontario by $27 per capita, Prince Edward Island by $29 per capita and Alberta by $163 per capita.
- *Welfare.* Only four provinces have reduced spending—Alberta by $143 per capita, and the next closest is New Brunswick by $65 per capita.
- *Education.* All but British Columbia have reduced their spending—Ontario by $143 per capita, Alberta by $138 per capita and all the others less than $77 per capita.

*—"Toward an Affordable Government,"*
Canada West Foundation, *October 1995*

"If countries are to 'live off their wits' in the 21st century, then the only solution is 'high levels of education for all,' not just those who supposedly can afford it."

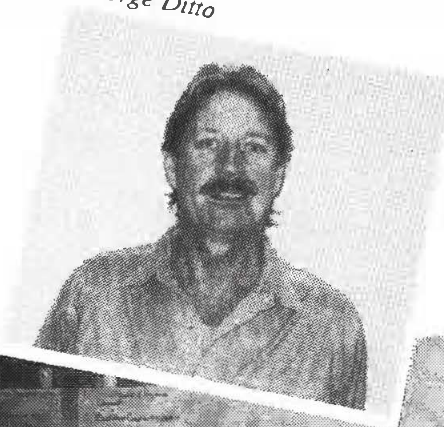
*—George Walden,*
*Conservative MP for Buckingham, England,*
*quoted in the* Times Educational Supplement,
*August 4, 1995.*

*Betty Morris*

# Potpourri of Photos
## '95 Thinkers Conference

(l to r) Bob Hart, Mike Stone, Art Jorgensen, Dick Kopan and Florence Glanfield

*George Ditto*

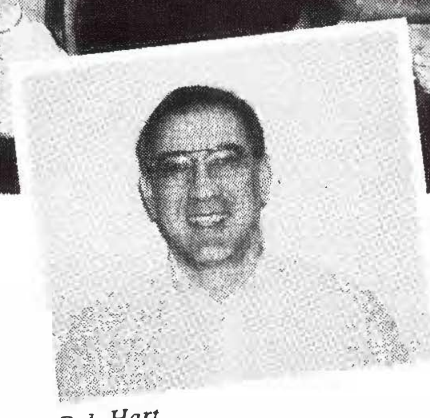


*Doug Weisbeck*

*(l to r) Graham Keogh and Bob Michie*

*(l to r) Mary Jo Maas, Florence Glanfield and Daryl Chichak*

*(l to r) Florence Glanfield and Donna Chanasyk*



*Bob Hart*

# Some Ideas on Teaching Data Management

*A. Craig Loewen*

The two most significant recent changes to Alberta's elementary mathematics program include the dramatic change in philosophy and the revisions which resulted in the new data management strand.[1]

## Why Teach Data Management?

The data management strand (Alberta Education 1994) involves all the concepts formerly contained within the graphing strand of the 1982 Alberta curriculum but goes much further to entail concepts in collecting, recording, organizing, displaying and interpreting data, as well as concepts in probability, including theoretical and experimental probabilities.[2] These additions to the math program are extremely positive for several reasons.

### Life Skills

The new data management topics represent a positive addition to the new program in that the skills addressed are in fact life skills. Our students need to know how to manipulate data, and, in an increasingly technological society, the ability to code, sort and interpret data has become paramount. It is perhaps arguable that data management skills are almost as important as number facts and number sense. The value of these skills justifies the increased amount of time spent in the data management strand.

### Relevance

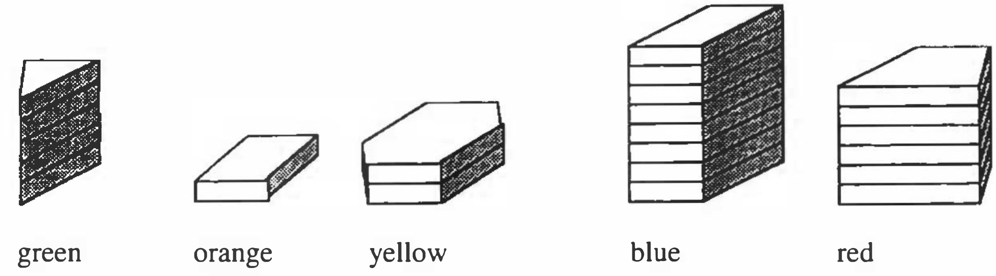
Mathematics has traditionally been criticized for its perceived lack of relevance at all grade levels. Even the simplest concepts have often been taught without context and without reference to any physical object. For example, it is possible to teach a simple addition fact, such as 2 + 1 = 3, without referring to concrete objects, but, when we contextualize it by talking about the joining of sets of fruit or children or bicycles, the concept takes on much more interest and meaning. In short, we forget at times that mathematics has a purpose: mathematics summarizes and represents patterns, events and regularities within our world. Mathematics describes the world around us. The data management strand has its strength in that it emphasizes that numbers and graphs represent information and as such are interpretable. This emphasis enhances the general sense of the relevance of the study of math.

### Integration

The data management strand is easily integrated with most other strands of the elementary curriculum. For example, one can easily visualize how collecting temperatures on given school days for a school year and charting them would address objectives within the data management and measurement strands. Whenever we strive to integrate strands, concepts or subjects, students are typically left with an enhanced sense of understanding and purpose of the study.

### Discussion

The topics in the data management strand lend themselves nicely to discussion. A familiar example involves having students select a pattern block of their favorite color. Once the selections are made, students are asked to come forward by color and stack their pattern blocks in towers of the same color. A concrete graph is created as follows:



Viewing the concrete graph, questions such as the following emerge: Do more people prefer red or blue? How many people are in the classroom? What is the least favorite color? The last question typically surprises students as it is not answerable from the graph alone. All possible colors are not represented in a set of pattern blocks—pattern blocks are painted red, green, yellow, blue, white or orange. Is it possible that brown or purple is the least favorite color? It is not difficult to see how an interesting discussion quickly emerges. Of course, interesting discussions are possible within any strand of the curriculum, but interpreting displayed data is a particularly rich source of such dialogue.

## Managing Data: An Example

The data management strand separates into three main topics: collecting and recording data, organizing and displaying data, and interpreting displayed data (Alberta Education 1994). The following example illustrates these components:

> My friend is opening a men's clothing store. He asked me to help him decide how many of each size belt he should stock.

### Collecting and Recording Data

To assist my friend, I agreed to stand on a busy street corner and ask the first 50 men who passed to tell me their belt size (a risky proposition, but I thought it might be interesting). I recorded the following numbers:

30 44 32 36 32 38 32 44 28 42 30 56 34 36 34 34 26
28 42 36 32 26 30 38 34 40 32 46 30 28 24 36 40 34
30 38 32 32 36 36 30 42 38 50 40 40 36 28 28 38

It is easy to see that the collection of data above has little obvious order. We might notice that the numbers are all even (belts are usually sold only in even sizes) and that several values are in the "30 range." Beyond these simple generalizations (which are not much help to my friend), the only order we know exists is that the numbers are listed according to the sequence in which they were collected. The lack of order forces us to consider how we might better manage or organize the data.

### Organizing and Displaying Data

Organizing and displaying the data involves many levels of sophistication. For example, a simple reorganization involves merely listing the numbers from lowest to highest:

24 26 26 28 28 28 28 28 30 30 30 30 30 30 32 32 32 32 32
32 32 34 34 34 34 34 36 36 36 36 36 36 36 38 38 38 38 38
40 40 40 40 42 42 42 44 44 46 50 56

Once the data is sorted and listed, information is much easier to find, such as, What is the largest belt size? What is the smallest belt size? What is the range of belt sizes? We may even (with a little more effort) be able to determine the most and least common belt sizes. With such a list, we can also determine the median belt size (by finding the middle value in the list) or the most common belt size (the mode).

However, our list is still somewhat visually intimidating. To find the most common belt sizes requires us to count items in our list, and it is hard to get a quick sense of how the values cluster. So, we may consider more sophisticated ways to display the data, such as tally charts and stem-and-leaf plots. The tally chart requires us to make some decisions, such as, do we wish to group together any ranges (or sizes) to create a more succinct presentation? For example, in the tally chart below, the data is presented in ranges of 24–28, 30–34, 36–40, 42–48 and 50–56. These ranges may approximate the categories small, medium, large, extra large and extra-extra large.

| | | |
|---|---|---|
| 24–28 | S | 𝍸 III |
| 30–34 | M | 𝍸 𝍸 𝍸 III |
| 36–40 | L | 𝍸 𝍸 II |
| 42–48 | XL | 𝍸 I |
| 50–56 | XXL | II |

Notice however that our grouping of data has caused us to lose some information: it is easier to see how many people fall in particular ranges, but we no longer know exactly how many people claimed to have a 36-inch waist. In comparison, the stem-and-leaf plot helps us maintain this information. In the stem-and-leaf plot, we define the stem as the digit in the tens place and the leaf as the digit in the ones or units place. We list all the leaves after the appropriate stem. Thus, our method of coding the data enables us to regenerate the entire list of raw data. A stem-and-leaf plot for our data is shown below.

| | |
|---|---|
| 2 | 46688888 |
| 3 | 000000222222244444666666688888 |
| 4 | 0000222446 |
| 5 | 06 |

In this plot, we can quickly see that more people are using a belt in the 30–38-inch range than in any other range. In comparison to the tally chart, in the stem-and-leaf plot, the means, median and mode are no less difficult to find; however, our ranges for grouping data are determined by place value, and in this case do not easily translate to the familiar method of sizing (S, M, L, XL, XXL). The primary advantages of the stem-and-leaf plot are that it is relatively easy to construct and that it can be easily created from unsorted raw data.

Though the stem-and-leaf plot is more visually interesting and appealing than the tally chart, other forms of graphs such as bar graphs, line graphs, histograms and pictographs are even more visually captivating. Figure 1 shows our data represented as a bar graph. This graph is easy to read but is significantly more difficult to construct than either the tally chart or stem-and-leaf plot. Creating a bar graph also requires making many decisions. We must decide what to place along our vertical and horizontal axes, not to mention vertical and horizontal spacing and what the spacing may represent. But, in a quick glance, one can easily determine the range, identify the most and least common measurements, compare frequencies of given measurements and get a strong sense of the distribution of the data.

### Interpreting Displayed Data

Whenever we try to get information from a graph or chart, we are interpreting displayed data. Many examples have already been given in the discussion above, but more difficult questions can also be posed. For example, Are any belt sizes sufficiently uncommon that you would not want to order for your store? Do any values in our graph seem surprising (for example, why is the 34-inch belt size less frequent than the 32-and 36-inch belt sizes when we would expect it to be more common)? How should we account for this anomaly in placing our order?

We may wish to experiment with our axes. For example, let's assume that my friend wishes to order approximately 500 belts for his store. By changing the values along the vertical axes (that is, multiplying each value by 10), we essentially scale the data and get a reasonable estimate of how many of each size belt should be ordered (Figure 2).

The example developed above helps us to understand the general flow involved in managing data. We begin by collecting the data (either first- or secondhand) and then proceed to the next step of sorting, organizing and displaying the data in different ways. Once displayed, our challenge is to interpret the data reasonably to solve the given problem or task. At each step along the way, the data becomes increasingly meaningful and useful.

## Sample Activities

Defining an imaginary problem or posing a question like the one above is an interesting and enjoyable class activity. A collection of other activities and problems emphasizing particular stages of the data-management process are presented below. The activities represent a broad range of difficulty from relatively easy (for example, Race to the Top) to very difficult (for example, Bar Graph Clues).

### Activity One: Find Me Five (Game)

*Topic*

Collecting Data

*Objective*

Formulate the questions and categories for data collection, and actively collect firsthand information (Alberta Education 1995, 54).

*Materials*

Paper, pencils

*Rules*

- Each player begins by selecting a question to which a person would answer *yes* or *no*. For example, the player may ask, "Did you have toast for breakfast this morning?"
- Once each player has selected a question, the players circulate around the room asking the question of 10 people.
- Players should keep lists of the names of people who answered *yes* and who answered *no*.
- The player who receives exactly five affirmative responses to his or her question wins.

*Discussion*

The difficulty of this game rests in trying to determine a question that would elicit exactly five affirmative responses. Asking students how they might adapt or change their question to achieve the goal may result in some interesting dialogue. The game also lends itself to discussions of how you may be able to bias your sample. For example, you could ask, "Are you wearing white socks?" and proceed to deliberately ask five people who you know are wearing white socks and five people you know are not. Answers to some questions are easier to predict than others.

### Activity Two: Tally Woe (Problem)

*Topic*

Collecting Data

*Objective*

Organize data, using such graphic organizers as diagrams, charts and lists (Alberta Education 1995, 54).

*Materials*

Paper, pencil

*Problem*

Anita has noticed that half the students in her class have blue eyes. In your class, do more than half of the students have blue eyes?

*Discussion*

The solver will need to set up some method for surveying his or her classmates to determine their eye color and develop some method for recording the results. Depending on the grade level at which this problem is introduced, the teacher may need to discuss the notion of one-half, specifically the number of students that would constitute one-half in that class. The teacher may also wish to discuss how tally charts are constructed and used. For example, does it make sense to collect information in a list or other form and later sort it into a tally chart, or are there benefits (and difficulties) associated with entering the data directly into a tally chart as it is collected?

## Activity Three: Race to the Top (Game)

*Topic*

Organizing and Displaying Data

*Objective*

Organize data, using such graphic organizers as diagrams, charts and lists. Describe the likelihood of an outcome using such terms as more likely, less likely, chance (Alberta Education 1995, 54 and 58).

*Materials*

Bar Graph Gameboard (Figure 3), pencil crayons (red, green, yellow, blue), Race to the Top Spinner Mats (Figure 4), overhead spinner

*Rules*

- This is a game for two players playing on the same gameboard (bar graph).
- The first player places the overhead spinner on any one of the spinner mats and twirls the spinner. The player colors a space at the bottom of the bar graph according to the results of the spin. For example, if the spinner points to red, the player would color the first block in the bar labeled red.
- The spinner in the bottom right-hand corner of the spinner mat has two possible outcomes: none and choice. If the player chooses that spinner, and the spinner points to none, then the player passes the turn to his or her opponent. If the spinner points to choice, the player must shade the next block in any one bar she or he chooses.
- The second player now chooses a spinner mat and twirls the spinner, recording the results of his or her spin.
- The players continue taking turns building up the bars of the bar graph until one bar reaches a height of 10.
- The player who completes any bar graph (that is, shades the tenth block in any bar) wins.

*Discussion*

The teacher may find it useful to discuss strategy with the students. Students should realize that turning over the bar graph to their opponents is not desirable while the height of any bar rests at nine. This strategy requires careful choice of the spinner mat on each turn. If working with younger students, the teacher may wish to let them each have their own bar graph and race each other to complete any bar or the entire graph. To add further variety to the game, the teacher can encourage students to create their own spinner mats. This game emphasizes organizing and displaying data in that students are essentially creating bar graphs to display the results of their experiment with the spinner.

## Activity Four: Bar Graph Clues (Problem)

*Topic*

Organizing and Displaying Data

*Objective*

Organize data, using such graphic organizers as diagrams, charts and lists (Alberta Education 1995, 54).

*Materials*

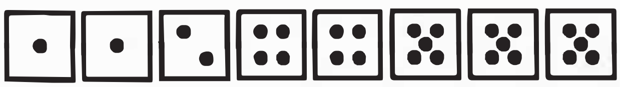
Paper, pencils, beans or small markers

*Problem*

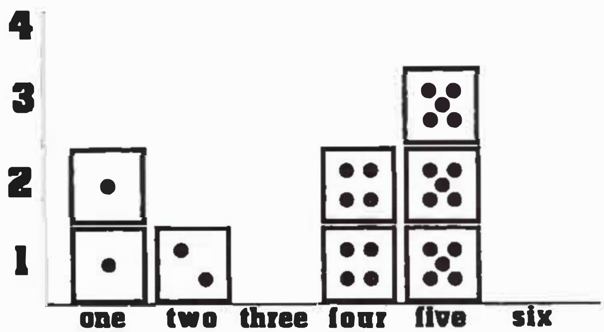
Gwenyth asked eight people to roll a die while she recorded the value they rolled (one through six). Gwenyth noticed more people rolled a 4, 5 or 6 than a 1, 2 or 3. She also noticed that more odd than even numbers were rolled and that the sum of the values rolled was 27. Construct a bar graph to show how many people rolled each of the values one through six.

*Discussion*

This somewhat difficult problem has four solutions. The teacher may wish to introduce the problem by distributing dice to the students and having them conduct the same experiment as Gwenyth. The students could talk about their firsthand data before being introduced to Gwenyth's secondhand data. There are many approaches to solving this problem, including making a list or chart and adjusting that list or chart until all conditions of the problem are met. The teacher may also consider having students model the problem using eight dice and a guess-and-check strategy. In this way, students can easily manipulate the dice until all conditions are met. See the following example:



The students can then rearrange the dice to create a concrete graph like the one below before constructing their bar graph to complete the problem.




## Activity Five: Bar Graph Battleship (Game)

***Topic***

Interpreting Displayed Data

***Objective***

Construct and label concrete/object graphs, pictographs and bar graphs. Generate new questions from displayed data (Alberta Education 1995, 54).

***Materials***

10 each of red, green, blue and yellow counters; pencil; Bar Graph Gameboard (Figure 3) for each player; paper bag

***Rules***

- This is a game for two players.
- Each player randomly draws a number of red, green, blue and yellow counters from a paper bag and creates a bar graph on the left-hand side of his or her gameboard to display the counters drawn. This bar graph is kept hidden from his or her opponent.
- Once both players have created their bar graphs, they take turns asking each other questions which can be answered with *yes* or *no*, attempting to determine how many of each color the other player drew. For example, the first player may ask, "Did you draw more red than blue?"
- The second player truthfully answers the question, and the first player notes the answer on the right side of his or her gameboard. The second player now asks a question.
- Players continue asking each other questions until one player announces, "I know how many of each color you have drawn." This player must construct/complete the bar graph she or he believes his or her opponent created at the beginning of the game.
- Players now compare the appropriate bar graphs. If the player has determined the correct number (and has constructed the correct bar graph), that player wins. If a mistake has been made, the other player wins.

***Discussion***

This game is not difficult for most students who are familiar with similar battleship games. Nonetheless, the teacher may wish to introduce the game using only one color, and over time introduce the remaining three colors. Some discussion time may be required to help students develop a system for recording opponents' answers. Teachers may also wish to discuss the elimination problem-solving strategy before starting the game. Younger students tend to ask simple direct questions such as, "Do you have five blue counters?" The teacher may wish to encourage other types of questions as well.

## Activity Six: Train of Thought (Problem)

***Topic***

Interpreting Displayed Data

***Objective***

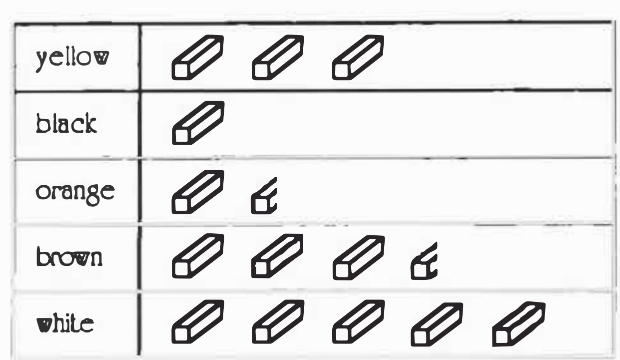
Discuss data, and draw and communicate appropriate conclusions (Alberta Education 1995, 54).

***Materials***

Paper, pencil, colored counting rods, metre stick

***Problem***

Tannis selected several colored counting rods and created a pictograph to show how many she picked of each color. If her rods were placed end to end, how long would her train be?




Scale: [rod symbol] = 2 blocks

## Discussion

A reasonable approach to this problem would be to find a set of colored counting rods and generate the set of rods represented by the given pictograph. These rods could then be placed end to end along the edge of a metre stick to create a train. Another solution would require finding one rod of each color mentioned, measuring it, and then multiplying and adding the lengths to determine the total length. Other interesting problems could be posed, such as How many of each color rod listed would you need to create a train exactly 1 m in length? What train lengths could you create if you must have the same number of each color of rod? Students may enjoy creating their own problems and graphs to challenge their friends and classmates.

## Notes

1. The concepts found in the data management strand of the October 1994 Alberta Program of Studies are contained under the statistics and probability strand in the Western Canada Protocol.

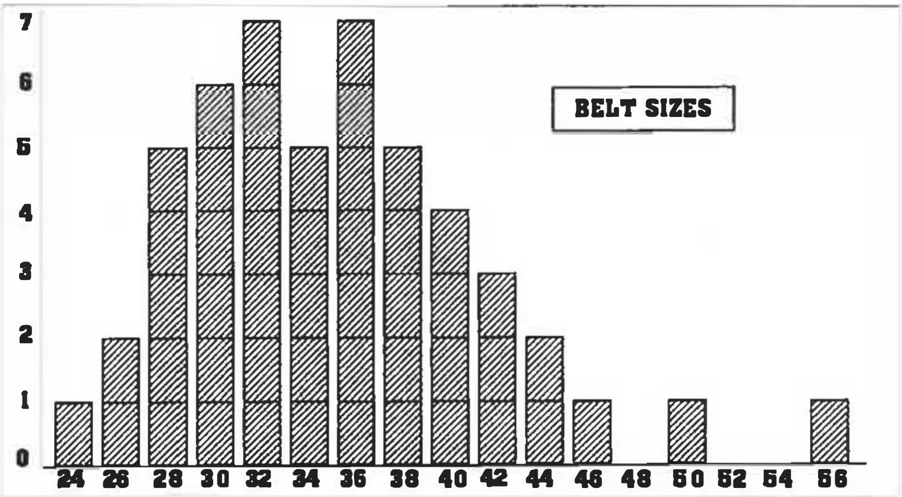
2. These topics are presented in two different strands in the Western Canada Protocol: statistics and probability (data analysis) and statistics and probability (chance and uncertainty).

## References


Alberta Education. *Program of Studies Elementary Schools: Mathematics (Interim Document)*. Edmonton: Author, 1994.

——. *The Common Curriculum Framework for K–12 Mathematics*. Edmonton: Author, 1995.


### Figure 1: Belt Sizes Data Presented as a Bar Graph




### Figure 2: Bar Graph with Vertical Axis Scale Adjusted for 500 Belts

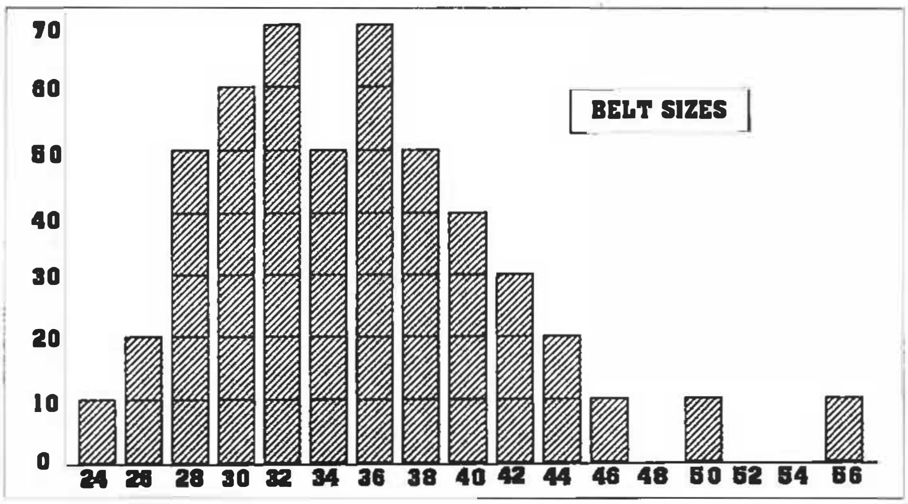


**Figure 3: Bar Graph Gameboard**

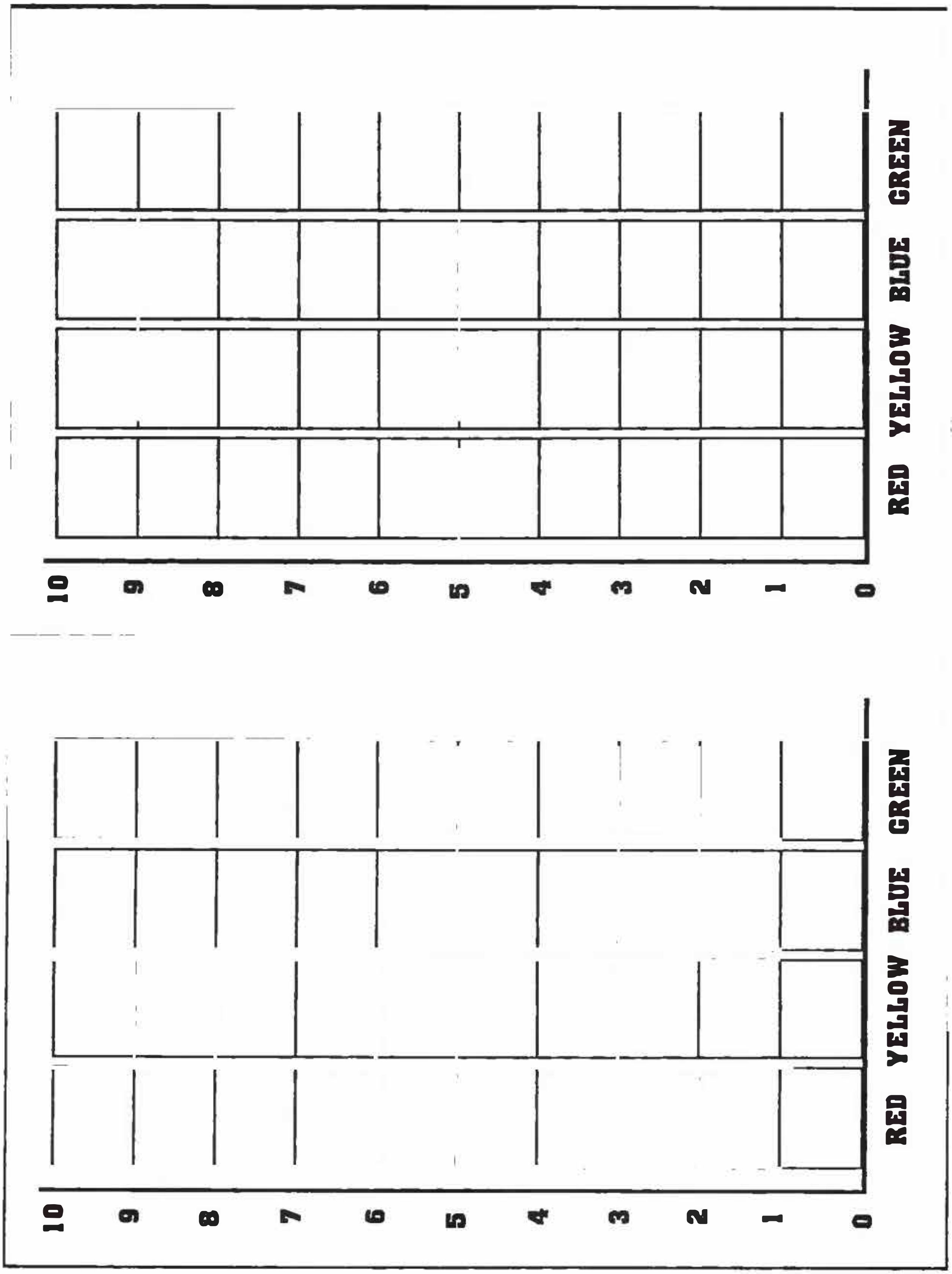
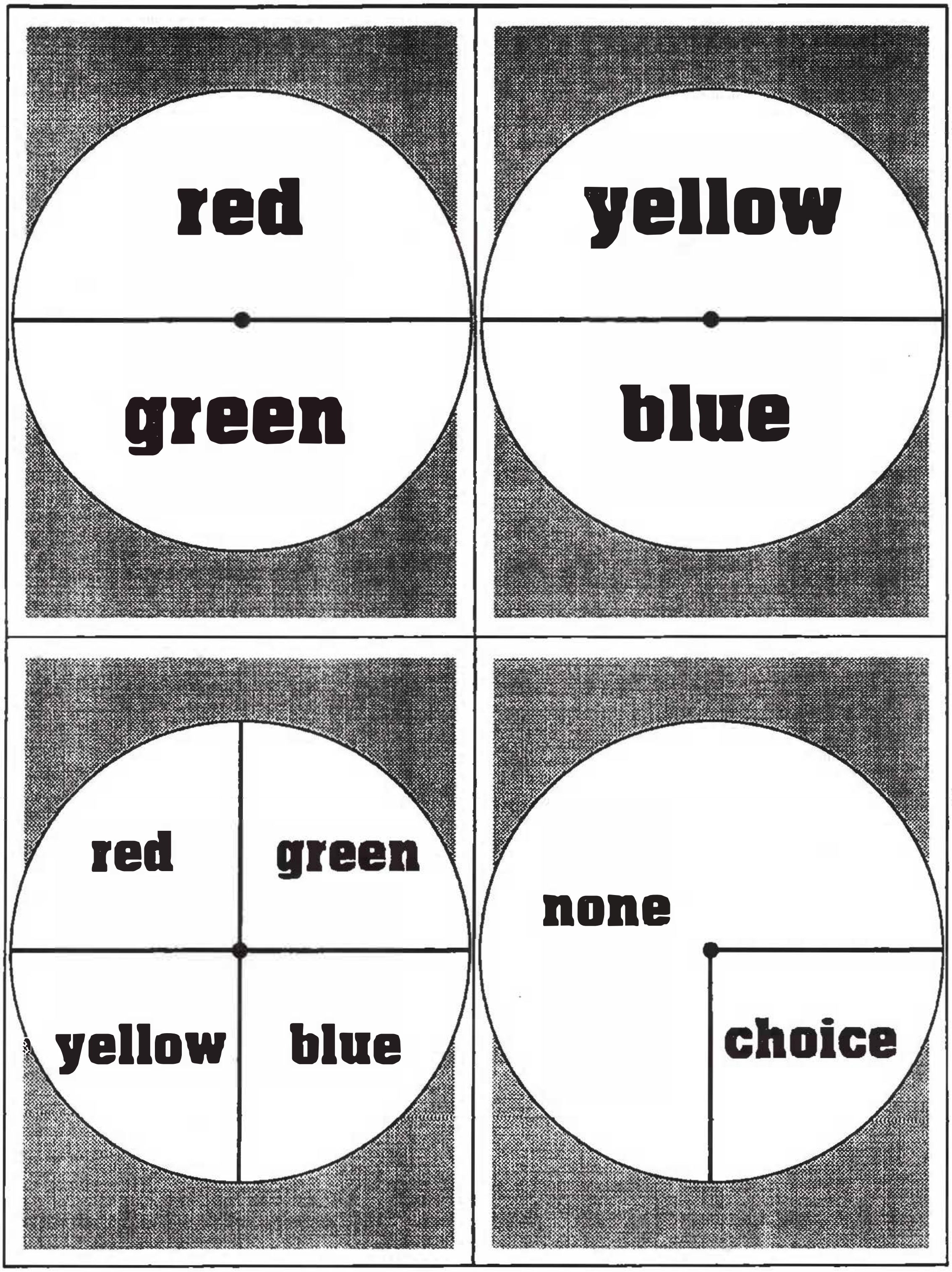


**Figure 4: Race to the Top Spinner Mats**



# Computers in Classrooms: Essential Learning Tool . . . Or Program for Disaster?

*Alison Dickie*

Jennifer was only seven when she started pestering her parents for a computer of her own. Her Grade 2 teacher had told her to ask her parents for one because all but three of the other children in her class were using home computers to complete their school projects. Jennifer's teacher believed that it was important for students to become comfortable with this technology and the sooner the better.

Teachers like Jennifer's are one reason sales of computers and software are booming. Between 1991 and 1992, sales of educational software jumped 50 percent. In the past three years, British Columbia schools spent $35 million on computers, and Ontario schools spent $140 million.

Ottawa and the other provinces have invested hundreds of millions more, and former Ontario Premier Bob Rae pledged $500 million for computers in schools over the next five years. And then there's the $475 million Canadians spent last year on video games—many people are under the impression the games have educational merit.

But not everyone is convinced that computers are effective learning tools. A growing number of parents and teachers are questioning the value of devoting so much scarce money and teacher time to a technology that is largely experimental. Some educators argue that most so-called educational software isn't good enough to introduce into the classroom.

"There is this idea that decent educational software is out there," says American learning theorist Roger Schank. "It isn't. The software you are seeing is a bad imitation of books. It's a good imitation of quizzes, but who wants quizzes as a form of instruction?"

Schank says the quality of most programs is poor because they have been designed by computer scientists who know nothing about how children really learn. As a result, they make programs Schank characterizes as "shoot the verb when it goes by."

Susan Kiil, a Toronto ecology consultant and author of several books on ecology for children, agrees. "Children are limited, not only by the parameters of the software but also by the creativity of the person who designed it," Kiil says.

## Bad Imitation of Books

Because much educational software involves fill-in-the-blank or drill-and-rote work, Kiil believes we may be creating a generation of vertical, not lateral, thinkers. At the same time, she is concerned that using this technology in the primary years subtly devalues a child's own handwriting and drawing. Not only does the computer create a homogenizing effect because every piece of work looks the same but also children are left with a feeling that work produced by hand is less worthy than work that comes from a machine. In the long run, she says, we could be creating kids with self-esteem problems.

"Children are first and foremost creatures of sense," says Roma Lupenec, a special education teacher and parent of a four-year-old.

Children are attracted to computers because of the strong visuals and the mechanical aspect of playing with the keyboard. But ultimately the computer is a one-sense experience. Real learning, she says, nourishes the child's imagination by engaging all the senses.

Lupenec is particularly sceptical of software programs that claim to teach eye-hand coordination. One example she cites is a game that simulates a handball court. But because young children don't play handball and are unlikely to have visited a court, the game has little meaning for them. Artistic activities, like painting, yarn work, cutting or sewing, are far more effective in teaching eye-hand coordination than any computer game.

"It is much healthier for a child to be standing using an art easel," Lupenec says, "than to be sitting almost motionless at a monitor."

Most progressive educators agree that early learning experiences should be child-centred and concrete, using examples drawn from a child's experience and appropriate to his or her level of development. To develop skill with numbers, for instance, children should first be given concrete objects like blocks or pebbles to play with.

Learning should always proceed from the concrete to the theoretical, says Valdemar Setzer, a math teacher and computer science lecturer. Computers, says Setzer in his book *Computers in Education*, force a child to work backward, moving from the abstract to the concrete.

For this reason, he is also highly critical of computer geography programs. The best way to teach geography, he believes, is by personal recognition—explore and describe the neighborhood surrounding the school before moving on to the complexities of map reading. Setzer believes that the best time to introduce computers into classrooms is when students are in high school and more capable of abstract thinking.

But if, as some proponents argue, computers can speed up the learning process by teaching children to read, write or do basic arithmetic more quickly, shouldn't they have access to them as early as possible?

Not necessarily. Pushing children into an activity too soon may force them to use a part of their brain that was meant for another function and can actually interfere with the learning process, says American psychologist Jane Healy.

Children's minds are already bombarded with too much fast-paced sensory stimulation from electronic sources, Healy argues. In her book *Endangered Minds: Why Children Don't Think, and What We Can Do About It*, Healy says that too much electronic stimulation may actually be changing the structure of children's brains.

## Bombard a Child

Television, computers and video games bombard a child with visual information, and do not leave enough time for quiet reflection, concentration or conversation. These quiet moments are essential, she says, if children are to develop into thoughtful people with the inner control necessary to manage their own lives.

Healy believes that electronic overstimulation, especially from computers and televisions, may be contributing to the rise in the incidence of learning disabilities, such as auditory-processing problems and attention deficit disorder. Children who are not taught to listen can easily develop habits that let them avoid exercising—and thus building—important auditory-processing connections in the brain.

"This very act of remembering lays down physical tracks in the brain, but children can quite easily avoid having to build these systems," she says.

Because some children now get more information from pictures than from talking, Healy argues that their brains are simply not trained to understand and retain language. She points out that teachers believe that the listening skills of children in schools today are much worse than those of previous generations.

Like Healy, Susan Kiil believes that computers spell trouble for children who are not visual learners. Furthermore, the pressure to embrace computer technology in the school system is siphoning funds from other areas, such as the creative arts, which are critical to the development of the whole child. Music, for instance, is known to have beneficial effects on math as well as literacy skills. Not only does the ability to play an instrument bring children joy and a sense of accomplishment but also it allows them to develop focus and discipline. Yet many school boards have cut back or eliminated music programs, only to spend money on computers. Kiil also fears that widespread use of technology is devaluing the creative teacher as well as the child.

Proponents of computer technology cite studies that seem to show that it can improve writing and arithmetic skills, sharpen critical thinking and motivate children who want to learn. However, critics have taken issue with these studies, which are often initiated by the companies that market the software.

When Henry Jay Becker of Johns Hopkins University analyzed the results of computer evaluation reports from elementary and middle grades, he concluded that some studies substantially overreported the effectiveness of computers.

"The poor quality of most evaluations, and the likely bias in what does get reported, all provide too weak a platform for district purchasing decisions," he concluded in an article in the 1992 *Journal of Educational Computing Research*.

More recently, a Japanese study explained the effectiveness of using microcomputers in teaching on a sample of 803 primary school children. The researchers concluded that computer use neither improved intellectual activities, such as creativity, nor motivated the Grades 1 and 2 children to study more. A 1994 American study designed to test the effectiveness of computer-based geography programs showed that students who used computers learned no more than their counterparts who used maps and atlases.

Susan Kiil worries that children who are given computers too early will grow up without an essential critical perspective on the very technologies they're using. If we want children to have an appreciation of the natural world and their place in it, we should give them more opportunities to experience this world, through regular field trips that take them out of the classroom. If the educational system persists in leaning so heavily on technology, she thinks we are in danger of losing the knowledge of how to use other, more imaginative technologies that may be more beneficial in teaching and learning.

Oh, and what about seven-year-old Jennifer, whose parents were told she needed a computer because all the other students had one?

Jennifer's mother went for advice to her sister, Charlene Watson, who is an office manager for a software developer. Watson knows enough about computers not to be impressed and advised her sister not to buy a computer the family couldn't afford and didn't need.

"Grade 2 is just too young to be using this technology," she said.

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## Smile

A smile costs nothing but gives much. It enriches those who receive without making poorer those who give. It takes but a moment, but the memory of it sometimes lasts forever. None is so rich or mighty that he can get along without it, and none is so poor but that he can be made rich by it.

A smile creates happiness in the home, fosters goodwill in business and is the countersign of friendship. It brings rest to the weary, cheer to the discouraged, sunshine to the sad and is nature's best antidote for trouble. Yet it cannot be bought, begged, borrowed or stolen, for it is something that is of no value to anyone until it is given away.

Some people are too tired to give you a smile, so give them one of yours, as none needs a smile so much as the person who has no more to give.

# Conic Aerobics

*Ken Harper*

When faced with questions involving the conic sections, many students have the impression that an infinite number of possible solutions exist, but only one acceptable to the teacher. With the gradual inclusion of graphic calculators, the laboriousness and many difficulties of the pencil-and-paper graphing approach have been removed. Other methods can assist students as they try to make sense of the topic for themselves. The activity that follows is a "low-tech" approach that has been used successfully to help Grades 11 and 12 students increase their awareness of how the different components of equations representing the conic sections affect the location and shape of their associated graphs. The methods used here do not resolve all issues, but they are easy to do, interesting, engaging for the students and contain mathematically worthwhile concepts in a memorable and enjoyable series of settings.

Conic aerobics has few requirements—a little space to move is essential and a willingness on the part of the teacher to be a bit of a ham certainly helps. A Richard Simmons wig and jogging suit might be appropriate—an imitation of the Coneheads might be going too far . . . then again . . .

The purpose of conic aerobics is to help more students remember how the various parts of the equations for parabolas, circles, ellipses and hyperbolas affect their appearance on the coordinate plane. The process described below has been used as an introduction and review of the conic sections. The process is based on the beliefs that many approaches to teaching and learning are more desirable than one and that the commitment to find and develop experiences has a strong impact on the students; one powerful enough that the students remember the approaches and can invoke them when they are needed. In this case, the intent is to give the students a feel for what happens as the components of equations representing the conic sections are altered.

The activities that follow have been tried in a classroom. They were introduced over several periods and were used along with calculators, textbooks and other materials as part of a complete mathematics program. The demonstrations and exercises are designed to lead students through each component of the conics and culminate in an activity where they work individually or in small groups to write the "script" for, and perform, a sequence of moves entitled An Invitation to the Conic Dance.

## Getting Started: The Standard Position

Figure 1 sets a standing figure positioned against a background composed of a vertical Cartesian Plane with the x-axis in line with the collarbone, the y-axis vertically dividing the body and the origin at the intersection of these. The arms are held upward in a gentle U-shaped curve representing the equation $y = x^2$ and is referred to as the standard position. All activities that follow start and end in this position.

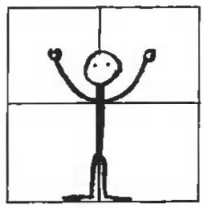


**Figure 1. The Standard Position**

## Opening and Closing the Curve: A Change in the Leading Coefficient Alters the "Tightness" of the Curve

While all students can move their arms up and down, making tighter and looser U-shapes, it is unlikely many of them have made a connection between this movement and the equations for various parabolas. It is likely the teacher will have to demonstrate that a slightly tighter U indicates a change from $y = x^2$ to $y = 2x^2$ as in Figure 2.

$y = x^2$ $y = 2x^2$

**Figure 2. Transforming from $y = x^2$ to $y = 2x^2$**

## Questions and Tasks

1. Model, draw or demonstrate $y = 4x^2$ and $y = 6x^2$. (The arms should be placed higher and tighter together as the coefficient increases.)
2. Demonstrate how you could show $y = \frac{1}{2}x^2$ or $y = \frac{1}{4}x^2$.
3. Show how a change in the coefficient in the written representation of the equation is reflected in the graph of the same equation. Do this through a series of sketches with equations and captions indicating the transformation.
4. Explore how a change in the leading coefficient moves circles or ellipses on the coordinate plane.

An interesting situation arises as a parabola opens up to become a straight line. In body language, the arms are outstretched while in the language of math textbooks the parabola degenerates to a straight line. In the conic aerobics exercise, this awareness can arise naturally from situations created in the activity. This experience can be used to reinforce earlier work focused on the horizontal quality of $y = 0$ in linear equations.

The next step is to investigate what happens when the coefficient is negative. An exaggerated imitation of the familiar "grunting he-man muscle-builder" posed with arms in an inverted "U" position would likely fix the image in the mind of most students.

**Figure 3. "He-Man" Negative Coefficient**

Representations so far have included

- the effects of various positive whole number coefficients,
- the effects of proper fraction coefficients between 0 and 1,
- the results when the coefficient is 0, and
- the views when the coefficient is less than 0.

Students have likely been exposed to enough information to allow them to create and label their own examples. One type can involve a series of sketches showing a flower opening ($y = 10x^2$) as it blooms ($y = 5x^2$ . . . $y = 2x^2$ . . . $y = 0x^2$) followed by the gradual transformation to death ($y = -10x^2$).

The equations or the drawings do not need to be exact representations. The goal is to help students develop an awareness or an ability to visualize how a particular change in the coefficient results in a specific change to the corresponding graphical representation of it.

Students should now have the tools to illustrate or construct a sequence of steps such as in Figure 4.

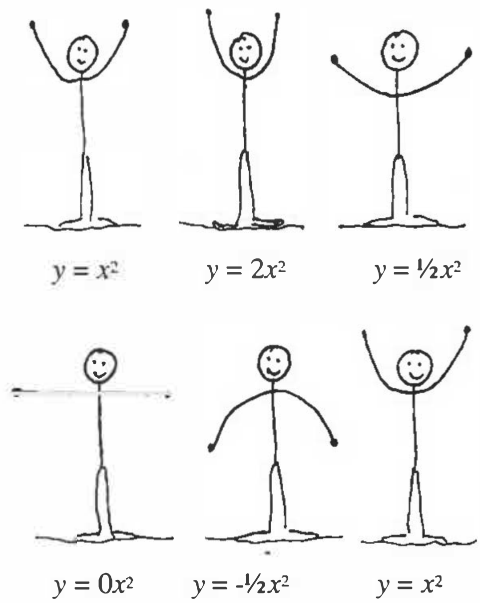



**Figure 4. Six Steps Beginning and Ending with $y = x^2$**

## Vertical Moves

Once students have mastered opening and closing the parabola by attending to the value of the coefficient, new steps can be added to their repertoire. The next series introduces the constant value or the last term usually found in a quadratic equation of this type.

Starting from the standard position, the teacher models or has a student demonstrate $y = x^2$, but standing with heels raised slightly off the floor. This represents $y = x^2 + 1$. The next position is created by returning to the standard position with both feet flat on the ground and then *slightly* bending the knees, lowering the torso and arms to a position that can be identified with $y = x^2 - 1$. With little imagination and practice, students could demonstrate (probably with exaggerated movements) $y = x^2 + 5$ or $y = x^2 - 10$.

Sequenced steps can now include focusing on only the up and down movements, but students may want to include "opening" and "closing" in addition to the "up" and "down." Now, it may be preferable to allow them to explore and write their own scripts. One restriction that may help to avoid mild chaos is to introduce a rule that restricts each move so that it only differs from the previous one by one characteristic. For example, if one step shows $y = 2x^2 + 3$, the next step could change the vertical orientation by replacing the 3 with 2 or 5. An alternative to this would be to change only the leading coefficient from 2 to 4 or -2.

In other words, it is not as important to restrict the extent to which each variable changes as it is to only change one at a time. More students seem able to work within these limits than are able to generate random examples. Working on a restricted sequence also focuses their attention on one detail at a time, probably helps them discriminate between the types of moves permitted and reinforces the mental pictures generated.

A possible sequence of moves is demonstrated in Figure 5.

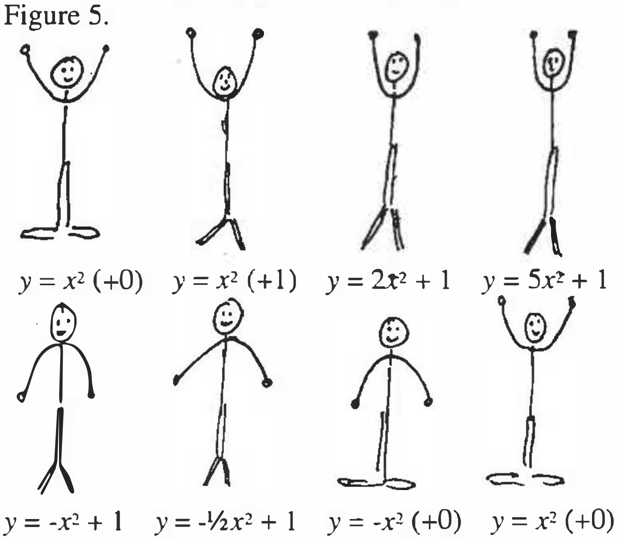



**Figure 5. Leading Coefficient and Constant Sequence**

## Horizontal Transformations

There is no one best order for introducing the components of equations for the conic sections. It is important, however, for students to realize that there are restrictions to the types of representations. The graphic representation can show students parabolas in a variety of opened or closed positions, upright (in standard form) or inverted, and as the special case when it becomes flat (or degenerates to a straight line). The parabola in any of its states can also be placed anywhere on the y-axis. In terms of keeping the same vertical orientation on a coordinate plane, there is only one more condition for consideration—movement to the left or right.

The process of introducing this component motion can be similar to that of the previous two transformations. Students generally do not have difficulty understanding that the movement to the left or right will be reflected in the equation, but they may seem uncomfortable with the idea that $y = (x + 1)^2$ indicates a shift to the "negative" side of the y-axis. One method of working through this dilemma is for the teacher to show an equation, demonstrate the move and then ask the students to make their own generalization, rule or mnemonic device to help them remember the direction they must move.

Once the three components are in place, increasingly complex settings can be created. Students may enjoy creating moves that are sequential and have a sense of rhythm. As they become knowledgeable and comfortable with the different variations, they also take more risks and move further away from the standard position and the origin of the coordinate plane.

At this stage, it may be ideal to introduce the group activity called An Invitation to the Dance. The template in Figure 6 can be used for small groups to design their own "dance" which then can be performed for the class. For one type of presentation, the completed template is placed on an overhead like the libretto at the opera. This allows the entire class to follow along. Designating one student as choreographer and announcer can help to keep the group moving together and serves as a reminder of the moves associated with the equations.

The "dance" commences at the standard position with each "step" having only one attribute different than its immediate predecessor. The moves continue in a similar way until it is resolved in step 12 where the dance ends in the same position it started.

## Conclusion

The conic aerobics activity is a simple idea that allows students to explore the variations and limits of allowable "moves" in graphing the conic sections. It is one more way that conics can come to be understood by students. Restricting the types of moves and requiring them to be presented in a sequence requires students to examine each step, discriminate between moves and assess which options are open for successive moves. This may be especially advantageous for long-term effect. Many textbook exercise sequences move quickly from simple to complex types of questions with little discernible rationale for the choice of each particular question in a set. In the sequence of tasks suggested as part of this article, students must consider each step in relation to the previous one. In a sense they are creating their own exercise and task analysis on this specific topic.

The suggested approach lends itself to the qualities of mathematics instruction promoted by the National Council of Teachers of Mathematics in the Curriculum and Evaluation Standards. Extensions to the activity are easily introduced. For example, arching horizontally suggests the graph of $x = y^2$. The parabola itself can be replaced with a circle created with the arms gracefully curved with fingers just touching. Many students are likely to suggest their own variations. On more than one occasion, the author encountered students presenting overacted representation of $y = x^3$ by imitating Steve Martin's gesticulating arms in his well-known King Tut sketch.

Activities such as these have other benefits. Students have opportunities to collaborate with each other in a lighthearted yet serious way. They can learn to appreciate the contributions of others. They can also become aware that serious mathematics can be enjoyable, countering the view that portrays mathematics as little more than a series of exercises to be completed in sequence following exact procedures in a solitary setting. Students are also allowed to ask questions that go beyond the current work. For example, what do the equations look like for parabolas symmetrical about the diagonal line represented by the equation $y = x$? How can physical rotations about an axis be accounted for in an equation? And isn't that what many of us want from our students—to become aware, to understand, to question, and to challenge themselves . . . and us?

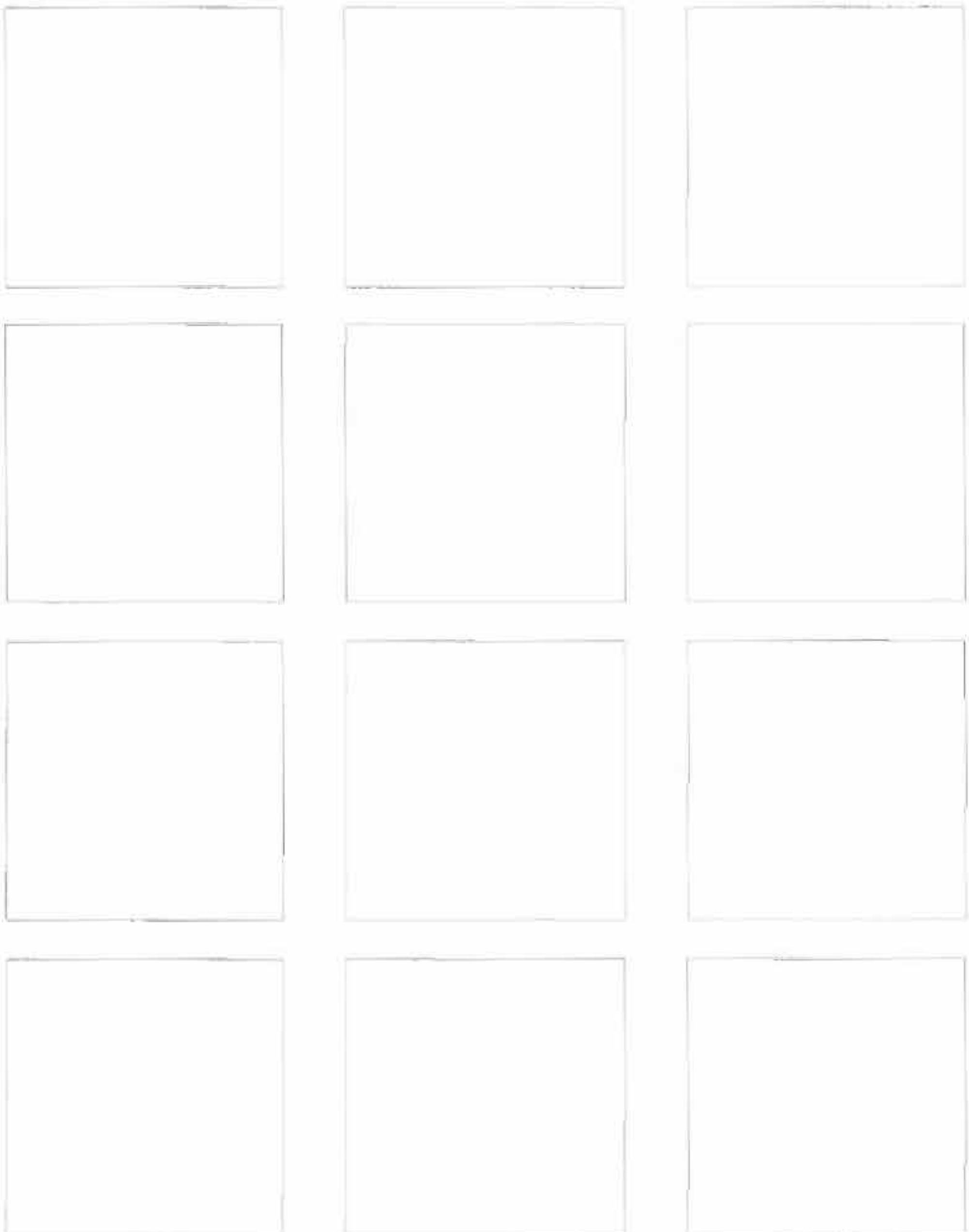


**Figure 6. An Invitation to the Conic Dance**

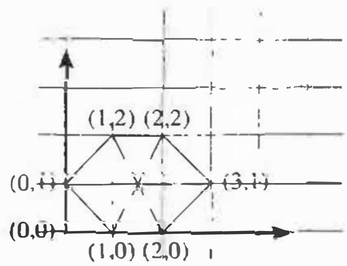
# Lattice Hexagons: Pattern Discovery Activities

*Bonnie H. Litwiller and David R. Duncan*

Teachers are always interested in mathematical activities that encourage students to find patterns and make generalizations leading to formulas. Figures drawn on a lattice (square graph paper) provide a rich setting for this type of activity.

Figure 1 displays a hexagon drawn on a coordinated lattice. The two horizontal sides are each one unit in length; the four diagonal sides are each $\sqrt{2}$ units. In addition, diagonals are drawn connecting opposite vertices.

**Figure 1**

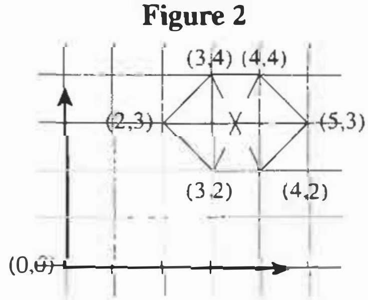



Observe the following:

1. The area of the hexagon is 4 square units.
2. The perimeter of the hexagon is $2 + 4\sqrt{2} = 2(1 + 2\sqrt{2})$ linear units.
3. Opposite vertices: (1,0) and (2,2); (2,0) and (1,2); (3,1) and (0,1).
   Sum of all coordinates: 5; 5; 5
   The sums are all constant.

Figure 2 displays the same size hexagon but is positioned differently on the lattice.

**Figure 2**




Observe the following:

1. The area and perimeter are, respectively, 4 units and $2(1 + 2\sqrt{2})$ units, the same as for the hexagon of Figure 1.
2. The opposite vertex coordinates again sum to a constant.

The patterns observed in Figures 1 and 2 are independent of the position of the hexagon.

Figure 3 displays a variety of hexagons drawn on a coordinated lattice.

**Figure 3**

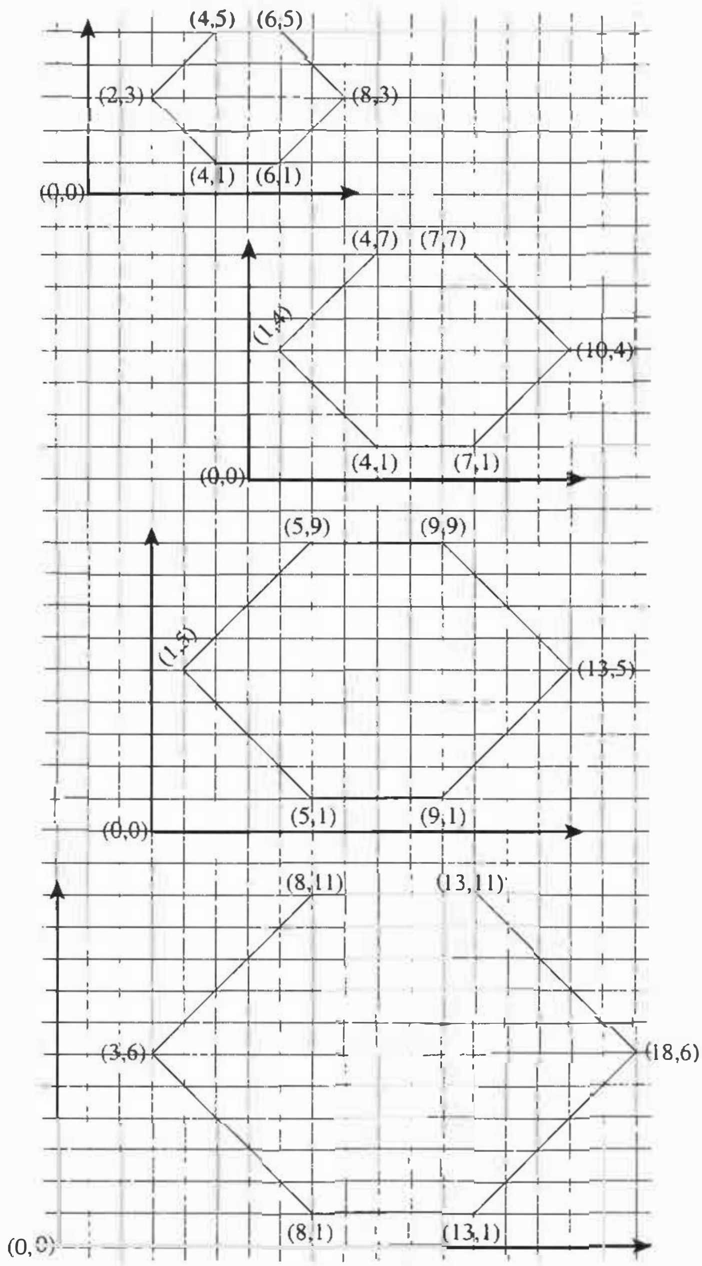


Table 1 reports the results of calculations for the hexagons of Figures 1 and 3.

**Table 1**

| Length of horizontal side | Area | Perimeter | Sum of each pair of opposite vertices |
|---|---|---|---|
| 1 | 4 | $2 + 4\sqrt{2}$ | 5 |
| 2 | 16 | $4 + 8\sqrt{2}$ | 16 |
| 3 | 36 | $6 + 12\sqrt{2}$ | 19 |
| 4 | 64 | $8 + 16\sqrt{2}$ | 24 |
| 5 | 100 | $10 + 20\sqrt{2}$ | 33 |

Several conjectures are suggested by Table 1:

1. Area
   Observe that the sequence of areas is composed of the squares of the consecutive even numbers; we conjecture that for a hexagon having horizontal sides of length $n$ (an $n$-hexagon), the area is $(2n)^2$.
2. The perimeters may be written as
   $2\ (1 + 2\sqrt{2})$
   $4\ (1 + 2\sqrt{2})$
   $6\ (1 + 2\sqrt{2})$
   $8\ (1 + 2\sqrt{2})$
   $10\ (1 + 2\sqrt{2})$

   The coefficients are the even natural numbers, so we conjecture that the perimeter of an $n$-hexagon is $2n(1 + 2\sqrt{2})$.
3. Although the actual sums depend on the placement of the hexagon, we conjecture that the constant opposite vertex sum property holds for any $n$-hexagon.

These activities connect algebra, geometry and arithmetic in a setting new to most students.

***Challenges***

1. Check the patterns by drawing hexagons in varying positions and sizes.
2. All figures were drawn in the first quadrant. Do the same patterns hold if other quadrants are used?
3. Examine other geometric figures drawn on a lattice. For example, try the octagon. Do similar patterns hold?
4. Can any of these patterns be replicated on an isometric lattice?
5. Find the other novel settings in which connections can be made and patterns discovered.

# The Seeds of Tomorrow: Iterating Functions

*J. Dale Burnett*

It's *déjà vu* all over again.

—*Yogi Berra*

The idea of something repeating itself is one of the strongest themes in mathematics. The idea lies at the heart of mathematics: the definition of the natural numbers 1, 2, 3. . . .

It also lies at the heart of computing science. Two of the most powerful ideas in the study of computing algorithms are the loop and recursion.

What happens when you combine mathematics and computing science? Bateson (1979) suggests that an alternative title for his book *Mind and Nature* could have been "The Pattern That Connects." It would also make an appropriate title for this article. The connections among the examples to be discussed may be envisioned at many levels. I am reminded of two books: *The Cerebral Symphony: Seashore Reflections on the Structure of Consciousness* (Calvin 1990) and *Descarte's Error* (Damasio 1994), which discuss our current thoughts about thinking. Hofstadter's (1979) *Gödel, Escher, Bach: An Eternal Golden Braid* and Pickover's *Computers, Pattern, Chaos and Beauty* (1990), *Computers and the Imagination* (1991) and *Chaos in Wonderland* (1994) represent additional readings that capture the spirit of this new era. A recent addition to my library is *Frontiers of Complexity* (Coveney and Highfield 1995). The foreword for this latter book is written by Baruch Blumberg, a nobel laureate in medicine, who says,

> My experience is that, in medicine, where observational science is crucial, the complexities of a phenomenon can be understood, at least in part, by repeated observations of a whole organism or a population of organisms under a wide variety of circumstances (p. xi).

That is the theme of this article.

One of my favorite books is Alfred North Whitehead's (1929) *The Aims of Education*. I find it still timely today. Quoting from the first page:

> In training a child to activity of thought, above all things we must beware of what I will call "inert ideas"—that is to say, ideas that are merely received into the mind without being utilised, or tested, or thrown into fresh combinations.

Extending this theme, my aim in this article is to provoke one into embarking on a personal voyage of exploration. I will mention two vehicles useful for the journey and even suggest a few interesting sites, but, if this is just an article to be read, then I will not have achieved my goal.

Computing has evolved to provide a variety of human-machine interfaces. It is no longer necessary to learn how to program to make use of the computational power of computer technology. Spreadsheets represent an important alternative for harnessing the computational power of the computer. Both approaches have appeal for one interested in exploring certain mathematical topics.

This article will explore a few fairly common mathematical functions using spreadsheets and a programming approach in part to compare the results, in part to compare the effort involved and in part to see what is noticeable from the different output displays. Although this article provides a comparison between two types of tool, the focus is on gaining a deeper understanding of the behavior of certain mathematical functions under a process of iteration.

Because a tool rightfully occupies a supporting role, the article will be organized around the exploration of different mathematical functions under the process of iteration. However, it is first necessary to get a feel for the tools, if only so they can recede into the background and leave us to focus on the product.

But even this latter statement misses an important feature. The process of using the tool, of thoughtfully engaging in the activity of creating a new personal understanding, is at least as enjoyable as examining the result. Csikszentmihalyi (1993) has captured this essence in his use of the term "flow": a metaphor that conveys the feeling of "being carried away by a current, everything moving smoothly without effort." He then points out, "Contrary to expectation, 'flow' usually happens not during relaxing moments of leisure and entertainment, but rather when we are actively involved in a difficult enterprise, in a task that stretches our physical or mental abilities" (p. xiii). This is an article about "flow."

# Two Vehicles for Mathematical Journeys

## Programming

Let's begin with an arbitrary linear function f(x) = 2x – 6 and calculate the first three values under repeated iteration. Suppose we begin with x = 1. Then

f(1) = 2 – 3 = -1
f(-1) = -2 – 3 = -5
f(-5) = -10 – 3 = -13

We could continue the process by hand. We could also continue it using a basic handheld calculator. We could also write a computer program, using a variety of computing languages. Here is one such program, written in LogoWriter, that computes the first 20 values:

```
To ITERATE1 :n :x
   repeat 20[make "n :n+1
             make "x 2*:x-3
             print se :n :x]
End
```

In English: this is a short procedure called ITERATE1 that takes two input values (*n*: this is a counter that keeps track of the number of iterations, and *x*: the initial value of the function as we begin the iterations). The procedure thus consists of three statements that are repeated 20 times. The first statement increments the counter by 1. The second statement calculates a new value for *x*, equal to 2 times the old value for *x*, minus 3. The third statement prints the values for *n* and *x*.

Here is the resulting output when one types ITERATEl 0 1

```
1 -1
2 -5
3 -13
4 -29
5 -61
6 -125
7 -253
8 -509
9 -1021
10 -2045
11 -4093
12 -8189
13 -16381
14 -32765
15 -65533
16 -131069
17 -262141
18 -524285
19 -1048573
20 -2097149
```

One can also write a LogoWriter procedure that uses the idea of recursion (that is, a procedure that calls itself).

```
To RECURSIVE1 :n :x
         if :n=21 [stop]
         print se :n :x
         RECURSIVE1 :n+1 2*:x-3
End
```

The logic of this algorithm is fundamentally different than the iterative approach described a moment ago. The first statement within the procedure is a stop criteria: if *n* is equal to 21 then stop the process. The second line is the familiar print statement. The third line is the essence: the procedure calls itself but alters the value of *x* to the new value 2*x* – 3.

Here is the output from typing RECURSIVE 0 1

[figure 4 goes here]

```
1 -1
2 -5
3 -13
4 -29
5 -61
6 -125
7 -253
8 -509
9 -1021
10 -2045
11 -4093
12 -8189
13 -16381
14 -32765
15 -65533
16 -131069
17 -262141
18 -524285
19 -1048573
20 -2097149
```

It is reassuring to see the results are identical!

One can use these two simple procedures to investigate almost any function. One need only alter one line in the procedure, the line that specifies the function.

## Spreadsheets

A variety of spreadsheet programs are available for all computers. I will use Excel, because it is one I use regularly. Here are the first two rows of the spreadsheet:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | 2x - 3 |
| 2 | 1 | 1 | -1 |
| 3 | | | |

The first row is just for typing in a heading for each column. The second row contains the initial values for each column. Thus *n* (the number of iterations) is 1. The second column contains the value of *x*. The third column is computed (using the formula =2*B2 – 3, which means 2 times the value in cell B2 minus 3).

The power of the spreadsheet begins to become apparent in the next row. Cell A3 is computed, using the formula =A2 + 1. That is, take the value in the cell above and add one. Cell B2 is also computed: the formula is =C2. Thus it takes the value that was computed in the previous row. Finally cell C3 is computed using the same formula as before: =2*B3 – 3, adjusted for row 3.

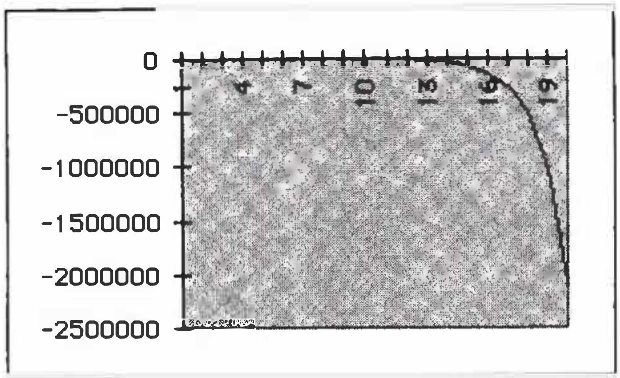
Here are the results:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | 2x - 3 |
| 2 | 1 | 1 | -1 |
| 3 | 2 | -1 | -5 |
| 4 | | | |

Now comes the action. One can use a command called Fill-Down and select the next 18 rows. The results are immediately displayed as follows:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | 2x - 3 |
| 2 | 1 | 1 | -1 |
| 3 | 2 | -1 | -5 |
| 4 | 3 | -5 | -13 |
| 5 | 4 | -13 | -29 |
| 6 | 5 | -29 | -61 |
| 7 | 6 | -61 | -125 |
| 8 | 7 | -125 | -253 |
| 9 | 8 | -253 | -509 |
| 10 | 9 | -509 | -1021 |
| 11 | 10 | -1021 | -2045 |
| 12 | 11 | -2045 | -4093 |
| 13 | 12 | -4093 | -8189 |
| 14 | 13 | -8189 | -16381 |
| 15 | 14 | -16381 | -32765 |
| 16 | 15 | -32765 | -65533 |
| 17 | 16 | -65533 | -131069 |
| 18 | 17 | -131069 | -262141 |
| 19 | 18 | -262141 | -524285 |
| 20 | 19 | -524285 | -1048573 |
| 21 | 20 | -1048573 | -2097149 |
| 22 | | | |

Once again, it is nice to see the results agree with the two LogoWriter procedures! The spreadsheet approach has another nice feature: graphing. It is a relatively easy matter to obtain the following graph:




This graphing feature may be useful later.

## Interesting Sites for Mathematical Journeys

Investigating a function under repeated iteration requires a few preliminary ideas. First, the iteration must begin with a specified initial value. This initial value is usually called the *seed.* Second, the successive values of the iterated function, for a particular seed, are called the *orbit* of that point. Thus each initial value of the function has its own orbit. The investigation of all the orbits of a given function is called *orbit analysis*.

Let's begin with a simple example, that of the linear function from the preceding section: $f(x) = 2x - 3$.

### A Linear Function: $f(x) = 2x - 3$

The previous section showed, using two approaches, that the orbit of the point $x = 1$ approached negative infinity. What about the orbits of other points? Do they also approach negative infinity?

A first venture into this question might proceed by simply trying a few disparate values of *x* for the seed, and seeing what happens. Let's try a few prototypical values: -100, -1, 0, 0.5, 10 and 100. Also let's look at $x = 3/2$, the root of the equation. Thus the first goal in our exploration is to develop a quick informal feel for the situation.

Here is the result of the first 20 iterations using the LogoWriter program ITERATE1 for the point $x = -100$:

```
1 -203
2 -409
3 -821
4 -1645
5 -3293
6 -6589
7 -13181
8 -26365
9 -52733
```

| | |
|---|---|
| 10 | -105469 |
| 11 | -210941 |
| 12 | -421885 |
| 13 | -843773 |
| 14 | -1687549 |
| 15 | -3375101 |
| 16 | -6750205 |
| 17 | -13500413 |
| 18 | -27000829 |
| 19 | -54001661 |
| 20 | -108003325 |

Each value is less than the preceding value. The orbit is approaching negative infinity.

Let's look at the orbit for $x = +100$:

| | |
|---|---|
| 1 | 197 |
| 2 | 391 |
| 3 | 779 |
| 4 | 1555 |
| 5 | 3107 |
| 6 | 6211 |
| 7 | 12419 |
| 8 | 24835 |
| 9 | 49667 |
| 10 | 99331 |
| 11 | 198659 |
| 12 | 397315 |
| 13 | 794627 |
| 14 | 1589251 |
| 15 | 3178499 |
| 16 | 6356995 |
| 17 | 12713987 |
| 18 | 25427971 |
| 19 | 50855939 |
| 20 | 101711875 |

This is different. The orbit for this value of $x$ tends to plus infinity. Are there other values that orbits for this function approach? Also, if not, where is the dividing point, such that for values of $x$ less than this value, all orbits tend to minus infinity, and for all values greater than this value, all orbits tend to plus infinity. Recall that the previous example showed the orbit for $x = 1$ was minus infinity.

Let's try the seed $x = 5$.

ITERATE1 shows that this orbit appears to tend to positive infinity. Let's try the root $x = 3/2$. The LogoWriter procedure shows this value tends to negative infinity.

As a result of trying a number of values, we can see that if the seed is less than 3 then the orbit moves to minus infinity, and if it is greater than 3, then the orbit moves to plus infinity. If the seed is 3, then the orbit consists of the single point 3.

This type of orbital analysis could be extended to any linear function. Is it always the case that orbits move to plus or minus infinity depending on the value of the seed?

## A Quadratic Function

Let's examine a slightly more complicated function: $f(x) = x^2 - 3$. What are the orbits for various starting points? This time, let's use a spreadsheet for a preliminary venture into the behavior of this function.

Let's try a seed of zero:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | x^2 - 3 |
| 2 | 1 | 0 | -3 |
| 3 | 2 | -3 | 6 |
| 4 | 3 | 6 | 33 |
| 5 | 4 | 33 | 1086 |
| 6 | 5 | 1086 | 1179393 |
| 7 | 6 | 1179393 | 1.391E+12 |
| 8 | 7 | 1.391E+12 | 1.9348E+24 |
| 9 | 8 | 1.9348E+24 | 3.7434E+48 |
| 10 | 9 | 3.7434E+48 | 1.4013E+97 |
| 11 | 10 | 1.4013E+97 | 1.964E+194 |

The orbit very quickly approaches infinity!

Let's use a spreadsheet to see what happens when the seed is -2, -1, 1 and 3. For $x = -2$:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | x^2 - 3 |
| 2 | 1 | -2 | 1 |
| 3 | 2 | 1 | -2 |
| 4 | 3 | -2 | 1 |
| 5 | 4 | 1 | -2 |
| 6 | 5 | -2 | 1 |
| 7 | 6 | 1 | -2 |
| 8 | 7 | -2 | 1 |
| 9 | 8 | 1 | -2 |
| 10 | 9 | -2 | 1 |
| 11 | 10 | 1 | -2 |
| 12 | 11 | -2 | 1 |
| 13 | 12 | 1 | -2 |
| 14 | 13 | -2 | 1 |
| 15 | | | |

This is surprising! Let's try the others. For $x = -1$:

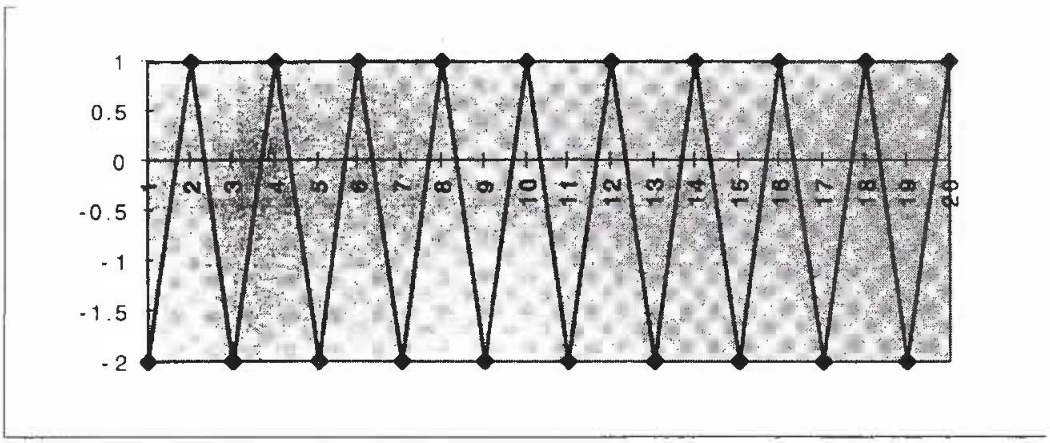
| | A | B | C |
|---|---|---|---|
| 1 | n | x | x^2 - 3 |
| 2 | 1 | -1 | -2 |
| 3 | 2 | -2 | 1 |
| 4 | 3 | 1 | -2 |
| 5 | 4 | -2 | 1 |
| 6 | 5 | 1 | -2 |
| 7 | 6 | -2 | 1 |
| 8 | 7 | 1 | -2 |

| | | | |
|---|---|---|---|
| 9 | 8 | -2 | 1 |
| 10 | 9 | 1 | -2 |
| 11 | 10 | -2 | 1 |
| 12 | 11 | 1 | -2 |
| 13 | 12 | -2 | 1 |
| 14 | 13 | 1 | -2 |
| 15 | | | |

For $x = 1$:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | x^2 - 3 |
| 2 | 1 | 1 | -2 |
| 3 | 2 | -2 | 1 |
| 4 | 3 | 1 | -2 |
| 5 | 4 | -2 | 1 |
| 6 | 5 | 1 | -2 |
| 7 | 6 | -2 | 1 |
| 8 | 7 | 1 | -2 |
| 9 | 8 | -2 | 1 |
| 10 | 9 | 1 | -2 |
| 11 | 10 | -2 | 1 |
| 12 | 11 | 1 | -2 |
| 13 | 12 | -2 | 1 |
| 14 | 13 | 1 | -2 |
| 15 | | | |

Here is the graph for these three cases:




For $x = 3$:

| | A | B | C |
|---|---|---|---|
| 1 | n | x | x^2 - 3 |
| 2 | 1 | 3 | 6 |
| 3 | 2 | 6 | 33 |
| 4 | 3 | 33 | 1086 |
| 5 | 4 | 1086 | 1179393 |
| 6 | 5 | 1179393 | 1.391E+12 |
| 7 | 6 | 1.391E+12 | 1.9348E+24 |
| 8 | 7 | 1.9348E+24 | 3.7434E+48 |
| 9 | 8 | 3.7434E+48 | 1.4013E+97 |
| 10 | 9 | 1.4013E+97 | 1.964E+194 |

These orbits for $x = -2$, -1 and 1 were a bit different—they all oscillate between just two values: -2 and 1.

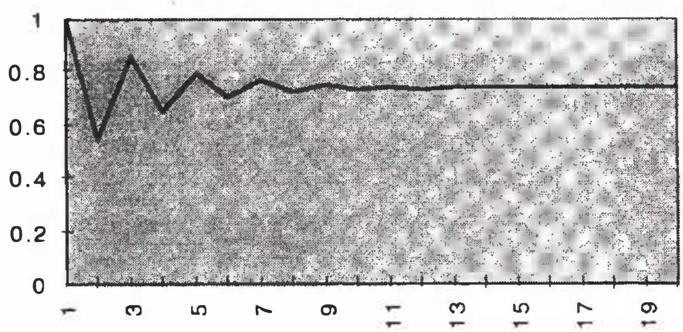
Although most orbits for this function tend to plus infinity, a few orbits have period 2, oscillating between just two values. The analysis has not been exhaustive. Are there other points that have interesting orbits? We will return to the case of quadratic functions again in a moment, but first let's have a quick look at a trigonometric function.

## The Trigonometric Function: cos x

We will begin with a seed of 1.

| | A | B | C |
|---|---|---|---|
| 1 | n | x | cos x |
| 2 | 1 | 1.0000 | 0.5403 |
| 3 | 2 | 0.5403 | 0.8576 |
| 4 | 3 | 0.8576 | 0.6543 |
| 5 | 4 | 0.6543 | 0.7935 |
| 6 | 5 | 0.7935 | 0.7014 |
| 7 | 6 | 0.7014 | 0.7640 |
| 8 | 7 | 0.7640 | 0.7221 |
| 9 | 8 | 0.7221 | 0.7504 |
| 10 | 9 | 0.7504 | 0.7314 |
| 11 | 10 | 0.7314 | 0.7442 |
| 12 | 11 | 0.7442 | 0.7356 |
| 13 | 12 | 0.7356 | 0.7414 |
| 14 | 13 | 0.7414 | 0.7375 |
| 15 | 14 | 0.7375 | 0.7401 |
| 16 | 15 | 0.7401 | 0.7384 |
| 17 | 16 | 0.7384 | 0.7396 |
| 18 | 17 | 0.7396 | 0.7388 |
| 19 | 18 | 0.7388 | 0.7393 |
| 20 | 19 | 0.7393 | 0.7389 |
| 21 | 20 | 0.7389 | 0.7392 |
| 22 | 21 | 0.7392 | 0.7390 |
| 23 | 22 | 0.7390 | 0.7391 |
| 24 | 23 | 0.7391 | 0.7391 |

In this case, the orbit appears to converge to a value of 0.7391.




From the examples we have looked at so far we have discovered orbits that

1. fly off to infinity,
2. exhibit periodic behavior, and
3. converge to a particular value.

## Another Quadratic Function

Earlier we considered the quadratic function $f(x) = x^2 - 3$

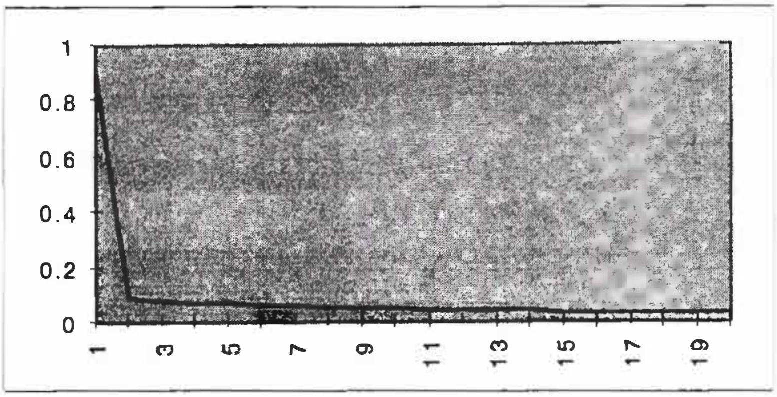
Here is another quadratic equation: $f(x) = cx(1 - x)$. This equation has a rich history, particularly in the life sciences, where it is used to model population dynamics. Because the value for $c$ is not specified, this equation represents a family of equations.

This time I want to hold the seed constant at $x = 0.9$ and see what happens to its orbit as I vary the value of $c$.

Case 1: $c = 1$

| | A | B | C |
|---|---|---|---|
| 1 | n | x | x(1 - x) |
| 2 | 1 | 0.90 | 0.09 |
| 3 | 2 | 0.09 | 0.08 |
| 4 | 3 | 0.08 | 0.08 |
| 5 | 4 | 0.08 | 0.07 |
| 6 | 5 | 0.07 | 0.06 |
| 7 | 6 | 0.06 | 0.06 |
| 8 | 7 | 0.06 | 0.06 |
| 9 | 8 | 0.06 | 0.05 |
| 10 | 9 | 0.05 | 0.05 |
| 11 | 10 | 0.05 | 0.05 |
| 12 | 11 | 0.05 | 0.05 |
| 13 | 12 | 0.05 | 0.04 |
| 186 | 185 | 0.01 | 0.01 |
| 187 | 186 | 0.01 | 0.01 |
| 188 | 187 | 0.01 | 0.00 |
| 189 | 188 | 0.00 | 0.00 |
| 190 | 189 | 0.00 | 0.00 |
| 191 | 190 | 0.00 | 0.00 |

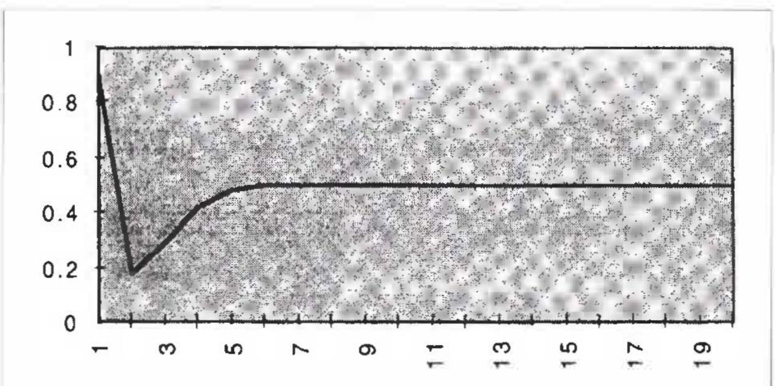
The orbit converges to zero.




Case 2: $c = 2$

| | A | B | C |
|---|---|---|---|
| 1 | n | x | 2x(1 - x) |
| 2 | 1 | 0.90 | 0.18 |
| 3 | 2 | 0.18 | 0.30 |
| 4 | 3 | 0.30 | 0.42 |
| 5 | 4 | 0.42 | 0.49 |
| 6 | 5 | 0.49 | 0.50 |
| 7 | 6 | 0.50 | 0.50 |
| 8 | 7 | 0.50 | 0.50 |
| 9 | 8 | 0.50 | 0.50 |
| 10 | 9 | 0.50 | 0.50 |
| 11 | 10 | 0.50 | 0.50 |
| 12 | 11 | 0.50 | 0.50 |
| 13 | 12 | 0.50 | 0.50 |
| 14 | 13 | 0.50 | 0.50 |

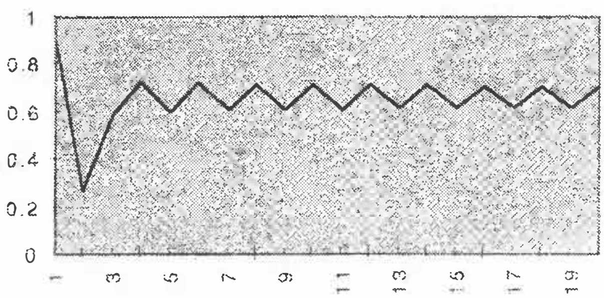
After a momentary drop, the orbit appears to converge to 0.5.



Case 3: $c = 3$

| | A | B | C |
|---|---|---|---|
| 1 | n | x | 3x(1 - x) |
| 2 | 1 | 0.90 | 0.27 |
| 3 | 2 | 0.27 | 0.59 |
| 4 | 3 | 0.59 | 0.72 |
| 5 | 4 | 0.72 | 0.60 |
| 6 | 5 | 0.60 | 0.72 |
| 7 | 6 | 0.72 | 0.60 |
| 8 | 7 | 0.60 | 0.72 |
| 9 | 8 | 0.72 | 0.61 |
| 10 | 9 | 0.61 | 0.72 |
| 11 | 10 | 0.72 | 0.61 |
| 12 | 11 | 0.61 | 0.71 |
| 13 | 12 | 0.71 | 0.61 |
| 14 | 13 | 0.61 | 0.71 |

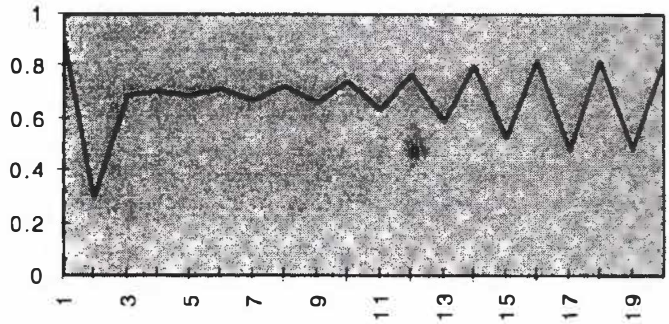
The orbit oscillates but appears to converge to two values.



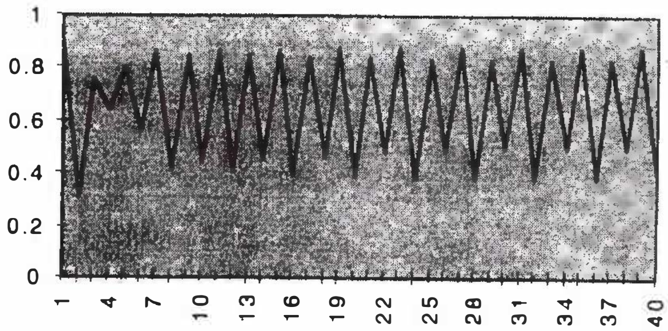

Case 4: $c = 3.3$

| | A | B | C |
|---|---|---|---|
| 1 | n | x | 3.3x(1 - x) |
| 2 | 1 | 0.90 | 0.30 |
| 3 | 2 | 0.30 | 0.69 |
| 4 | 3 | 0.69 | 0.71 |
| 5 | 4 | 0.71 | 0.68 |
| 6 | 5 | 0.68 | 0.71 |
| 7 | 6 | 0.71 | 0.67 |
| 8 | 7 | 0.67 | 0.73 |
| 9 | 8 | 0.73 | 0.66 |
| 10 | 9 | 0.66 | 0.74 |
| 11 | 10 | 0.74 | 0.63 |
| 12 | 11 | 0.63 | 0.77 |
| 13 | 12 | 0.77 | 0.59 |
| 14 | 13 | 0.59 | 0.80 |
| 15 | 14 | 0.80 | 0.53 |
| 16 | 15 | 0.53 | 0.82 |
| 17 | 16 | 0.82 | 0.48 |
| 18 | 17 | 0.48 | 0.82 |
| 19 | 18 | 0.82 | 0.48 |
| 20 | 19 | 0.48 | 0.82 |

Once again, the orbit converges to two values.

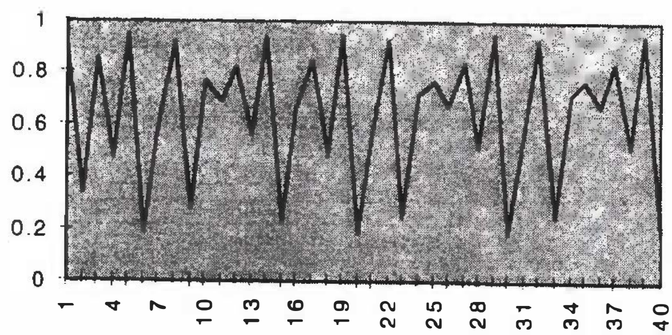



Extending the number of cycles gives:




Case 5: $c = 3.8$

| | | | |
|---|---|---|---|
| **296** | 295 | 0.63 | 0.89 |
| **297** | 296 | 0.89 | 0.37 |
| **298** | 297 | 0.37 | 0.89 |
| **299** | 298 | 0.89 | 0.38 |
| **300** | 299 | 0.38 | 0.89 |
| **301** | 300 | 0.89 | 0.36 |
| **302** | 301 | 0.36 | 0.87 |
| **303** | 302 | 0.87 | 0.42 |
| **304** | 303 | 0.42 | 0.93 |
| **305** | 304 | 0.93 | 0.26 |
| **306** | 305 | 0.26 | 0.69 |



There is no noticeable pattern! This orbit is called chaotic.

A review of this last example has illustrated how a seemingly simple idea, repeated iteration of a relatively simple equation, can give rise to genuine unpredictability. It also introduces a new branch of mathematics, chaos theory.

The above examples illustrate how existing software (programming languages and spreadsheets) can be used to explore new topics in mathematics, topics that are only accessible with computer support.

As we moved from the pencil to the calculator, there was spirited debate about the potential implications. Much of this discussion was predicated on assumptions about how people learn and, in particular, about how they learn mathematical ideas. It is sobering to realize that after 50 years of research, we still have only a vague sense of how people learn or of the conditions that support such learning.

It is also sobering to read about the many crosscultural studies of students' mathematical performance and of the relative standing of most North American students in these studies. Such findings do not support a position of complacency. There is a growing realization that we need to emphasize a deeper understanding of the ideas and concepts that constitute mathematics. However, agreeing on the destination and getting there are different things.

In conclusion, I would like to emphasize three points. First, this article has suggested that we should consider some new destinations for mathematics education. I would like to see new topics inserted into the curriculum that reflect the enthusiasm of practising mathematicians. Second, in considering pedagogical issues, the mathematics curriculum of the near future needs to shift from its dominant behaviorist position with its emphasis on practice, to a more cognitively oriented position that compares, interprets and discusses different mathematical situations. Finally, we must learn to incorporate new tools, such as computers, into the curriculum—tools that permit us to journey further into mathematical domains.

The self is a repeatedly reconstructed biological state. . . .

—*Antonio R. Damasio* (1994, 227)

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# An Intuitive Meaning for the Number *e*

*Murray L. Lauber*

Two irrational numbers that crop up continually in mathematics are $\pi$ and $e$. It is easy to attach an intuitive meaning to $\pi$: the ratio of the circumference of a circle to its diameter. But in the minds of the majority of students, the number $e$ has no such meaning. And yet, as is the case for many mathematical concepts, attaching an intuitive meaning to $e$ might enrich students' conceptual frameworks and enable them to make conjectures about how expressions involving $e$ should behave.

## The Number *e* in Terms of Continuously Compounding Interest

One such meaning for $e$ grows out of continuously compounding interest. It is easiest to make the connection with $e$ if one begins with the problem of calculating the amount after 1 year of $1 invested at 100 percent per year compounded annually. One can then consider how the amount changes when the interest is compounded quarterly, monthly, weekly, daily and hourly. The logical extension is the case of compounding continuously. Using the formula $A=P(1+i)^n$ where $P$ represents the principle, $n$ the number of compounding periods, $i$ the interest rate per compounding period and $A$ the amount, one may compute the amounts as in the following chart. In this problem, $P=1$. The values of $n$ and $i$ depend on how often the interest is compounded each year. For compounding annually, $n=1$ and $i=1$ (100 percent); for compounding quarterly, $n=4$ and $i=¼=0.25$ (corresponding to 25 percent per quarter for 4 quarters); for compounding monthly, $n=12$ and $i=1/12$; and so on. A calculator can be used to calculate the decimal approximations in the last column, although the limits of the calculator become apparent as one proceeds to shorter compounding periods. For example, the value of $A$ in the last row and column is suspect.

| Compounding Period | Number of Periods, $n$ | Interest per Period, $i$ | Amount $A$ after 1 year | Decimal Approximation of $A$ (Using Calculator) |
|---|---|---|---|---|
| Yearly | $n=1$ | $i=1$ | $(1+1)^1$ | 2.0000000 |
| Quarterly | $n=4$ | $i=¼$ | $(1+¼)^4$ | 2.4414063 |
| Monthly | $n=12$ | $i=1/12$ | $(1+1/12)^{12}$ | 2.6130353 |
| Weekly | $n=52$ | $i=1/52$ | $(1+1/52)^{52}$ | 2.6925969 |
| Daily | $n=365$ | $i=1/365$ | $(1+1/365)^{365}$ | 2.7145677 |
| Hourly | $n=8760$ | $i=1/8760$ | $(1+1/8760)^{8760}$ | 2.7181209 |
| By the Minute | $n=525600$ | $i=1/525600$ | $(1+1/525600)^{525600}$ | 2.7180100[1] |

So, in general, the amount after 1 year of $1 at 100 percent per year compounded $n$ times in the year is $\left(1+\frac{1}{n}\right)^n$. Further, the amount after 1 year of $1 at 100 percent per year compounded continuously should be the limit of this expression as $n$ becomes arbitrarily large, that is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. One of the definitions for $e$ is precisely this limit. Thus we have an intuitive meaning for $e$ in terms of continuously compounding interest: the amount after 1 year of 1 unit of currency compounded continuously at 100 percent per year.

## Using this Concept of *e* to Make Conjectures About Limits

One can extend this reasoning to express the amounts in more general compound interest problems in terms of $e$. Changing the amount of the principle $P$ while keeping other factors constant changes the amount $A$ by the same proportion as $P$ is changed. For a $1 principle, let's consider the amount after several years.

### Example 1

Consider the case where we wish to determine the amount after 2 years of $1 invested at 100 percent per year compounded hourly, and then compounded continuously.

***Solution***

We first use an iterative process to determine the amount with hourly compounding. The number of compounding periods or hours per year is $n$=8760 and the interest rate per hour is $i$=1/8760. The amount after 1 year is $\left(1+\frac{1}{8760}\right)^{8760}$. Using this as the principle for the second year of investment and substituting in the formula $A=P(1+i)^n$ with $n$=8760 (the number of hours in the second year) and $i=\frac{1}{8760}$, we obtain

$$A=\left(1+\frac{1}{8760}\right)^{8760}\left(1+\frac{1}{8760}\right)^{8760}=\left(1+\frac{1}{8760}\right)^{17520}\approx 7.3881769.$$

Alternatively, we could use the interest rate of $i=\frac{1}{8760}$ per hour and the fact that there are 17,520 hours in 2 years to obtain $A=\left(1+\frac{1}{8760}\right)^{17520}$ directly.

Using an iterative process to determine the amount with continuous compounding, we first note that the amount after 1 year is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Using this as the principle for the second year of investment, we obtain

$$A=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2=$$

$$\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^2=e^2\approx 7.3890561$$

Alternatively, we could use an interest rate of $i=\frac{1}{n}$ where there are $n$ compounding periods per year, or $2n$ compounding periods altogether to obtain $A=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2$ directly.

Consideration of problems with differing yearly interest rates can prompt conjectures about related limits. This is illustrated in the next example. Here a conjecture is made employing the notion of continuously compounding interest and then proved using l'Hospital's Rule.

## Example 2

Consider the case where we wish to determine the amount after 1 year of $1 invested at 200 percent per year compounded hourly, and then compounded continuously.

***Solution***

First consider the case of compounding hourly. The number of compounding periods (hours) in the year is $n$=8760 and the interest rate per compounding period is $i=\frac{2}{8760}$. Thus we obtain

$$A=\left(1+\frac{2}{8760}\right)^{8760} \approx 7.3873531$$

Note that the decimal approximation here is close to but not the same as the decimal value in Example 1 for the amount after 2 years of $1 invested at 100 percent/year compounded hourly.

The amount for continuous compounding is then given by $A=\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$. Intuitively, compounding continuously at 200 percent per year for 1 year should be the same as compounding continuously at 100 percent per year for 2 years (since the interest rate for the total period is 200 percent in both cases). Thus the amount for continuous compounding should be the same in examples 1 and 2. This leads to the conjecture that

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2 \text{ or } e^2.$$

This can be proven using l'Hospital's Rule as follows.

Let $y = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

$= e^{\ln\left[\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right]}$

$= e^{\lim_{n\to\infty}\left[\ln\left(1+\frac{2}{n}\right)^n\right]}$

Let $x = \lim_{n\to\infty}\left[\ln\left(\frac{n+2}{n}\right)^n\right]$

Then $y = e^x$

$x = \lim_{n\to\infty}\left[\ln\left(\frac{n+2}{n}\right)^n\right]$

$= \lim_{n\to\infty}\left[n\ln\left(\frac{n+2}{n}\right)\right]$

$= \lim_{n\to\infty}\frac{\ln\left((n+2)/n\right)}{1/n}$

Because both numerator and denominator of this limit approach 0, we may apply l'Hospital's Rule. However, it is convenient to modify the numerator before differentiating the numerator and denominator.

$x = \lim_{n\to\infty}\frac{\ln(n+2)-\ln n}{1/n}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+2} - \frac{1}{n}}{-1/n^2} \qquad \text{l'Hospital's Rule}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n(n+2)} = 2$$

Then y $= e^x = e^2$. Q.E.D.

One can make conjectures about limits similar to those in Examples 1 and 2 by thinking of them in terms of continuously compounding interest. The following examples lead to conjectures that can be proved using l'Hospital's Rule by a process analogous to that used in Example 2.

## Example 3

Find $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n$

This is the amount of $1 after 1 year compounded continuously at 300 percent per year. It should equal $\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^3$, the amount after 3 years of $1 compounded continuously at 100 percent per year. This leads to the conjecture that $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^3$.

## Example 4

Find $\lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{2n}$

This is the amount of $1 compounded continuously at 500 percent per year for 2 years. It should equal $\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{10}$, the amount after 10 years of $1 compounded continuously at 100 percent per year. This leads to the conjecture that $\lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{2n} = e^{10}$.

## Example 5

Find $\lim_{n\to\infty} \left(\frac{2n+5}{2n+1}\right)^{4n}$

It may not be immediately apparent how to make a direct association between this limit and continuously compounding interest, but by changing its form and making a substitution we may do so. First we change its form:

$$\lim_{n\to\infty} \left(\frac{2n+5}{2n+1}\right)^{4n} = \lim_{n\to\infty} \left(\left(1+\frac{4}{2n+1}\right)^{2n}\right)^2.$$

Now let $m = 2n + 1$ or $2n = m - 1$.

Then as $n \to \infty$, $2n + 1 \to \infty$ and $2n \to \infty$, that is $m \to \infty$ and $m - 1 \to \infty$. Thus, the above limit should be equivalent to

$$\lim_{m\to\infty} \left(\left(1+\frac{4}{m}\right)^{m-1}\right)^2$$

$$= \lim_{m\to\infty} \left(\left(1+\frac{4}{m}\right)^{m}\right)^2$$

This should equal $(e^4)^2 = e^8$, which leads to the conjecture that

$$\lim_{n\to\infty} \left(\frac{2n+5}{2n+1}\right)^{4n} = e^8.$$

Proofs of the conjectures in Examples 3, 4 and 5 are left to the reader. Conjectures about similar limits can often be made easily by a quick inspection and thus serve as an efficient check for finding their limits by the more laborious methods that employ l'Hospital's Rule.

## Example 6 (A More General Limit)

Make a conjecture about the value of $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$ and prove it using l'Hospital's Rule.

### *Solution*

First, the formation of the conjecture $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} \left(\left(1+\frac{a}{x}\right)^{x}\right)^b$

This should equal $\lim_{x\to\infty} \left(\left[\left(1+\frac{1}{x}\right)^{x}\right]^a\right)^b$

$= (e^a)^b$ or $e^{ab}$

Proof by l'Hospital's Rule:

Let $y = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$

Then $y = e^{\ln\left[\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}\right]}$

$$= e^{\lim_{x\to\infty}\left[\ln \left(1+\frac{a}{x}\right)^{bx}\right]}$$

Let $u = \lim_{x\to\infty}\left[\ln \left(1+\frac{a}{x}\right)^{bx}\right] = \lim_{x\to\infty}\left[\ln \left(\frac{x+a}{x}\right)^{bx}\right]$

Then $y = e^u$. We proceed first to find $u$, using the methods employed in Example 2.

$$u = \lim_{x \to \infty} \left[ bx \left( \ln \frac{x+9}{x} \right) \right]$$

$$= b \lim_{x \to \infty} \left[ \frac{\ln \left( \frac{x+a}{xx} \right)}{1/x} \right]$$

The numerator and denominator of this limit both approach 0, so we may apply l'Hospital's Rule. As in Example 2, we change the form of the numerator before differentiating the numerator and denominator.

$$u = b \lim_{x \to \infty} \left[ \frac{\ln(x+a) - \ln x}{1/x} \right]$$

$$= b \lim_{x \to \infty} \left[ \frac{\frac{1}{x+a} - \frac{1}{x}}{-\frac{1}{x^2}} \right]$$

$$= b \lim_{x \to a} \frac{a\ x^2}{(x+a)(x)}$$

$= ba$

Thus $y = e^u = e^{ba}$. Q.E.D.

## Continuously Compounding Interest and Exponential Growth

Now let us return to a more general problem in continuously compounding interest. Suppose that $P is invested at a rate of $k$ (or $100k$ percent) per year. With continuous compounding, what should be the amount after $t$ years?

We can find the amount here with the aid of the formula $A = P\ (1 + i)^n$ and the results of Example 6. Assuming $n$ compounding periods per year, or $nt$ compounding periods altogether, and noting that the interest rate per compounding period is then $\frac{k}{n}$, the amount after $t$ years is given by

$$A = P \left( 1 + \frac{k}{n} \right)^{nt}$$

The amount after $t$ years with continuous compounding is given by

$$A = \lim_{n \to \infty} P \left( 1 + \frac{k}{n} \right)^{nt}$$

$$= P \lim_{n \to \infty} \left( 1 + \frac{k}{n} \right)^{nt}$$

From the results of Example 6, this equals $Pe^{kt}$.

Thus we have $A = Pe^{kt}$. This is the formula for exponential growth encountered by students first at the high school level. The results should not be surprising because continuous compounding and exponential growth are two descriptions for the same phenomenon.

## Conclusion

Perhaps the attachment of an intuitive meaning to *e* is illustrative of what can happen with many mathematical concepts. Relating *e* to continuously compounding interest can, on the one hand, serve as an efficient means for making conjectures about limits—conjectures which may then be subjected to more rigorous inspection using tools such as l'Hospital's Rule. Or, conversely, the conceptual framework for making conjectures provided by this meaning for *e* can be used as an efficient check for results found by methods that, despite the advantages of their rigor, may be fraught with opportunities for errors.

As with *e*, so with many mathematical concepts. Although it may not always be possible or desirable to attach intuitive meanings to mathematical concepts, often attaching an intuitive meaning, or several intuitive meanings, to a mathematical concept, enriches one's understanding of the concept. It enables one, on one hand, to make conjectures that one would not be able to make otherwise and, on the other hand, to detect errors in results arrived at by more formal arguments.

## Note

1. There is a significant rounding error in this figure due to the limitations of the calculator in storing the number 1 + 1/525600.

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# Mathematics as Problem Solving—A Japanese Way

*Daiyo Sawada*

We know from many international studies that Japanese students (and Asian students in general) do quite well in mathematics on knowledge-level questions and on deeper conceptual and problem-solving items (Stevenson and Stigler 1992; Stedman 1994 and many others). A question arises: How are mathematical concepts and problem solving approached in Japanese elementary schools?

My response to this question was largely formed during a research visit to the University of Michigan in Ann Arbor at the invitation of Harold W. Stevenson during spring/summer 1993.[1] His project team was in the midst of analyzing voluminous data gathered from classrooms in Taipei, Beijing, Sendai and Chicago. Dr. Stevenson kindly made it possible for me to immerse myself in a set of classroom observations of 160 lessons taught by 40 teachers of Grades 1 and 5 in Sendai, Japan.

To share with you what I found, I do not want to provide a "once over lightly" treatment of a few general characteristics of problem solving in Japanese elementary schools. Rather I have selected a typical Grade 5 lesson taught by a typical Grade 5 teacher with typically 40 students in her classroom. This teacher does not have many striking characteristics that would mark her as exceptional among the set of 40 teachers included in the data set. I would describe her as competent, careful and caring. That description covers most of the other teachers as well. I present the lesson in some detail to provide readers with a feel for the problem. I then identify and comment on some aspects that were particularly striking to me. I close the article with a question and a partial answer to the main question. [Note: Parenthetical material is enclosed in square brackets.]

## Lesson Description (Grade 5)

The teacher, Ms. Sato (a pseudonym), introduces the lesson with a discussion of "crowdedness" (a notion quite meaningful for people of Japan). After pointing out that one cannot say that a place is crowded simply because it has a lot of people, she focuses discussion on a particular comparison: "Which is more crowded, Hokkaido or Okinawa?" She reminds the class of problems they have already worked on in previous classes and raises one of them again: "Which tatami room is more crowded, the one that measures 10 mats and has 1 person, or the room that measures 10 mats and has 10 persons?" [Note: The area of a tatami room is measured in terms of the number of tatami mats it takes to cover the floor. The floor space of a tatami room is always designed as a particular tessellation of tatami mats. A tatami mat itself is 1 m by 2 m.] A student volunteers a response: "The second room." Ms. Sato agrees and explains that this problem is easy because the rooms are the same size: "But in our case, Hokkaido and Okinawa have different areas." She then asks if the two areas can be compared. A student suggests figuring out how many times bigger Hokkaido is than Okinawa. The teacher explains that this method is similar to the one used in the "quantity of salt experiment" they worked on earlier, and that this too can be a good method.

Another student suggests that one can "turn over a certain amount of area from Hokkaido to Okinawa to make them the same size." Other students object, suggesting that this method is too cumbersome to use here. Another student volunteers, "Like what we did with the tatami room problem, we should try to figure out crowdedness for the same amount of area first" (instead of messing around with exchanging area, as in the previous suggestion). Ms. Sato takes up this suggestion asking, "What then shall be done with the tatami room case?" A student responds, "Compare number of persons per mat." The teacher writes this suggestion on the board while repeating it orally.

Ms. Sato recalls the figures from the tatami mat problem from a previous lesson (Room A—9 persons in a 6-mat room; Room B—15 persons in a 15-mat room) and calculates the number of persons per mat for each tatami room. She then asks, "Which room is more crowded?" She confirms the answer by drawing two diagrams on the board showing one and one-half bodies per mat for Room A and one body per mat for Room B. She summarizes saying that we calculated "quantity per unit."

Turning to the Hokkaido-Okinawa problem, she asks what unit to use. A student responds with "one square kilometre." Accepting this, the teacher points out again that they can base the calculation on how many persons per square kilometre. She then summarizes with "population divided by area = population per square kilometre" and writes this on the board. Students copy this statement into their notebooks. The teacher explains further that by dividing population by area, one figures out how many live in the same amount of space.

In preparation for the calculations, the teacher reminds students about the number of significant digits to keep when rounding. A student responds, "Round off to significant digits when doing calculations" and explains that the numbers given in the problem are already rounded off to two significant digits, and the teacher confirms. She tells students to solve the problem starting with Hokkaido and helps by having students identify the rounded numbers they should use and writes them on the board. Students work on the calculations at their desks. Ms. Sato circulates. She then asks for an answer and gets "70.8" from a student. She tells him to round it off and then states that Hokkaido has about 71 people per square kilometre.

Students then calculate using the numbers for Okinawa, and the teacher again reminds them to use two significant digits when calculating and not to round off until after they find the answer. A student volunteers "478," and Ms. Sato asks, "Rounding this number to two significant digits we have?" A student responds with "480." The teacher then asks which island is more crowded, receives the answer that Okinawa is more crowded and confirms this by reiterating that Hokkaido has 71 people per square kilometre whereas Okinawa has 480 per square kilometre.

Ms. Sato directs students to look at page 79 in the textbook and asks, "What do we call this "crowdedness" that we are trying to figure out?" In a choral manner, students respond, "Population density." The teacher confirms, saying, "We have just learned to figure out population density today."

At this point, 35 minutes have elapsed. During the remaining 7 minutes, students do two more problems involving population density. [Two days later, students were still working on density problems but with some variation: the opening problem involved iron and silver—250 cc of iron weighing 1,975 g; 350 cc of silver weighing 3,675 g. This problem took 25 minutes to solve and discuss. The next day, the concept of average was developed as a particular case of density. The problem consisted of comparing the output per factory of two kinds of production where the procedures for handling density generated a quantity which could be called the "average production" (per factory).]

## Looking Back

On reflection, these points seem worthy of notice:

- A deliberate effort was made to connect the present problem to *previous problems* and solutions.
- *Simpler problems* were used.
- Teacher functioned as a *guide with a definite agenda*.
- Crowdedness provided a meaningful context for *embedding the concept* of population density.
- In turn, the concept of density provided a context for embedding the concept of average.
- More than 35 minutes were spent working on *one problem*.
- *Student contributions* were used to determine the content of the lesson as well as its flow.
- The notion of rounding and *significant digits* was used consistently when doing the computations.
- The problem was rich in *social studies* content.
- *Formulaic statements* did not enter the lesson until they could be used as summary confirmations.
- The lesson was not followed up by assigning several *application exercises*. Instead, two related problems were discussed and solved, each widening the interpretation of the idea of density.

## Commentary

The comments to follow also reflect my study of the 159 other lessons taught by Sendai teachers in the data set.

### Embedding

Problem solving occurs *pervasively* in this lesson. Problem-solving techniques are used not only in solving the problem about density but also in the approach to the lesson structure itself. I have used the term *embedded* to describe this more pervasive use of problem solving in teaching mathematics in which problem solving is the *medium* for mathematics teaching and learning. In the lesson described, there is as much emphasis on the medium (problem solving process) as there is on the message (the particular mathematical content to be learned). This emphasis on the medium in many Grade 5 lessons in Sendai often takes precedence over the message so that instructional decisions (how to respond to mathematical "errors," for example) are made with the health of the medium as much in mind as the need to assess the correctness of the message. (For example, a so-called "error" is often discussed at length and thereby contributes to the diversity of the "solution space" and therefore to the health of the medium, and thus is valued even if not particularly "correct" as a message.) Indeed, such contributions, because they do enrich the solution space are actively sought by teachers through such simple leads as, "Any other methods?" None of the responses to such questions are rejected at the outset, and this decision not to reject indicates that the health of the medium is at least as important as the correctness of the message. This priority on the medium occurred in the lesson in many particular as well as general ways.

From a mathematical viewpoint,

- crowdedness was used as a medium for (to embed) density;
- density was used as a medium for rate/ratio; and
- in its turn, rate/ratio was used as a medium for average.

From a pedagogical viewpoint,

- the general problem situation was a medium for the structure of the lesson;
- the particular problem was a medium for the mathematical concept; and
- the multiple solutions generated by students were a medium for the "correct" solution.

From a learning viewpoint,

- the process of solving the problem was a medium for learning of mathematics;
- right or wrong, student answers de facto were the medium for exploring the solution space; and
- generating and discussing solutions was the medium for arriving at and critiquing answers.

## Spending 35 Minutes or More on One Problem

Spending nearly the whole period on one problem might seem an inefficient and drawn-out way to teach a mathematical concept. On the other hand, if only four or five minutes were spent on a nonroutine problem, I would have grave doubts that a problem-solving approach was being used at all. For a rich problem-solving process to occur, there needs to be time for investigating the nature of the problem, generating and proposing multiple solutions, discussing and critiquing various types of solutions, assessing and comparing the relative merits of each solution within the set of solutions, reflecting on what one has learned and what else could be learned. This sounds like a lot of rhetoric, but it happens regularly in most Grade 5 classes in Sendai (statistically speaking, 86 percent of Sendai Grade 5 teachers use problem solving to embed mathematical concepts, while the comparable figure for Chicago elementary schools is 14 percent), and it means that the whole period will often be spent on only one problem.

## Acting Out the Problem

Although the students in the lesson described did not act out the solution process, in other Sendai classrooms this happened quite often. For example, the teacher would bring cushions to simulate tatami mats, children would select which one(s) they would prefer to sit on and a problem of density would be enacted. After this, rooms with so many mats would be drawn on the board with students selecting which room they preferred, and the density problem thus enacted would be ready to be solved.

## Multiple Solutions

Multiple solutions are the "rice and fish" of the problem-solving approach used in Sendai. If only one solution exists, it would have to be the correct one so nothing would function as, nor need to function as, a medium for anything else. The correct solution would be demonstrated or illustrated for all to learn. Multiple solutions, and establishing the conditions for generating, articulating, understanding, comparing and critiquing multiple solutions are required components to the way problem solving happens in Sendai classrooms.

## Interesting Problems to Solve

While the problem of the relative population density of Hokkaido and Okinawa may be of interest to Japanese students, as a generalization, most problems are rather mundane. Many of them remind me of typical word problems occurring by the hundreds in Canadian texts. For the classrooms in Sendai, the problems per se are not different; the way these problems become the medium of instruction and of learning is contrastive.

## Use of Manipulative Aids

The NCTM publication *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan* (Stevenson et al. 1990) reports a surprising finding: When manipulatives are introduced into American lessons, the amount of talk decreases, while in Japanese elementary classrooms it increases. This finding may have been surprising, but, when problem solving is the pervasive mode of teaching and learning, introducing manipulative aids contributes immensely to the variety of multiple solutions to be generated, assessed, compared and so on, thus providing many more opportunities for talk.

## Interpretation Rather Than Application

Currently in North America, we talk about developing problem-solving strategies and skills and then applying them. In this sense, problem solving is split into two rather distinct parts:

1. The learning of the concepts and skills (perhaps through problem solving)
2. The use or application of these concepts and skills in similar situations

This second part is often taken to be the full extent of problem solving. It is a matter of applying concepts learned, not for learning concepts and skills. Our textbooks are organized this way: concepts are taught (often by demonstration or explanation, as well as problem solving), and then students are given a collection of similar "problems" to do. Because of this practice, lessons in North America are likely to end with students working at their desks. The contrast in Sendai is striking. Classes often end with discussion. And when children are working at their desks at the end of the class, they are not only applying the concepts just learned but also *interpreting* problem situations that extend the ideas beyond the initial circumstances. In the lesson presented, the problem-solving approach is not two parts but just one. Problem solving becomes two parts when the concepts learned (the messages) become so important that they need to be separated and dealt with differently (as applications). On the other hand, if we keep the problem-solving process intact and pervasive, the messages learned will never dominate the medium that created them.

## Conclusion

Several other aspects about problem solving in Sendai deserve mention, such as the deliberate making of errors (by the teacher) or the occurrence of memorization, but I think the important points have been made. I close this article with a question and a partial answer. Why has the problem-solving approach as medium taken hold so pervasively in Japanese elementary schools but remains largely problematic in North American schools despite a decade in which nothing in mathematics education received more attention than problem solving? While not wishing to oversimplify, I should like to suggest that, in Japan, accepting ways of doing things as being as important as the things themselves is a familiar and comfortable stance. We see it in martial arts, flower arranging, the tea ceremony, pottery making and in teaching, too. Problem solving as a medium for teaching mathematics is another of these important ways.

## Note

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# Principles for Fair Student Assessment Practices for Education in Canada

*Principles for Fair Student Assessment Practices for Education in Canada* contains a set of principles and related guidelines generally accepted by professional organizations as indicative of fair assessment practice within the Canadian educational context. Assessments depend on professional judgment; the principles and related guidelines presented in this document identify the issues to consider in exercising this professional judgment and in striving for fair and equitable assessment of all students.

Assessment is broadly defined in the *Principles* as the process of collecting and interpreting information that can be used

- to inform students, and their parents/guardians where applicable, about the progress they are making toward attaining the knowledge, skills, attitudes and behaviors to be learned or acquired and
- to inform the various personnel who make educational decisions (instructional, diagnostic, placement, promotion, graduation, curriculum planning, program development, policy) about students.

Principles and related guidelines are set out for developers and users of assessments. Developers include people who construct assessment methods and people who set policies for particular assessment programs. Users include people who select and administer assessment methods, commission assessment development services or make decisions on the basis of assessment results and findings. The roles may overlap, as when a teacher or instructor develops and administers an assessment instrument and then scores and interprets the students' responses, or when a ministry or department of education or local school system commissions the development and implementation of an assessment program and scoring services and makes decisions on the basis of the assessment results.

The *Principles* is the product of a comprehensive effort to reach consensus on what constitutes sound principles to guide the fair assessment of students. The principles and their related guidelines should be considered neither exhaustive nor mandatory; however, organizations, institutions and professionals who endorse them are committing themselves to endeavor to follow their intent and spirit so as to achieve fair and equitable assessments of students.

### Organization and Use of the Principles

The principles and their related guidelines are organized in two parts. Part A is directed at assessments carried out by teachers at the elementary and secondary school levels. It is also applicable at the postsecondary level with some modifications, particularly with respect to whom assessment results are reported. Part B is directed at standardized assessments developed external to the classroom by commercial test publishers, provincial and territorial ministries, departments of education and local school jurisdictions (boards, boroughs, counties and school districts).

Five general principles of fair assessment practices are provided in each part. Each principle is followed by a series of guidelines for practice. In Part A, where no prior sets of standards for fair practice exist, a brief comment accompanies each guideline to help clarify and illuminate the guideline and its application.

The Joint Advisory Committee recognizes that in the field of assessment some terms are defined or used differently by different groups of people. To maintain as much consistency in terminology as possible, an attempt has been made to employ generic terms in the *Principles*.

## A. Classroom Assessments

Part A is directed toward the development and selection of assessment methods and their use in the classroom by teachers. Based on the conceptual framework provided in the *Standards for Teacher Competence in Educational Assessment of Students* (American Federation of Teachers 1990), it is organized around five interrelated themes:

1. Developing and choosing methods for assessment
2. Collecting assessment information
3. Judging and scoring student performance
4. Summarizing and interpreting results
5. Reporting assessment findings

The Joint Advisory Committee acknowledges that not all guidelines are equally applicable in all circumstances. However, consideration of the full set of principles and guidelines within Part A should help to achieve fairness and equity for the students to be assessed.

## Developing and Choosing Methods for Assessment

*Assessment methods should be appropriate for and compatible with the purpose and context of the assessment.*

*Assessment method* is used here to refer to the various strategies and techniques teachers might use to acquire assessment information. These strategies and techniques include, but are not limited to, observations, text- and curriculum-embedded questions and tests, paper-and-pencil tests, oral questioning, benchmarks or reference sets, interviews, peer- and self-assessments, standardized criterion referenced and norm-referenced tests, performance assessments, writing samples, exhibitions, portfolio assessment, and project and product assessments. Several labels have been used to describe subsets of these alternatives, with the most common being direct assessment, authentic assessment, performance assessment and alternative assessment. However, for the purpose of the *Principles*, the term *assessment method* has been used to encompass all the strategies and techniques that might be used to collect information from students about their progress toward attaining the knowledge, skills, attitudes or behaviors to be learned.

➤1. Assessment methods should be developed or chosen so inferences drawn about the knowledge, skills, attitudes and behaviors possessed by each student are valid and not open to misinterpretation.

Validity refers to the degree to which inferences drawn from assessment results are meaningful. Therefore, development or selection of assessment methods for collecting information should be clearly linked to the purposes for which inferences and decisions are to be made. For example, to monitor the progress of students as proofreaders and editors of their own work, it is better to assign an actual writing task, to allow time and resources for editing (dictionaries, handbooks and so on) and to observe students for evidence of proofreading and editing skill as they work than to use a test containing discrete items on usage and grammar that are relatively devoid of context.

➤2. Assessment methods should be clearly related to the goals and objectives of instruction and be compatible with the instructional approaches used.

To enhance validity, assessment methods should be in harmony with the instructional objectives to which they are referenced. Planning an assessment design at the same time as planning instruction will help integrate the two in meaningful ways. Such joint planning provides an overall perspective on the knowledge, skills, attitudes and behaviors to be learned and assessed, and the contexts in which they will be learned and assessed.

➤3. When developing or choosing assessment methods, consideration should be given to the consequences of the decisions to be made in light of the obtained information.

Some assessment outcomes may be more critical than others. For example, misinterpretation of the level of performance on an end-of-unit test may result in incorrectly holding a student from proceeding to the next instructional unit in a continuous progress situation. In such "high-stake" situations, every effort should be made to ensure the assessment method will yield consistent and valid results. "Low-stake" situations, such as determining if a student has correctly completed an in-class assignment, can be less stringent. Low-stake assessments are often repeated during the course of a reporting period using a variety of methods. If the results are aggregated to form a summary comment or grade, the summary will have greater consistency and validity than its component elements.

➤4. More than one assessment method should be used to ensure comprehensive and consistent indications of student performance.

To obtain a more complete picture or profile of a student's knowledge, skills, attitudes or behaviors, and to discern consistent patterns and trends, more than one assessment method should be used. Student knowledge might be assessed using completion items. Process or reasoning skills might be assessed by observing performance on a relevant task. Evaluation skills might be assessed by reflecting on the discussion with a student about what materials to include in a portfolio. Self-assessment may help to clarify and add meaning to the assessment of a written communication, science project, piece of art work or an attitude. Use of more than one method will also minimize inconsistency brought about by different sources of measurement error (for example, poor performance because of an "off-day"; lack of agreement among items included in a test, rating scale or questionnaire; lack of agreement among observers; instability across time).

➤5. Assessment methods should be suited to the backgrounds and prior experiences of students.

Assessment methods should be free from bias brought about by student factors extraneous to the purpose of the assessment. Possible factors to consider include culture, developmental stage, ethnicity, gender, socioeconomic background, language, special interests and special needs. Students' success in answering questions on a test or in an oral quiz, for example, should not depend on prior cultural knowledge, such as understanding an allusion to a cultural tradition or value, unless such knowledge falls within the content domain being assessed. All students should be given the same opportunity to display their strengths.

➢6. Content and language that would generally be viewed as insensitive, sexist or offensive should be avoided.

The vocabulary and problem situation in each test item or performance task should not favor or discriminate against any group of students. Steps should be taken to ensure that stereotyping is not condoned. Language that might be offensive to particular groups of students should be avoided. A judicious use of different roles for males, females and minorities and the careful use of language should contribute to more effective and fairer assessments.

➢7. Assessment instruments translated into a second language or transferred from another context or location should be accompanied by evidence that inferences based on these instruments are valid for the intended purpose.

Translation of an assessment instrument from one language to another is a complex and demanding task. Similarly, adopting or modifying an instrument developed in another country is often not simple and straightforward. Care must be taken to ensure that the results from translated and imported instruments are not misinterpreted or misleading.

## Collecting Assessment Information

*Students should be provided with a sufficient opportunity to demonstrate the knowledge, skills, attitudes or behaviors being assessed.*

Assessment information can be collected in various ways (observations, oral questioning, interviews, oral and written reports, paper-and-pencil tests). The guidelines that follow are not all equally applicable to each of these procedures.

➢1. Students should be told why assessment information is being collected and how this information will be used.

Students who know the purpose of an assessment are in a position to respond in a manner that will provide information relevant to that purpose. For example, if students know that their participation in a group activity is to be used to assess cooperative skills, they can be encouraged to contribute to the activity. If students know the purpose of an assessment is to diagnose strengths and weaknesses rather than to assign a grade, they can be encouraged to reveal weaknesses as well as strengths. If students know the purpose is to assign a grade, they are well advised to respond in a way that will maximize strength. This is especially true for assessment methods that allow students to make choices, such as with optional writing assignments or research projects.

➢2. An assessment procedure should be used under conditions suitable to its purpose and form.

Optimum conditions should be provided for obtaining data from and information about students to maximize validity and consistency. Common conditions include such things as proper light and ventilation, comfortable room temperature and freedom from distraction (for example, movement in and out of the room, noise). Adequate work space, sufficient materials and adequate time limits appropriate to the purpose and form of the assessment are also necessary. For example, if the intent is to assess student participation in a small group, adequate work space should be provided for each student group, with sufficient space between subgroups so the groups do not interfere with or otherwise influence one another. This gives the teacher the same opportunity to observe and assess each student within each group.

➢3. In assessments involving observations, checklists or rating scales, the number of characteristics to be assessed at one time should be small enough and concretely described so that the observations can be made accurately.

Student behaviors often change so rapidly that it may not be possible simultaneously to observe and record all the behavior components. In such instances, the number of components to be observed should be reduced and the components should be described as concretely as possible. One way to manage an observation is to divide the behavior into a series of components and assess each component in sequence. By limiting the number of components assessed at one time, the data and information become more focused, and time is not spent observing later behavior until prerequisite behaviors are achieved.

➢4. The directions provided to students should be clear, complete and appropriate for their ability, age and grade level.

Lack of understanding of the assessment task may prevent maximum performance or display of the behavior called for. In the case of timed assessments, for example, teachers should describe the time limits, explain how students might distribute their time among parts for those assessment instruments with parts and describe how students should record their responses. For a portfolio assessment, teachers should describe criteria to be used to select materials to be included, who will select these materials and, if more than one person will be involved in the selection process, how judgments will be combined. Where appropriate, sample material and practice should be provided to increase the likelihood that instructions will be understood.

➤5. In assessments involving selection items (for example, true-false, multiple-choice), the directions should encourage students to answer all items without threat of penalty.

A correction formula is sometimes used to discourage guessing on selection items. The formula is intended to encourage students to omit items for which they do not know the answer rather than to guess the answer. Because research evidence indicates the benefits expected from the correction are not realized, use of the formula is discouraged. Students should be encouraged to use whatever partial knowledge they have when choosing their answers and to answer all items.

➤6. When collecting assessment information, interactions with students should be appropriate and consistent.

Care must be taken when collecting assessment information to treat all students fairly. For example, when oral presentations by students are assessed, questioning and probes should be distributed among the students so all students have the same opportunity to demonstrate their knowledge. While writing a paper-and-pencil test, a student may ask to have an ambiguous item clarified, and, if warranted, the item should be explained to the entire class.

➤7. Unanticipated circumstances that interfere with the collection of assessment information should be noted and recorded.

Events such as a fire drill, an unscheduled assembly or insufficient materials may interfere in the way in which assessment information is collected. Such events should be recorded and subsequently considered when interpreting the information obtained.

➤8. A written policy should guide decisions about the use of alternative procedures for collecting assessment information from students with special needs and students whose proficiency in the language of instruction is inadequate for them to respond in the anticipated manner.

It may be necessary to develop alternative assessment procedures to ensure a consistent and valid assessment of those students who, because of special needs or inadequate language, are not able to respond to an assessment method (for example, oral instead of written format, individual instead of group-administered, translation into first language, providing additional time). Use of alternative procedures should be guided by a written policy developed by teachers, administrators and other jurisdictional personnel.

## Judging and Scoring Student Performance

*Procedures for judging and scoring student performance should be appropriate for the assessment method used and be consistently applied and monitored.*

Judging and scoring refers to the process of determining the quality of a student's performance, the appropriateness of an attitude or behavior or the correctness of an answer. Results derived from judging and scoring may be expressed as written or oral comments, ratings, categorizations, letters, numbers or as some combination of these forms.

➤1. Before an assessment method is used, a procedure for scoring should be prepared to guide the process of judging the quality of a performance or product, the appropriateness of an attitude or behavior or the correctness of an answer.

To increase consistency and validity, properly developed scoring procedures should be used. Different assessment methods require different forms of scoring. Scoring selection items (true-false, multiple-choice, matching) requires the identification of the correct or, in some instances, best answer. Guides for scoring essays might include factors such as the major points to be included in the best answer or models or exemplars corresponding to different levels of performance at different age levels and against which comparisons can be made. Procedures for judging other performances or products might include specification of characteristics to be rated in performance terms and, to the extent possible, clear descriptions of different levels of performance or quality of a product.

➤2. Before an assessment method is used, students should be told how their responses or the information they provide will be judged or scored.

Informing students about scoring procedures to be followed prior to the use of an assessment method should help ensure similar expectations are held by both students and their teachers.

➤3. Care should be taken to ensure results are not influenced by factors not relevant to the purpose of the assessment.

Various errors occur in scoring, particularly when a degree of subjectivity is involved (for example, marking essays, rating a performance, judging a debate). For example, if the intent of a written communication is to assess content alone, the scoring should not be influenced by stylistic factors such as vocabulary and sentence structure. Personal bias errors are indicated by a general tendency to rate all students in approximately the same way (too generously or too severely). Halo effects can occur when a rater's general impression of a student influences the rating of individual characteristics or when a previous rating influences a subsequent rating. Pooled results from two or more independent raters (teachers, other students) will generally produce a more consistent description of student performance than a result obtained from a single rater. In combining results, personal biases of individual raters tend to cancel one another.

➤4. Comments formed as part of scoring should be based on responses made by the students and presented in a way that students can understand and use.

Comments, in oral and written form, are provided to encourage learning and to point out correctable errors or inconsistencies in performance. Comments can also be used to clarify a result. Such feedback should be based on evidence pertinent to the learning outcomes being assessed.

➤5. Any changes made during scoring should be based on a demonstrated problem with the initial scoring procedure. The modified procedure should then be used to rescore all previously scored responses.

Anticipating the full range of student responses is a difficult task for several forms of assessment. There is always the danger that unanticipated responses or incidents relevant to the purposes of the assessment may be overlooked. Consequently, scoring should be continuously monitored for unanticipated responses and these responses should be taken into account.

➤6. A process students may use to appeal a result should be described to them at the beginning of each school year or course of instruction.

Situations may arise where a student believes a result incorrectly reflects his or her level of performance. A procedure by which students can appeal such a situation should be developed and made known to them. This procedure might include, for example, checking for addition or other recording errors or judging or scoring by a second qualified person.

## Summarizing and Interpreting Results

*Procedures for summarizing and interpreting assessment results should yield accurate and informative representations of a student's performance in relation to the goals and objectives of instruction for the reporting period.*

Summarizing and interpreting results refers to the procedures used to combine assessment results in the form of summary comments and grades which indicate both a student's level of performance and the valuing of that performance.

➤1. Procedures for summarizing and interpreting results for a reporting period should be guided by a written policy.

Summary comments and grades, when interpreted, serve a variety of functions. They inform students of their progress. Parents, teachers, counsellors and administrators use them to guide learning, determine promotion, identify students for special attention (honors, remediation) and to help students develop future plans. Comments and grades also provide a basis for reporting to other schools in the case of school transfer and, in the case of senior high school students, postsecondary institutions and prospective employers. They are more likely to serve their many functions and those functions are less likely to be confused if they are guided by a written rationale or policy sensitive to these different needs. This policy should be developed by teachers, school administrators and other jurisdictional personnel in consultation with representatives of the audiences entitled to receive a report of summary comments and grades.

➤2. The way in which summary comments and grades are formulated and interpreted should be explained to students and their parents/guardians.

Students and their parents/guardians have the right to know how student performance is summarized and interpreted. With this information, they can make constructive use of the findings and fully review the assessment procedures followed.

Some aspects of summarizing and interpreting are based on a teacher's best judgment of what is good or appropriate. This judgment is derived from training and experience and may be difficult to describe specifically in advance. In such circumstances, examples might be used to show how summary comments and grades were formulated and interpreted.

➤3. The individual results used and the process followed in deriving summary comments and grades should be described in sufficient detail so the meaning of a summary comment or grade is clear.

Summary comments and grades are best interpreted in the light of an adequate description of the results on which they are based, the relative emphasis given to each result and the process followed to combine the results. Many assessments conducted during a reporting period are of a formative nature. The intent of these assessments (for example, informal observations, quizzes, text-and-curriculum embedded questions, oral questioning) is to inform decisions regarding daily learning and to inform or otherwise refine the instructional sequence. Other assessments are of a summative nature. It is the summative assessments that should be considered when formulating and interpreting summary comments and grades for the reporting period.

➤4. Combining disparate results into a single summary should be done cautiously. To the extent possible, achievement, effort, participation and other behaviors should be graded separately.

A single comment or grade cannot adequately serve all functions. For example, letter grades used to summarize achievement are most meaningful when they represent only achievement. When they include other aspects of student performance such as effort, amount (as opposed to quality) of work completed, neatness, class participation, personal conduct or punctuality, not only do they lose their meaningfulness as a measure of achievement but also suppress information concerning other important aspects of learning and invite inequities. Thus, to more adequately and fairly summarize different aspects of student performance, letter grades for achievement might be complemented with alternate summary forms (checklists, written comments) suitable for summarizing results related to other behaviors.

➤5. Summary comments and grades should be based on more than one assessment result to ensure adequate sampling of broadly defined learning outcomes.

More than one or two assessments are needed to adequately assess performance in multifaceted areas such as reading. Underrepresentation of such broadly defined constructs can be avoided by ensuring that the comments and grades used to summarize performance are based on multiple assessments, each referenced to a particular facet of the construct.

➤6. The results used to produce summary comments and grades should be combined in a way that ensures each result receives its intended emphasis or weight.

When the results of a series of assessments are combined into a summary comment, care should be taken to ensure the actual emphasis placed on various results matches the intended emphasis for each student.

When numerical results are combined, attention should be paid to differences in the variability, or spread, of different sets of results and appropriate account taken where such differences exist. If, for example, a grade is to be formed from a series of paper-and-pencil tests, and if each test is to count equally in the grade, then the variability of each set of scores must be the same.

➤7. The basis for interpretation should be carefully described and justified.

Interpretation of the information gathered for a reporting period for a student is a complex and, at times, controversial issue. Such information, whether written or numerical, will be of little interest or use if it is not interpreted against some pertinent and defensible idea of what is good and what is poor. The frame of reference used for interpretation should be in accord with the type of decision to be made. Typical frames of reference are performance in relation to prespecified standards, performance in relation to peers, performance in relation to aptitude or expected growth and performance in terms of the amount of improvement or amount learned. If, for example, decisions are to be made as to whether a student is ready to move to the next unit in an instructional sequence, interpretations based on prespecified standards would be most relevant.

➤8. Interpretations of assessment results should take account of the backgrounds and learning experiences of the students.

Assessment results should be interpreted in relation to a student's personal and social context. Among the factors to consider are age, ability, gender, language, motivation, opportunity to learn, self-esteem, socioeconomic background, special interests, special needs and test-taking skills. Motivation to do school tasks, language capability or home environment can influence learning of the concepts assessed, for example. Poor reading ability, poorly developed psychomotor or manipulative skills, lack of test-taking skills, anxiety and low self-esteem can lead to lower scores. Poor performance in an assessment may be attributable to a lack of opportunity to learn because required learning materials and supplies were not available, learning activities were not provided or inadequate time was allowed for learning. When a student performs poorly, the possibility that one or more factors such as these might have interfered with the response or performance should be considered.

➤9. Assessment results to be combined into summary comments and grades should be stored in a way that ensures their accuracy at the time they are summarized and interpreted.

Comments and grades and their interpretations, formulated from a series of related assessments, can be no better than the data and information on which they are based. Systematic data control minimizes errors which would otherwise be introduced into a student's record or information base and protects confidentiality.

➢10. Interpretations of assessment results should be made with due regard for limitations in the assessment methods used, problems encountered in collecting the information and judging or scoring it, and limitations in the basis used for interpretation.

To be valid, interpretations must be based on results determined from assessment methods relevant and representative of the performance assessed. Administrative constraints, the presence of measurement error and limitations of the frames of reference used for interpretation also need to be accounted for.

## Reporting Assessment Findings

*Assessment reports should be clear, accurate and of practical value to the audiences for whom they are intended.*

➢1. The reporting system for a school or jurisdiction should be guided by a written policy. Elements to consider include such aspects as audiences, medium, format, content, level of detail, frequency, timing and confidentiality.

The policy to guide the preparation of school reports (reports of separate assessments; reports for a reporting period) should be developed by teachers, school administrators and other jurisdictional personnel in consultation with representatives of the audiences entitled to receive a report. Cooperative participation not only leads to more adequate and helpful reporting but also increases the likelihood the reports will be understood and used by those for whom they are intended.

➢2. Written and oral reports should contain a description of the goals and objectives of instruction to which the assessments are referenced.

The goals and objectives that guided instruction should serve as the basis for reporting. A report will be limited by a number of practical considerations, but the central focus should be on the instructional objectives and the types of performance that represent their achievement.

➢3. Reports should be complete in their descriptions of strengths and weaknesses of students, so strengths can be built on and problem areas addressed.

Reports can be incorrectly slanted toward faults in a student or toward giving unqualified praise. Both biases reduce the validity and utility of assessment. Accuracy in reporting strengths and weaknesses helps reduce systematic error and is essential for stimulating and reinforcing improved performance. Reports should contain information to assist and guide students, their parents/guardians and teachers to take relevant follow-up actions.

➢4. The reporting system should provide for conferences between teachers and parents/guardians. Whenever appropriate, students should participate in these conferences.

Conferences scheduled at regular intervals and, if necessary, on request provide parents/guardians and, when appropriate, students with an opportunity to discuss assessment procedures. Conferences can help clarify and elaborate their understanding of the assessment results, summary comments and grades, reports and, where warranted, to work with teachers to develop relevant follow-up activities or action plans.

➢5. An appeal process that may be used to appeal a report should be described to students and their parents/guardians at the beginning of each school year or course of instruction.

Situations may arise where a student and the parents/guardians believe the summary comments and grades inaccurately reflect the student's level of performance. A procedure by which they can appeal such a situation should be developed and made known to them (for example, in a school handbook or newsletter provided to students and their parents/guardians at the beginning of the school year).

➢6. Access to assessment information should be governed by a written policy consistent with applicable laws and basic principles of fairness and human rights.

A written policy, developed by teachers, administrators and other jurisdictional personnel, should be used to guide decisions regarding the release of student assessment information. Assessment information should be available to those to whom it applies—students and their parents/guardians, teachers and other educational personnel obligated by profession to use the information constructively on behalf of students. In addition, assessment information might be made available to others who justify their need for the information (postsecondary institutions, potential employers, researchers). Issues of informed consent should also be addressed in this policy.

➢7. Transfer of assessment information from one school to another should be guided by a written policy with stringent provisions to ensure maintenance of confidentiality.

To make a student's transition from one school to another as smooth as possible, a clear policy should be prepared indicating the type of information to go with the student and the form in which it will be reported. Such a policy, developed by jurisdictional and ministry personnel, should ensure the information transferred will be sent and received by the appropriate people within the sending and receiving schools respectively.

## B. Assessments Produced External to the Classroom

Part B applies to the development and use of standardized assessment methods used in student admissions, placement, certification and educational diagnosis, curriculum and program evaluation. These methods are primarily developed by commercial test publishers, ministries, departments of education and local school systems.

The principles and accompanying guidelines are organized in four areas:

1. Developing and selecting methods for assessment
2. Collecting and interpreting assessment information
3. Informing students being assessed
4. Implementing mandated assessment programs

The first three areas of Part B are adapted from the *Code of Fair Testing Practices for Education* (Joint Committee on Testing Practices 1988) developed in the United States. The principles and guidelines as modified in these three sections are intended to be consistent with the *Guidelines for Educational and Psychological Testing* (Canadian Psychological Association 1986). The fourth area has been added to contain guidelines particularly pertinent for mandated educational assessment and testing programs developed and conducted at the national, provincial and local levels.

### Developing and Selecting Methods for Assessment

*Developers of assessment methods should strive to make them as fair as possible for use with students who have different backgrounds or special needs. Developers should provide the information users need to select methods appropriate to their assessment needs.*

***Developers' Responsibilities***

➤ 1. Define what the assessment method is intended to measure and how it is to be used. Describe the characteristics of the students with which the method may be used.

➤ 2. Warn users against common misuses of the assessment method.

➤ 3. Describe the process by which the method was developed. Include a description of the theoretical basis, rationale for selection of content and procedures, and derivation of scores.

➤ 4. Provide evidence the assessment method yields results that satisfy its intended purpose.

➤ 5. Investigate the performance of students with special needs and students from different backgrounds. Report evidence of the consistency and validity of the results produced by the assessment method for these groups.

*Users should select assessment methods that have been developed to be as fair as possible for students who have different backgrounds or special needs. Users should select methods appropriate for the intended purposes and suitable for students to be assessed.*

***Users' Responsibilities***

➤ 1. Determine the purpose for assessment and the characteristics of students to be assessed. Then select an assessment method suited to that purpose and type of student.

➤ 2. Avoid using assessment methods for purposes not specifically recommended by the developer unless evidence is obtained to support the intended use.

➤ 3. Review available assessment methods for relevance of content and appropriateness of scores with reference to the intended purpose and characteristics of students to be assessed.

➤ 4. Read independent evaluations of methods being considered. Look for evidence supporting claims of developers with reference to the intended application of each method.

➤ 5. Ascertain whether the content of the assessment method and the norm group or comparison group are appropriate for the students to be assessed. For assessment methods developed in other regions or countries, look for evidence that the characteristics of the norm group or comparison group are comparable to the characteristics of students to be assessed.

➤ 6. Provide potential users with representative samples or complete copies of questions or tasks, directions, answer sheets, score reports, guidelines for interpretation and manuals.

➤ 6. Examine specimen sets, samples or complete copies of assessment instruments, directions, answer sheets, score reports, guidelines for interpretation and manuals. Judge their appropriateness for the intended application.

➤7. Review printed assessment methods and related materials for content or language generally perceived to be insensitive, offensive or misleading.

➤ 7. Review printed assessment methods and related materials for content or language that would offend or mislead students to be assessed.

➤ 8. Describe the specialized skills and training needed to administer an assessment method correctly and the specialized knowledge to make valid interpretations of scores.

➤ 8. Ensure all individuals who administer the assessment method, score the responses and interpret the results have the necessary knowledge and skills to perform these tasks (learning assistance teachers, speech and language pathologists, counselors, school psychologists, psychologists).

➤ 9. Limit sales of restricted assessment materials to persons who possess the necessary qualifications.

➤9. Ensure access to restricted assessment materials is limited to people with the necessary qualifications.

➤10. Provide for periodic review and revision of content and norms and, if applicable, passing or cut-off scores. Inform users.

➤10. Obtain information about the appropriateness of content, the recency of norms, and, if applicable, the appropriateness of cut-off scores for use with students to be assessed.

➤11. Provide evidence of the comparability of different forms of an instrument where the forms are intended to be interchangeable, such as parallel forms or the adaptation of an instrument for computer administration.

➤11. Obtain information about the comparability of interchangeable forms, including computer adaptations.

➤12. Provide evidence that an assessment method translated into a second language is valid for use with that language. This information should be provided in the second language.

➤12. Obtain evidence about the validity of the use of an assessment method translated into a second language.

➤13. Advertise an assessment method in a way that states it can be used only for the purposes for which it was intended.

➤13. Verify advertising claims made for an assessment method.

## Collecting and Interpreting Assessment Information

*Developers should provide information to help users administer an assessment method correctly and interpret assessment results accurately.*

*Users should follow directions for proper administration of an assessment method and interpretation of assessment results.*

***Developers' Responsibilities***

➤1. Provide clear instructions for administering the assessment method and identify the qualifications people who should administer the method should have.

➤2. When feasible, make available appropriately modified forms of assessment methods for students with special needs or whose proficiency in the original language of administration is inadequate to respond in the anticipated manner.

***Users' Responsibilities***

➤1. Ensure the assessment method is administered by qualified personnel or under their supervision.

➤2. When necessary and feasible, use appropriately modified forms of assessment methods with students who have special needs or whose proficiency in the original language of administration is inadequate to respond in the anticipated manner. Ensure instruments translated from one language to another are administered by people proficient in the translated language.

- ➤ 3. Provide answer keys and describe procedures for scoring when scoring is to be done by the user.
- ➤ 4. Provide score reports or procedures for generating score reports that describe assessment results clearly and accurately. Identify and explain possible misinterpretations of scores yielded by the scoring system (grade equivalents, percentile ranks, standard scores) used.
- ➤ 5. Provide evidence of the effects on assessment results of such factors as speed, test-taking strategies and attempts by students to present themselves favorably in their responses.
- ➤ 6. Warn against using published norms with students who are not part of the population from which the norm or comparison sample was selected or when the prescribed assessment method has been modified in any way.
- ➤ 7. Describe how passing and cut-off scores, where used, were set and provide evidence regarding rates of misclassification.
- ➤ 8. Provide evidence to support the use of any computer scoring or computer-generated interpretations. The documentation should include the rationale for such scoring and interpretations and their comparability with the results of scoring and interpretations made by qualified judges.

- ➤ 3. Follow procedures for scoring as set out for the assessment method.
- ➤ 4. Interpret scores taking into account the limitations of the scoring system used. Avoid misinterpreting scores on the basis of unjustified assumptions about the scoring system (grade equivalents, percentile ranks, standard scores) used.
- ➤ 5. Take into account the effects of such factors as speed, test-taking strategies and attempts by students to present themselves favorably in their responses.
- ➤ 6. Take into account major differences between the norm group(s) or comparison group(s) and the students being assessed. Consider discrepancies between recommended and actual procedures and differences in familiarity with the assessment method between norm group(s) and students being assessed. Examine the need for local norms and, if called for, develop these norms.
- ➤ 7. Explain how passing or cut-off scores were set and discuss the appropriateness of these scores in terms of rates of misclassification. Examine the need for local passing or cut-off scores and, if called for, reset these scores.
- ➤ 8. Ensure any computer administration and computer interpretations of assessment results are accurate and appropriate for the intended use. If necessary, ensure relevant information not included in computer reports is also considered.
- ➤ 9. Observe jurisdictional policies regarding storage of and subsequent access to the results. Ensure computer files are not accessible to unauthorized users.
- ➤ 10. Ensure all copyright and user agreements are observed.

## Informing Students Being Assessed

*Direct communication with those being assessed may come from either the developer or the user of the assessment method. In either case, the students being assessed and, where applicable, their parents/guardians should be provided with complete information presented in an understandable way.*

***Developers' or Users' Responsibilities***

- ➤ 1. Develop materials and procedures for informing the students being assessed about the content of the assessment, types of question formats used and appropriate strategies for responding.
- ➤ 2. Obtain informed consent from students or, where applicable, their parents/guardians in the case of individual assessments to be used for identification or placement purposes.
- ➤ 3. Provide students or their parents/guardians with information to help them decide whether to participate in the assessment when participation is optional.
- ➤ 4. Provide information to students or their parents/guardians of alternate assessment methods where available and applicable.

*Control of results may rest with either the developer or user of the assessment method. In either case, the following steps should be followed.*

***Developers' or Users' Responsibilities***

➤1. Provide students or their parents/guardians with information as to their rights to copies of instruments and completed answer forms, to reassessment, to rescoring or to cancellation of scores and other records.

➤2. Inform students or their parents/guardians of the length of time assessment results will be kept on file and of the circumstances under which the assessment results will be released and to whom.

➤3. Describe the procedures students or their parents/guardians may follow to register concerns about the assessment and try to have problems resolved.

## Implementing Mandated Assessment Programs[1]

*Under some circumstances, administration of an assessment method is required by law. In such cases, the following guidelines should be added to the applicable guidelines outlined in the first three sections of Part B.*

***Developers' and Users' Responsibilities***

➤1. Inform all persons with a stake in the assessment (administrators, teachers, students, parents/guardians) of the purpose of the assessment, how results will be used and who has access to the results.

➤2. Design and describe procedures for developing or choosing the methods of assessment, selecting students where sampling is used, administering the assessment materials and scoring and summarizing student responses.

➤3. Interpret results in light of factors that might influence them. Important factors to consider include characteristics of the students, opportunity to learn and comprehensiveness and representativeness of the assessment method in terms of the learning outcomes to be reported on.

➤4. Specify procedures for reporting, storing, controlling access to and destroying results.

➤5. Ensure reports and explanations of results are consistent with the purposes of the assessment, the intended uses of results and planned access to results.

➤6. Provide reports and explanations of results that can be readily understood by the intended audiences. If necessary, employ multiple reports designed for different audiences.

## Note

1. The Joint Advisory Committee wishes to point out it has not taken a position on the value of mandated assessment and testing programs. Rather, given the presence of these programs, the intent of the guidelines presented in this section, when combined with applicable guidelines in the first three sections of Part B, is to help ensure fairness and equity for the students being assessed.

## References


American Federation of Teachers, National Council on Measurement in Education and National Educational Association. *Standards for Teacher Competence in Educational Assessment of Students*. Washington, D.C.: Author, 1990.

Canadian Psychological Association. *Guidelines for Educational and Psychological Testing*. Ottawa: Author, 1986.

Joint Committee on Testing Practices. *Code of Fair Testing Practices for Education*. Washington, D.C.: Author, 1988.


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Principles for Fair Student Assessment Practices for Education in Canada, 1993. *Edmonton, Alberta: Joint Advisory Committee. (Mailing address: Joint Advisory Committee, Centre for Research in Applied Measurement and Evaluation, 3-104 Education Building North, University of Alberta, Edmonton T6G 2G5.)*

# Calendar Math

*Art Jorgensen*

This activity is for Grades 3–6 to do in June 1996.

1. A side of a lot is 50 m. If posts are placed a metre apart, how many posts will be required for the fence?
2. If it takes three minutes to boil one egg, how long will it take to boil three eggs?
3. A jury is made up of 12 people. If a particular jury has 4 more women than men, how many men and women are on the jury?
4. What is half of $2\frac{1}{2}$?
5. Tom can build a model car in 2 days. Susan can build a model car in 3 days. Together how many model cars can they build in 12 days?
6. When you write the numbers from 1 to 100, how many times do you write the digit 5?
7. If oranges cost 10¢ and grapefruits cost 20¢, what possible fruit arrangements can I buy for $2?
8. How many days until your next birthday?
9. Find the sum of the even numbers between 1 and 10.
10. The following numbers are all "Busy Bees." What do they have in common?

    96, 825, 339, 762, 2256

    Write two more "Busy Bees."
11. If June 6 falls on a Thursday, what day does June 20 fall on?
12. The straight line passing from one corner of a rectangle to the diagonally opposite corner is called a ______.
13. What is a four-sided figure with only two parallel sides called?
14. Graph the shoe sizes of the students in your class.
15. Continue the pattern: 2, 7, 12, 17 . . .
16. If you left home at 8:30 a.m. to go to school and did not reach home for seven hours, what time did you reach home?
17. Among your friends, six have only a dog as a pet, seven have only a cat as a pet, and four have a dog and a cat as pets. You have two dogs and one cat as pets. How may pets are there altogether?
18. Tim's mother made 24 cookies. He ate one-half of them, and his sister ate one-third of them. How many cookies were left?
19. Helen has $6, made up of an equal number of quarters, dimes and nickels. How many of each coin does she have?
20. The numbers 464; 1,001; 66 and 2,442 are called palindromes. Why? Write three more palindromes.
21. Use each of the following numbers 1, 2, 3, 4 one time to end with an answer of 10. You might add, subtract, multiply and divide. Is there more than one solution?
22. If Tom can run a kilometre in 5 minutes and Ann can run a kilometre in 280 seconds, who is the faster runner?
23. When you begin school at 9 a.m. in Alberta, what time is it in Toronto?
24. On Tuesday, I worked for seven hours and received $4.50 an hour. On Thursday, I worked five hours and received $5.00 an hour. How much money did I earn altogether.
25. I put half the money I earned in the bank. How much money did I keep?
26. Hamburger costs $3.50 per kg, and steak costs $8.00 per kg. Mary has $20.00 in her pocket. Does she have enough money to buy 2 kg of hamburger and steak? How much does she have left over, or how much is she short?
27. On Sunday, Michael eats one chocolate; on Tuesday, he eats two chocolates; and on Wednesday, he eats three chocolates. If he continues with the same pattern, how many chocolates will he eat in a week?
28. Using only nickels, dimes and quarters, how many ways can you make change for $1?
29. Canada became an independent nation on July 1, 1867. How old will Canada be on July 1, 1996?
30. Graph the birth dates of your classmates according to the month in which they were born. Which month has the most birthdays? The fewest?
31. Find 3 words that each contain 10 letters.

Many of these problems with minor adaptations can be made easier or more difficult depending on the students' ability and interests. Encourage students to make adaptations.

# UPCOMING CONFERENCES

## '96 MCATA Conference

Attend the next Math Council annual conference in Red Deer November 1–3, 1996. The theme of the conference is "Math: Making Connections." Sessions will be held involving connections between grade divisions as well as connections to postsecondary education, business and industry. Special emphasis will be placed on implementing the Western Canada Protocol in the classroom. Sessions on assessment will also be given. Possible keynote speakers include the following:

- Pat Rogers from York University will speak on issues confronting math teachers in education. A workshop on integrated problem solving will also be addressed.
- Katherine Heinrich from Simon Fraser University will speak on making math connections. A second session on putting women in the equation will also be offered.

Plan to attend this important conference to keep up-to-date on new developments in math education and to share ideas with other teachers. Other sessions at all levels will be offered by Alberta educators.

Preregistration and an informal social take place Friday night. Sessions start Saturday, with a luncheon involving one of the keynote speakers. On Saturday, take in The Acme Theatre production and social to follow at the Capri. Sessions continue Sunday with the conference wrapping up early Sunday afternoon. All sessions and entertainment will be held at the Capri Convention Centre with excellent convention rates for the conference. Watch for updates in newsletters and on the Internet. For more information, contact Margaret Anne Stroth, University of Calgary, Conference Management Service, 1833 Crowchild Trail NW, Calgary T2M 4S7; phone 220-6229, fax 284-4184.

## NCTM Canadian Regional Conference

NCTM's Canadian Regional Conference will be held August 22–24, 1996, at Hotel Vancouver, Vancouver, B.C. With more than 150 sessions offered, attendees will be certain to find just the right menu of topics to fit their interests and needs. What a way to start the new school year! Following is a sampling of topics:

- Super Math for the Classroom, Naturally!
- Adding Librarians to Mathematics
- Active Participation in Mathematics
- Innovative Techniques to Motivate
- Geometry in Elementary Optics: The Math and Science of It
- Fifty Manipulatives to Motivate the Learning Process in Mathematics
- Fractal Cards: A Space for Exploration
- Geometry and the Art of M.S. Escher
- Measuring Dinosaurs: Real World Mathematics Through Science
- A Taste of Mathematical Elegance

Publishers' displays, exhibitors and NCTM products will also be available.

Registration and program information were sent to NCTM members in the spring. For more information, contact NCTM at (703) 620-9840.

## WHAT'S NEW?

The following resources are available from the NCTM Customer Services Department at 1-800-235-7566, e-mail orders@nctm.org. NCTM members receive a 20-percent discount off the purchase price.

- *Mathematics for Every Child*, an eight-minute video, highlights the positive changes taking place in mathematics classrooms and espouses the leadership role NCTM had played in shaping today's mathematics education.

  NCTM representatives of each affiliated group and teacher educators placing orders for starter kits or mini-subscriptions received complimentary copies of the video for use in presentations and membership recruitment programs. The video is also available to all mathematics educators for U.S.$10.

- A new collection of six colorful posters to display and discuss in your class is available. They offer brainteasing problems in addition, subtraction, algebra, geometry and palindromic numbers. Each poster is 45.72 by 60.96 cm and costs U.S.$8.
  - Which Triangles Have the Same Area? Grades 4–9, 609AGN
  - Palindromic Numbers Grades 3–8, 607AGN
  - How Many Triangles, Squares, etc.? Grades 3–10, 606AGN
  - Find Numbers on the Chart (odd, prime, multiples) Grades 3–8, 604AGN
  - How Many Nails? (algebraic thinking) Grade 3–9, 605AGN
  - What Happens If You? (subtraction) Grades 3–8, 607AGN

- *NCTM's Reissues—Tools for Today's Math Teacher.* These five yearbooks were selected as special reissues because they are treasures that enriched math education in a unique way. The issues they focused on are still relevant. Each book is 15.24 by 22.86 cm and is list-priced at U.S.$15.
  - *1st Yearbook: A General Survey of Progress in the Last Twenty-Five Years, 1926*, 210 pp., ISBN 0-87353-396-8, 565AGN
  - *6th Yearbook: Mathematics in Modern Life, 1931*, 195 pp., ISBN 0-87353-398-4, 567AGN
  - *11th Yearbook: The Place of Mathematics in Modern Education, 1936*, 258 pp., ISBN 0-87353-400-X, 569AGN
  - *13th Yearbook: The Nature of Proof, 1938*, 146 pp., ISBN 0-87353-402-6, 571AGN
  - *21st Yearbook: The Learning of Mathematics—Its Theory and Practice, 1953*, 353 pp., ISBN 0-87353-404-2, 573AGN
  - All five reissued yearbooks package price U.S.$60. ISBN 0-87353-406-9, 575AGN

- *Algebra in a Technological World: Addenda Series, Grades 9–12* by Kathleen Heid, Jonathan Choate, Charlene Sheets and Rose Mary Zbiek. Addresses high school algebra in light of the NCTM Standards and the dramatic changes brought about by graphing calculators and computer software. Many classroom-tested activities using algebraic functions and mathematical modeling to explore real-world situations. 21.59 by 27.94 cm, softcover, 168 pp., ISBN 0-87353-326-7, 467AGN, U.S.$15 list price.

- NCTM's newest Standards document, *Assessment Standards for School Mathematics*, answers the prevailing question, How can educators effectively assess students' performance and progress? Assessment takes many forms, and this book will update you on alternative assessment techniques.

  An important theme of the *Assessment Standards* is that the assessment of students' achievement should be based on information obtained from a variety of sources—and that much of this information should be gathered by teachers during the instruction process.

  This excellent guide will show you that
  - teachers can set high expectations that every student can achieve;
  - different performances can and will meet agreed-on expectations; and
  - teachers can be fair and consistent judges of students' performance.

  The *Assessment Standards* is the third book in the Standards trilogy. It complements the *Curriculum and Evaluation Standards for School Mathematics* and the *Professional Standards for Teaching Mathematics*. 21.59 by 27.94 cm, 112 pp., ISBN 0-87353-419-0, 593AGN, U.S.$15. All three Standards books for U.S.$60, ISBN 0-87353-420-4, 613AGN

- *Seventy-Five Years of Progress: Prospects for School Mathematics*, edited by Iris M. Carl and sponsored by the Mathematics Education Trust, details the critical reform issues facing K–16 mathematics education and the prospects for their advancement into the 21st century. 15.25 by 22.86 cm hardback, 340 pp.; ISBN 0-87353-418-2, 036AGN, U.S.$20 list price.

The following publications are available from the Grand Valley Mathematics Association. Orders under $50 must be paid in advance. Please forward your order with payment or a purchase order (for no less than $50) to Patty Mah, Faculty of Mathematics, University of Waterloo, Waterloo, ON N2L 3G1; fax (519) 746-6592. Make cheques payable to Grand Valley Mathematics Association.

- ★ *An Outcomes Based Grade 9 Mathematics Program (1995), $10*
  This new 35-page publication includes outcomes, tests, an exam, portfolio ideas, report cards, assessment strategies and much more.
- ★ *OAC Algebra and Geometry Sample Final Exams* (1994), $5
- ★ *OAC Calculus Sample Final Exams—II* (1993), $5
- ★ *OAC Finite Mathematics Sample Final Exams* (1989), $5
- ★ *MAT3A1 Sample Examination* (1991), $2
- ★ *MAT4A1 Sample Examination* (1990), $2
- ★ *Computers and Graphing Calculators in the Math Classroom* (1993), $12
- ★ *Teaching in the Transition Years* (1990), $12

## 10 Guidelines for Math Teachers

1. Believe in yourself and your ability to *think* and *reason*. Risk new methods, approaches, horizons and math experiences, and give your students opportunities for the same. Know and be interested in your subject.
2. Self-confidence is the foundation of knowledge and growth. Create activities and experiences that will build student confidence in themselves and their ability to understand and "do" math.
3. Connect mathematics to the lives of the students you teach. Make the problems real, down to earth and practical. Also link math concepts to other areas of the curriculum.
4. Realize that the best way to learn anything is to discover it. Give your students the "right" to make mistakes. Encourage students to make an honest attempt at solving each problem, without the fear of being laughed at or made fun of by others.
5. Use problems and questions that have multiple answers. Allow them to learn by guessing. Place more emphasis on the process than on the answer. Expect your students to think out their work and explain their reasons for the methods they chose as well as for the answers to their math problems. Reasoning is the essence of what mathematical teaching is all about.
6. Encourage group work, cooperation, team work, verbal communication: a real sharing of ideas among your students. Suggest . . . do not force. (Allow bright students to help and encourage slow students, offering some challenge to both groups.)
7. Make math communication (written and oral) an essential part of the math curriculum.
8. Use manipulatives as integral to mathematics instruction at all grade levels. These tangibles should be available to your students to ensure a deeper understanding of mathematical processes and concepts.
9. Allow ample opportunity for creativity, curiosity, exploration of the unconventional and room for the personal interests of the students to "interrupt" and complement the curriculum. (Remember that the NCTM recommends calculators as essential to fostering new areas and eliminating the drudgery of long, complex calculations.)
10. Prepare your class well. Be willing to explore areas of math that you, personally, are less sure of. Experiment with new methods and approaches of teaching. Remain "open" to ideas and directions that you have not been previously exposed to.

Think positively!

Remember: You are a facilitator, a coach, a guide, a leader—not a dispenser of knowledge or a "sage on the stage."

*Louise M. Lataille*

# GUIDELINES FOR MANUSCRIPTS

*delta-K* is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies; and
- a specific focus on technology in the classroom.

## Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. All contributors are encouraged to submit their manuscripts on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. Letters to the editor or reviews of curriculum materials are welcome.
7. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. Send manuscripts to Arthur Jorgensen, 4411 5 Avenue, Edson T7E 1B7; fax 723-2414.

### On the Lighter Side

Disappointment should always be taken as a stimulant and never viewed as discouragement.

Psychiatrists say it's not good to keep too much to yourself. The government also thinks the same.

People who don't know whether they're coming or going are usually in a big hurry to get there.

# MCATA Executive 1995–96

**President**
George Ditto — Res. 289-1709
610 18 Avenue NW — Bus. 294-8709
Calgary T2M 0T8 — Fax 294-8116
gditto@cbe.ab.ca

**Past President**
Wendy Richards — Res. 482-2210
505-12207 Jasper Avenue NW — Bus. 453-1576
Edmonton T5N 3K2 — Fax 455-7605

**Vice Presidents**
Richard Kopan — Res. 254-9106
72 Sunrise Crescent SE — Bus. 777-7520
Calgary T2X 2Z9 — Fax 777-7529
rjkopan@cbe.ab.ca

Florence Glanfield — Res. 489-0084
8215 169 Street NW — Fax 483-7515
Edmonton T5R 2W4 — glanfiel@gpu.srv.ualberta.ca

**Secretary**
Donna Chanasyk — Res. 455-3562
13307 110 Avenue NW — Bus. 459-4405
Edmonton T5M 2M1 — Fax 459-0187

**Treasurer**
Doug Weisbeck — Res. 434-1674
208-11325 40 Avenue NW — Bus. 434-9406
Edmonton T6J 4M7 — Fax 434-4467

**Communications Director**
Betty Morris — Res. 466-0539
10505 60 Street NW — Bus. 441-6104
Edmonton T6A 2L1 — Fax 425-8759

**Publications Coeditors**
Art Jorgensen — Res. 723-5370
4411 5 Avenue — Fax 723-2414
Edson T7E 1B7

Klaus Puhlmann — Res. 795-2568
PO Box 6482 — Bus. 723-4471
Edson T7E 1T9 — Fax 723-2414
klaupuhl@gyrd.ab.ca

**Conference Director**
Cynthia Ballheim — Res. 278-2991
612 Lake Bonavista Drive SE — Bus. 228-5810
Calgary T2J 0M5 — Fax 229-9280

**1996 Conference Chair**
Graham Keogh — Res. 347-5113
37568 Range Road 275 — Bus. 342-4800
Red Deer County T4S 2B2 — Fax 343-2249

**1997 Conference Chair**
Margaret Marika — Res. 433-0692
9756 73 Avenue NW — Bus. 429-8227
Edmonton T6E 1B4 — Fax 426-0098

**Alberta Education Representative**
Kathleen Melville — Res. 458-6725
Student Evaluation Branch — Bus. 427-0010
11160 Jasper Avenue NW, Box 43 — Fax 422-3206
Edmonton T5K 0L2 — kmelville@edc.gov.ab.ca

**NCTM Representative**
TBA

**Faculty of Education Representative**
Dale Burnett — Res. 381-1281
Faculty of Education — Bus. 329-2416
University of Lethbridge — Fax 329-2252
Lethbridge T1K 3M4 — burnett@hg.uleth.ca

**Mathematics Representative**
Michael Stone — Res. 286-3910
Room 472 Math Sciences Bldg. — Bus. 220-5210
University of Calgary — Fax 282-5150
2500 University Drive NW
Calgary T2N 1N4

**Membership Director**
Daryl Chichak — Res. 450-1813
1826 51 Street NW — Bus. 463-8858
Edmonton T6L 1K1 — Fax 469-0414

**Education Director**
Cindy Meagher — Res. 539-1209
8018 103 Street — Bus. 539-0950
Grande Prairie T8W 2A3 — Fax 539-4706

**Public Relations Director**
Linda Brost — Res. 239-3432
151 Country Club Lane — Bus. 294-8616
Calgary T2M 4L4 — Fax 294-6301
lgbrost@cbe.ab.ca

**Regional Activities Director**
Sandra Unrau — Res. 284-2642
11 Hartford Place NW — Bus. 777-6110
Calgary T2K 2A9 — Fax 777-6112
sunrau@cbe.ab.ca

**PEC Liaison**
Carol D. Henderson — Res. 256-3946
521-860 Midridge Drive SE — Bus. 938-6666
Calgary T2X 1K1 — Fax 256-3508

**ATA Staff Advisor**
David L. Jeary — Bus. 265-2672
SARO — or 1-800-332-1280
200-540 12 Avenue SW — Fax 266-6190
Calgary T2R 0H4

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Barnett House
11010 142 Street NW
Edmonton AB  T5N 2R1