History - A Great Source of Challenging Problems

Maria de Losada

Finding challenging but "elementary" problems for young people can be a challenge in itself. Many creative mathematicians generate new and exciting problems, some specializing from their research, some recreating personal problem-solving experiences, and others simply producing great new ideas for young people.

Problems not requiring advanced ideas but that test problem-solving abilities or require creative thought, are the lifeblood of mathematical challenges and enrichment programs in general. While the sources we have already mentioned are vital, many such problems can be found in the research questions of the past.

For experts dedicated to mathematical challenges, searching historical material for inspiration is an established practice. In the Hungarian Problem Books (English translation published by the MAA), the use of historical references made routinely by József Kürschák is particularly striking. Although the majority of the methods and theorems cited there form part of the mainstream of mathematical thought and are, therefore, quite well-known, it is clear that the history of mathematics was indeed one of his most prized sources.

In what follows we will try to show a variety of ways in which problems from the past can make good material for enrichment and mathematical challenges, be these in the style of multiple choice, short-answer or requiring full solution and proof. Our remarks will show how some of the historical material can be or is being used. However in order to find problems tailor-made for the students' interests and the lines of research that they naturally would wish to follow, teachers must rely on firsthand contact with the history of mathematics.

Some suggestions of sources for multiple-choice type problems

Some early recreational problems that can be found in the Greek Anthology and the Liber Abaci of Fibonacci are still a source of good material for enrichment and popular challenges, although many have filtered into everyday methods and textbooks (the famous "rule of three," for example) and become commonplace.

Well suited to multiple-choice competition problems are those involving a variety of diophantine equations, based on vintage problems such as these:

- Diophantus (II-III Century a.d., Book I, Problem 16). Find three numbers such that when they are added in pairs, the sums are 20, 30 and 40.

Teachers should certainly try to invent their own version of this problem. It has appeared, and continues to appear, in various guises in mathematical challenges. The following is an especially fanciful British version, posed in the First UK School Mathematics Challenge:

- Weighing the baby at the clinic was a problem. The baby would not keep still and caused the scale to wobble. So I held the baby and stood on the scale while the nurse read off 78 kg. Then the nurse held the baby while I read off 69 kg. Finally, I held the nurse while the baby read off 137 kg. What would be the combined weight, in kilograms, of nurse, baby and me?

  (a) 142 (b) 147 (c) 206 (d) 215 (e) 284

- Christoff Rudolff (1526). Find the number of men, women and children in a company of 20 persons, if together they pay 20 coins, each man paying 3, each woman 2 and each child ½.

The following problems, although they come from three different mathematical cultures and traditions nevertheless address the same question.

- Yen Kung (1372). We have an unknown number of coins. If you make 77 strings of them, you are 50 coins short; but if you make 78 strings it is exact. How many coins are there?
- Bhaskara (1150). What number when divided by 6 leaves a remainder of 5, when divided by 5 leaves a remainder of 4, divided by 4 leaves a remainder of 3, and divided by 3 leaves a remainder of 2?

- Fibonacci (1202). Find a multiple of 7 having remainders 1, 2, 3, 4, 5 when divided by 2, 3, 4, 5, 6.

The details of the solutions of some of these problems make them long and messy. Thus a numerical answer included among the various solutions makes checking easy. A slight modification in the question, such as asking for the sum of the two smallest numbers with the given property, requires the development of an essentially complete solution, making the problem especially suitable for multiple-choice or short-answer challenges. Below is the approach taken in the 1985 Australian Mathematics Competition, Junior Division:

- The smallest positive integer which, when divided by 6 gives remainder 1 and when divided by 11 gives remainder 6, is in the range

(a) 115 to 120 (b) 90 to 95 (c) 125 to 130
(d) 60 to 65 (e) 35 to 40

Although somewhat overworked at present, these are still nice little problems for beginners.

Problems suitable for short answers or easy proofs

A class of problems appropriate for a short-answer format are those that concern spending money in intricate ways. The following example can be found in Fibonacci. A merchant doing business in Lucca doubled his money there and then spent 12 denarii. On leaving he went to Florence, where he also doubled his money and then spent 12 denarii. Returning home to Pisa, he there doubled his money and again spent 12 denarii, nothing remaining. How much did he have at the beginning?

This is representative of problems in which the well-known strategy of working backwards is tested and promoted among young scholars and "aficionados" of mathematics.

Many results in geometry from the past constitute excellent moderately difficult problems, that can be adapted to short-answer challenges. The following deals with the harmonic mean $\frac{2ab}{a+b}$ of two quantities a and b.

In his Mathematical Collection, Pappus constructed the harmonic mean of two segments PQ and PR as follows. On the perpendicular to PR at R construct and let the perpendicular to PR at Q meet PS at V. Draw VT and let X be the point of intersection of VT and PR. Then X is the harmonic mean of PR and PQ.

This construction can form the basis of a problem that asks the direct question of expressing PX in terms of PQ and PR, or it can be tailored to other questions such as proving that X is independent of the length of RT.

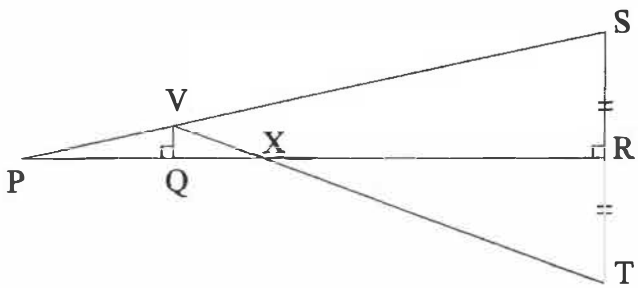

Figure 1

It is known (as told in the *Theatetus*) that the mathematics tutor of Plato, Theodorus of Cyrene, proved the irrationality of $\sqrt{3}, \sqrt{5}, \ldots, \sqrt{17}$. Theodorus had shown how to construct a segment of length $\sqrt{n}$ as the leg of a right triangle with hypotenuse $\frac{n+1}{2}$ and second leg $\frac{n-1}{2}$. And it is but a short step from here to Plato's formulas for Pythagorean triples. Now this can be used to construct a series of right triangles with a common vertex such that the length of the leg opposite the common vertex in each case is 1. The hypotenuse of the nth triangle in this sequence has length $\sqrt{n+1}$.

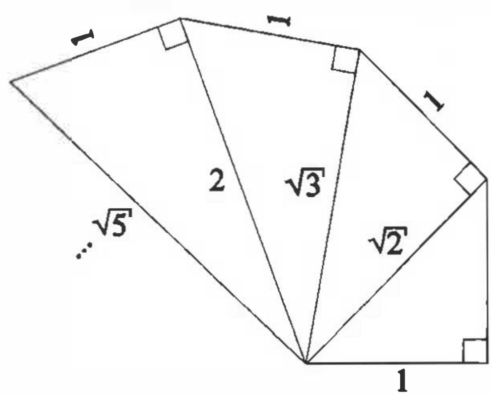

Figure 2

Questions that can be posed relative to the situation described abound. It can even be used to explain why Theodorus stopped his proofs of irrationality at $\sqrt{17}$. We invite our readers to think up two or three.

The classical problem of finding the side a of a cube with twice the volume of a given cube of side b may be reduced to that of finding two mean proportionals c and d between a and b, in other words, $\frac{a}{c}=\frac{c}{d}=\frac{d}{b}$. This is attributed to Hypocrates of Chios. Many of the attempts to produce a ruler and compass construction based on this idea lead to figures that can be used to pose good problems.

Consider a construction attributed to Plato. Let ABC and DBC be two right triangles lying on the same side of their common side BC. Furthermore let hypotenuses AC and BD meet at right angles at point P such that $AP = a$ and $DP = 2a$. It is clear that BP and CP are two mean proportionals between a and $2a$.

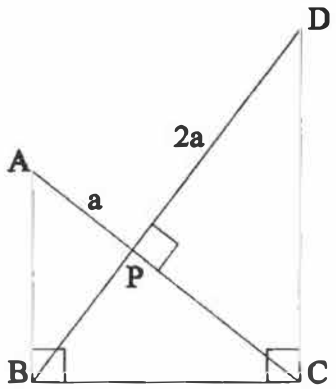

Figure 3

Now, given the construction as described, we can ask students to prove this fact, or we can elaborate on the problem in a variety of ways. For example, we may ask them to find the area of triangle BPC.

To solve the quadratic equation $x^2 + ax = b^2$, Descartes used the following construction. Draw a line segment AB of length b and a circle of diameter a tangent to AB at A. Let O be the center of the circle. Finally draw BO cutting the circle in points E and D. It is clear that the length x of the segment BE satisfies the given equation.

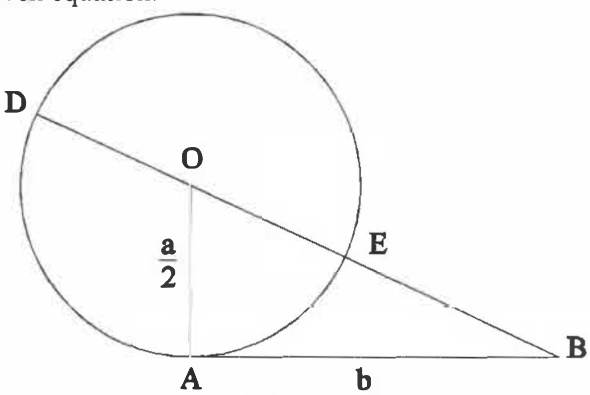

Figure 4

We can ask the student to prove this fact, a very easy task indeed. Or we can start from the geometric construction and ask the student to construct segments of lengths a and b so that the length x of a given segment BE is a root of the quadratic equation $x^2 + ax + = b^2$.

The point in all of these problems is to start with a relatively unknown but interesting construction of certain importance at one time in the history of mathematics. We then formulate questions that require the student to explore the special properties of the construction.

A good problem is adaptable

History gives us an enormous store of suitable problems in geometry that can be adapted to almost any challenge format or more generally to an enrichment program; and there is no doubt that Archimedes is a great source for challenges. Consider, for example, Archimedes' result regarding the "broken" chord. If AB and BC make up any "broken" chord in a circle (where $BC > AB$), and if M is the midpoint of the arc ABC and MF the perpendicular to the longer chord, then F is the midpoint of the broken chord. That is, $AB + BF = FC$.

An easy problem related to this result would ask for the measure of angle BMF, given the measures of angles CBM and BAM.

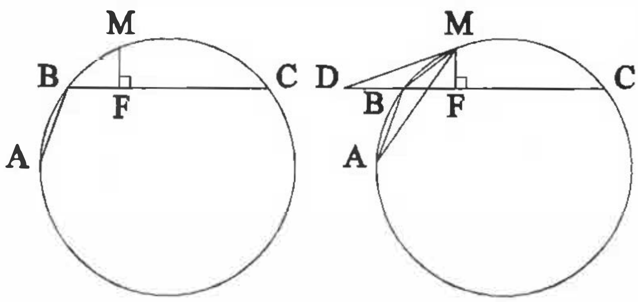

Figure 5

Now Archimedes proved his result by extending chord CB to D with $BD = BA$. A somewhat harder problem would give the lengths of AB and BF and ask for the length of CF. The observation that is required here is in essence the proof of Archimedes' result. Asking for that proof is also a good option (and has a nice solution using rotations). Finally, a problem can be formulated by stating several tentative properties of the figure and asking a question about which combinations of these are true (I and II only, I and III only, etc.).

Another problem studied by Archimedes concerns the "arbelos". This means the "shoemaker's knife", and refers to the curvilinear region in Figure 6 bounded by the three semicircles. Problems related to this figure, given on the cover of the late Sam Greitzer's journal of the same name, take this form:

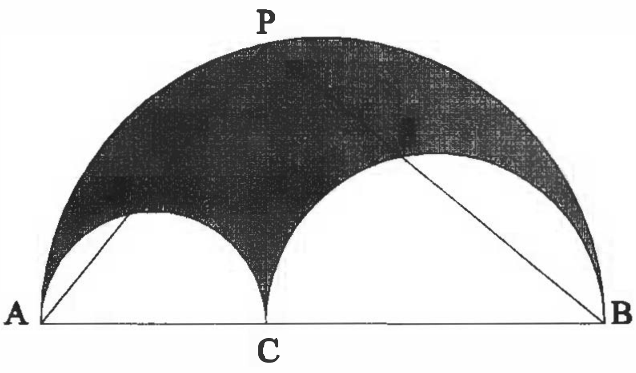

Figure 6

Segment AB is divided in two parts at C. On the same side of AB semicircles with diameters AB, AC and CB are drawn. If PC is perpendicular to AB at C and if R and S are the points of intersection of AP and BP with the respective semicircles, then show that

- RS is a common tangent of the two smaller semi-circles;
- PC and RS bisect one another;
- the total area of the arbelos is equal to the area of the circle with diameter PC;
- the circles inscribed in 'segments' ACP and BCP have equal radii.

Original problems start with Archimedes' results but go beyond them. For example, another problem concerning the arbelos can be found in the International Competition held in Luxembourg in 1980 (a year when there was no IMO) and attended by Luxembourg, Belgium, the Netherlands, Great Britain and the former Yugoslavia. There the problem posed was that of expressing the ratio of the areas of the triangles PRS and PAC in terms of the radii of the two smaller semicircles.

How can a teacher use this material? Obviously we can pose a problem giving numerical values for the radii and ask students to determine the length of PC, the area of triangle PAB, the area of the trapezoid with vertices R, S and the centres of the two smaller semicircles, etc. Or we can ask our students to prove one of the general properties of the arbelos or even to state and prove a property that has not been mentioned among our results. There is much room for interesting and even provocative mathematics here.

As a last Archimedean example, let us consider the following problem taken from the Book of Lemmas which we find appropriate as a problem requiring a full proof from the students.

Let AB be a diameter of a circle and t the tangent to the circle at B. From point P on t draw a second line tangent to the circle at D. Let F be the foot of the perpendicular dropped from D to AB and E the intersection of P and DF. Prove that $DE = EF$.

Simple problems from the past help solve/invent difficult problems for the present

In his book, *Liber Quadratorum*, Fibonacci posed this very simple problem: given the squares of three successive odd numbers, show that the largest square exceeds the middle square by eight more than the middle square exceeds the smallest.

And yet the result can be used to solve in part, or to invent, an olympiad problem at the international level like this one from the 1986 IMO in Poland.

Let d be any positive integer not equal to 2, 5 or 13. Show that it is possible to find two different numbers a and b belonging to the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

We leave it to our readers to discover the link between these two problems.

Methods lost

Other sources of inspiration for Olympiad problems include specialized tools originally developed to solve specialized problems, but never integrated into the mainstream of mathematical thought. Formerly important tools, like continued fractions, finite differences and many geometric methods, which have now been almost forgotten (at least in the basic secondary curriculum) can also serve as sources of inspiration.

The following is an example of a specialized tool. Consider the formula known as the "bloom of Thymaridas" given in a first-century manuscript of Iamblichus and judged to be copied from an ancient source. We have n unknown quantities $x_1, x_2, \ldots, x_n$ to which we add one more unknown quantity x. The following sums are given:

$$\begin{aligned} x + x_1 + x_2 + \cdots + x_n &= S \\ x + x_1 &= a_1 \\ x + x_2 &= a_2 \\ \cdots &= \cdots \\ x + x_n &= a_n \end{aligned}$$

Iamblichus gives a general solution to the problem that is equivalent to the formula

$$x = \frac{(a_1+a_2+...+a_n) - S}{n - 1}$$

As van der Waerden tells us, this solution can be found using a well-known method of Diophantus which requires us to set $x = s$ and then observe that the remaining unknowns are $a_1 - s, a_2 - s, \ldots, a_n - s$ so that the sum S is given by

$$(a_1 + \ldots + a_n) - (n - 1)s.$$

Iamblichus uses this solution in a clever way to solve (diophantine) equations of somewhat different appearance, such as the system:

$$\begin{aligned} x + y &= 2(z + u) \\ x + z &= 3(y + u) \\ x + u &= 4(y + z) \end{aligned}$$

which constitutes an interesting problem; one of its solutions starts by putting these into the same form as the bloom of Thymaridas.

The given equations are equivalent to

$$\begin{aligned} x + y + z + u &= 3(u + v) \\ x + y + z + u &= 4(y + u) \\ x + y + z + u &= 5(y + z) \end{aligned}$$

Now the sum of the four numbers must be divisible by 3, 4 and 5. So Iamblichus sets $S = 120$ and arrives at the same equations as given in the bloom of Thymaridas as follows.

$$\begin{array}{lllll} x + y + z + u & & = & S & = 120 \\ x + y & = 2(z + u) & = & \frac{2}{3} S & = 80 \\ x + z & = 3(y + u) & = & \frac{3}{4} S & = 90 \\ x + u & = 4(y + z) & = & \frac{4}{5} S & = 96 \end{array}$$

At this point the formula of Thymaridas can be applied. (Undoubtedly common elimination procedures will also do the job, but may not be as much fun nor as well focused.) This same problem reappeared in Fibonacci's *Liber Abaci* and was used this year in the Colombian Olympiad's final round for students from sixth to eighth grades, as transcribed below.

Consider this problem also from *Liber Abaci*. Three men, *A*, *B* and *C*, each of whom already has some gold coins, found a purse of gold coins. *A* said: If you give me all of the coins in the purse, I will have twice as many coins as *B* and *C* put together. *B* said: If you give me all of the coins in the purse, I will have three times as many coins as *A* and *C* together. Finally *C* said: If you give me all of the coins in the purse, I will have four times as many coins as *A* and *B* together. What is the least number of coins that can be in the purse?

A final thought

Ideas are our most important, and (fortunately) renewable resources. They can be used over and over, on varying levels of difficulty, and never be fully exhausted.

We hope that these few examples illustrate how ideas of the past can be put to use in the formation of young mathematics students of today and that teachers looking for challenging and intriguing problems will be able to discover in the history of mathematics a vast reserve that awaits their exploration.

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