Appendix V: Answers and Solutions

A. Supplementary Problems in Appendix I

An Imaginary Postal Service:

The only perfect design with the five-stamp sets is the one shown in the diagram, apart from the switching of the one and two cent stamps.

1	3
8	2
7	

Dissecting Rectangular Strips into Dominoes:

A recurrence relation is $f_n = f_{n-2}$ with initial conditions $f_0 = 1$, $f_1 = 4$, $f_2 = 2$ and $f_3 = 3$.

A Space Interlude:

1. We already know that if each Space Cannon is divided into 3 parts, we need nine Space Pods. With only eight available, we divide the Space Cannons into A, B, C and D. The Space Pods carry (A,B), (A,B), (A,C), (B,C), (C), (D), (D) and (D) respectively. To get all four parts, the Space Octopus must grab one of the last three, but will not get all of A, B and C by grabbing only one other.

2. With seven Space Pods, we divide the Space Cannons into A, B, C, D and E. A working scheme has the Space Pods carrying (A,B,C), (A,B,D), (A,C,D), (B,C,D), (E), (E) and (E) respectively. Suppose we divide the Space Cannons into A, B, C and D only. Clearly, none of them should be carrying 3 parts or more. Let one of them carry A and B. Then none of them can carry C and D. Thus three other Space Pods carry C and the remaining three carry D. We still have 2 A parts and 2 B parts. We may assume that one of the Space Pods carrying C also carries A. Then none of those carrying D can carry B, so that the 2 B parts go to the other two Space Pods carrying C. Now the remaining A part must go to one of the Space Pods carrying D, but this allows the Space Octopus to get all parts by grabbing just two Space Pods.

3. With six Space Pods, we divide the Space Cannons into A, B, C, D, E and F. A working scheme has the Space Pods carrying (A,B,C), (A,B,D), (C,D,E), (C,D,F), (E,F,A) and (E,F,B) respectively. Suppose we divide the Space Cannons into A, B, C, D and E only. Then there are 15 parts in all. Hence some Space Pod must carry at least 3 parts. Certainly, it should not carry 4 or more. Hence we may assume that it carries A, B and C. We have 6 D and E parts to be carried by the other five Space Pods, so that one of them must carry both of them. Hence it is possible for the Space Octopus to grab just two Space Pods and get all 5 parts.

How to Flip without Flipping:

There are three such triangles which are not similar to one another. The measures of their angles are (20°, 40°, 120°), ($22\frac{1}{2}$°, 45°, $112\frac{1}{2}$°) and ($25\frac{5}{7}$°, $51\frac{3}{7}$°, $102\frac{6}{7}$°).

A Tale of Two Cities:

Obviously, such a negative circle must be in the interior of the inner chain. It has a neighbor of each kind if and only if it has an integration point.

B. Contest Questions in Appendix II B

These are worked out by **Daniel Robbins**, a Grade 12 student at École Secondaire Beaumont.

1967	ecede	abeda	ccdcd	baebb	adbec
1968	dbcca	dedec	ebadb	beaca	aedeb
1969	ceaea	cedeb	cdabc	cbdce	dbced
1970	cbadb	deebc	acadd	bcbba	edacd
1971	baadd	cdcbb	debce	abecd	
1972	ddaca	ecabb	dbcde	cebae	

1973	cbaca	cdbcd	eacbc	cdbcb
1974	cccbd	bedda	becbe	bcdba
1975	dadcc	cbdea	dabec	bddab
1976	cedba	beaec	bcedc	dddaa
1977	ddbbb	accbe	daeda	ecbca
1978	ccbad	babdb	beead	deece
1979	bdcbe	edaec	acdbe	dbaed
1980	bbddb	ccaab	aedab	caecd
1981	aecbc	ccbaa	eadae	ebbeb
1982	bccac	bbaec	caad	abdac
1983	abece	bcccd	cbabc	eccba

C. Sample Contest Problems in Appendix II C

1. More generally, suppose there are 2^n friends. After n rounds, the most anyone can learn are 2^n pieces of gossip. Hence n rounds are necessary. We now prove by induction on n that n rounds are also sufficient. For $n = 1$, the result is trivial. Suppose the result holds up to $n - 1$ for some $n > 2$. Consider the next case with 2^n friends. Have them call each other in pairs in the first round. After this, divide them into two groups, each containing one member from each pair who had exchanged gossip. Each group has 2^{n-1} friends who know all the gossip among them. By the induction hypothesis, $n - 1$ rounds are sufficient for everyone within each group to learn everything. This completes the induction argument. In particular, with 64 friends, 6 rounds are both necessary and sufficient.

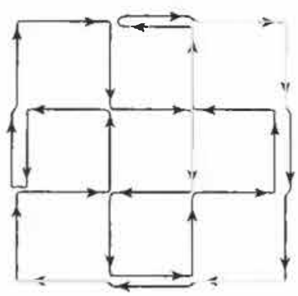
2. The "sheep" player wins. Place one of the 50 sheep on each of the lines $y = 3m$, $1 < m < 50$, so that initially, no sheep is within 1 metre of the wolf. The sheep will stay on their respective lines, which are 3 metres apart. Since the wolf's maximum speed is 1 metre per move, it can threaten at most 1 sheep at a time. In a one-to-one scenario, the wolf cannot run down the sheep even if the sheep is confined to move along a fixed line.

3. Construct a graph G in which each city is represented by a vertex and each direct air-route by an edge. Let G' be the graph obtained from G by removing M, the vertex representing the capital, and all edges incident with it. By hypothesis, G is a connected graph, but G' may consist of a number of components. However, each component must contain at least one vertex connected to M in G. In G', such vertices have degree 9 while all others have degree 10. However, each component must have an even number of vertices with odd degree. Hence at least two vertices in each component are connected to M in G. Since M has degree 100, the number of components in G' is at most 50. Hence we can reconnect G' by restoring M and one edge connecting it to each component of G'. This is just a different way of saying that we can remove from G at least 50 edges incident with M without disconnecting it.

4. The answer is no. Divide the 3.5 hours into 7 periods each of 0.5 hours. The pedestrian walks at 6 kilometres per hour in periods 1, 3, 5 and 7 and at 4 kilometres per hour in periods 2, 4 and 6. For any 1 hour interval, the pedestrian walked at each speed for exactly 0.5 hours. Hence the distance covered is exactly 5 kilometres. However, the total distance covered is 18 kilometres, yielding an average speed of more than 5 kilometres per hour.

5. Let $b_1, b_2 \ldots b_{15}$ be the heights of the boys and $g_1, g_2 \ldots g_{15}$ be the heights of the girls. Suppose for some k, $1 < k < 15$, $|b_k - g_k| > 10$. Without loss of generality, we may assume $g_k - b_k > 10$. Then $g_i - b_j > 10$ for all i and j where $1 < i < k$ and $k < j < 15$. Consider the boys of height b_j, $k > j > 15$ and the girls of height g_i, $1 < i < k$. By the Pigeonhole Principle, some two of these 16 must form a couple in the original lineup. However, $g_i - bj > 10$ contradicts the hypothesis.

6. The diagram shows a closed tour of length 28 with fourfold symmetry. We claim that it has minimum length. Each of the four corners is incident with two roads and requires at least one visit. Each of the remaining twelve intersections is incident with three or four roads and requires at least two visits. Hence the minimum is at least $4 + 12 \times 2 = 28$.



7. Let the musicians be A, B, C, D, E and F. Suppose there are only three concerts. Since each of the six must perform at least once, at least one concert must feature two or more musicians. Say both A and B perform in the first concert. They must still perform for each other. Say A performs in the second concert for B and B in the third for A. Now C, D, E and F must all perform in the second concert, since it is the only time B is in the audience. Similarly they must all perform in the third. The first concert alone is not enough to allow C, D, E and F to perform for one another. Hence we need at least four concerts. This is sufficient, as we may have A, B and C in the first, A, D and E in the second, B, D and F in the third and C, E and F in the fourth.

8. Note that the total number of chameleons is divisible by 3. When divided by 3, the initial numbers of the three kinds of chameleons leave remainders of 0, 1 and 2. Of course, the sum of these three remainders will always be divisible by 3, so that their collective values must be one of (0, 0, 0), (1, 1, 1), (2, 2, 2) and (0, 1, 2). In a multi-color meeting, all three remainders change value. Hence one of the remainders is not 0, one of them is not 1 and one is not 2. It follows that they must always be 0, 1 and 2 in some order, meaning that there are chameleons of at least two different colors at any time.

9. More generally, we show that $3n - 2$ weighings are sufficient for 2n coins. We first divide the coins into n pairs, and use n weighings to sort them out into a "heavy" pile and a "light" pile. The heaviest coin is among the n coins in the "heavy" pile. Since each weighing eliminates 1 coin, $n - 1$ weighings are necessary and sufficient for finding it. Similarly, $n - 1$ weighings will locate the lightest coin in the "light" pile. Thus the task can be accomplished in $3n - 2$ weighings. For $2n = 68$, $3n - 2 = 100$.

10. There are six permutations of the grasshoppers: 123 (132) 312 (321) 231 (213). They are arranged so that each alternate one is in brackets. A jump changes a permutation into either one of its neighbors, where the first and last permutations are also considered as neighbors of each other. Suppose that the initial permutation is not one in brackets. Note that 1985 is an odd number. Then after an odd number of jumps, the resulting permutation must be in brackets. Hence after 1985 jumps, the grasshoppers cannot regain even their initial relative positions.

11. Of the 8 teams, there must be a champion who has won the most games. We denote that team by A. This team must have won at least 4 games. We denote such teams which A has defeated by B, C, D and E. Among these teams, there must also be a champion that has won at least 2 games, for example team B, which beats teams C and D. By symmetry, we may assume that C beats D. This yields the desired ordering.

12. Construct a graph with 20 vertices representing the 20 teams. Two vertices are joined by a red edge if the two teams they represent play each other on the first day, and by a blue edge if they play on the second day. Since each vertex is incident with one red edge and one blue edge, each component of the graph is an even cycle. By taking every other vertex in each cycle, we have 10 independent vertices, representing 10 teams, no two of which have yet played each other.

13. The key observation is that from a non-square formation, one can always leave a square formation, but from a square formation, one must leave a non-square formation. Since the starting formation is non-square and the winning move consists of leaving a 1 by 1 square formation, the first player has a sure win by leaving a square formation on every move.

14. Each use of the machine increases the total number of coins by 4, an even number. Since Peter starts with 1 coin, he will always have an odd number of coins. Thus it is impossible for him to have an equal number of nickels and pennies.

15. (a) Note that 5 of the pawns start on black squares while the other 4 start on white squares. All permitted moves preserve square color. It is thus impossible for the pawns to be moved to the upper left hand corner as this contains 5 white squares and 4 black squares.

(b) If we number the columns starting from the left, 6 pawns commence on odd-numbered columns while the other 3 commence on even-numbered columns. All permitted moves preserve the parity of the column number. It is impossible for the pawns to be moved to the upper right hand corner as this contains 6 squares in even-numbered columns and 3 squares in odd-numbered columns.

16. The first player wins. The sequence 1987, 993, 496, 248, 124, 62, 31, 15, 7 and 3 is obtained by dividing each term by 2, ignoring any remainder. We claim that these are winning positions. By the rules, 1987 is a winning position. Suppose $2k$ or $2k + 1$ is a winning position. We claim that so is k. On the opponent's next move, the largest number that can be chosen is $2k - 1$, which falls short of $2k$. The smallest number that can be chosen is $k+1$, after which $2k+1$ can be reached. This justifies our claim. Since the initial number is 2, the first player can win by choosing 3 and the next winning position thereafter.

17. Assume that the task is impossible. Then the total number of baskets is not more than 99, as otherwise we could leave 1 apple in each of 100 baskets and remove the rest. Now the largest basket has at most 99 apples. The second largest has at most 49, as otherwise we leave 2 baskets each with 50 apples. Similarly, the next largest four have at most 33, 24, 19 and 16 respectively. Even if each of the remaining 93 has 16 apples each, the total is at most $99 + 49 + 33 + 24 + 19 + 16 \times 94 = 1828$, which is a contradiction.

18. Let a, b, c and d denote the respective numbers of people with neither blue eyes nor fair hair, those with blue eyes but not fair hair, those with fair hair but not blue eyes, and those with blue eyes and fair hair. Then the proportion of people with fair hair among people with blue eyes is $\frac{c}{b+c}$ and the proportion of people with fair hair among all people is $\frac{c+d}{a+b+c+d}$. We are given that $\frac{c}{b+c} \times \frac{c+d}{a+b+c+d}$, so that $\frac{c}{c+d} \times \frac{c+d}{a+b+c+d}$. It follows that the proportion of people with blue eyes among people with fair hair is more than the proportion of people with blue eyes among all people.

19. The key to this problem is to consider each square of the chess board to be one of a set of four, symmetric about a horizontal and a vertical line through the centre of the board. For example, the four central squares form such a set, as do the four corner squares. The first player wins regardless of the starting position, by simply placing the pawn in the opposite square of the current "set of four". The second player is always forced to move to an outer "set of four" and the first player responds by placing the pawn in the opposite square of that set. The second player 2 must ultimately place the pawn in a corner square whereupon the first player wins by placing the pawn in the opposite corner.

20. (a) Given any distribution of the 300 stars, consider a set S of 100 rows containing the highest number of them. We claim that this number is at least 200. Otherwise, there is a row in S containing at most 1 star, and a row not S containing at least 2 stars. This contradicts the maximality of S. It follows that if we remove the 100 rows in S, at most 100 stars are left. We can make them disappear by removing at most 100 columns.

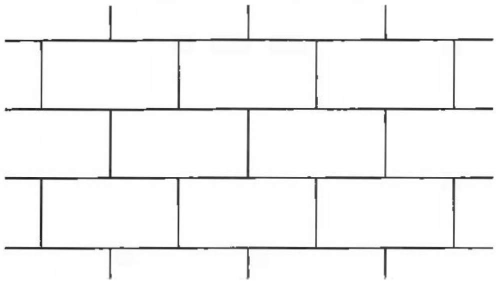
    (b) Place 200 stars on the main diagonal, 100 stars on the diagonal immediately above in rows 1 to 100, and the last star on the first square of the 101st row. Denote by S_1 the 99 rows each with 1 star, and by S_2 the 101 rows each with 2 stars. Suppose we remove k-1 rows from S_1 and 101-k rows from S_2. The 100-k stars in remaining rows in S_1 occupy 100-k columns with no other stars. The $2k$ stars in the remaining rows in S_2 occupy at least k+1 columns collectively. Thus we cannot get rid of all the stars remaining by removing any 100 columns.

D. Sample Problems in Appendix III

1. Consider a particular person. Suppose she has more acquaintances than strangers among the other five. Then she must be acquainted with at least three of them. If no two of these three are acquainted, then they form a triangle of mutual strangers. If some two of them are acquainted, then they form with the first person a triangle of mutual acquaintances. The case where the first person has fewer acquaintances than strangers among the other five can be handled in the same way.

2. The second player can force a draw with the following strategy. Divide the infinite board into dominoes as shown in the diagram. Whenever

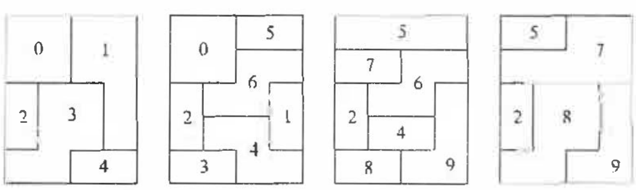
the first player takes a cell, the second player takes the other cell of the same domino. Since every compact 2×2 configuration must contain a domino, the first player cannot win.



3. If every guest moves to the next room, then the first room is available for the new guest. Since the hotel is infinite, there is no other "end" from which a guest would have been expelled.

4. Your opponent will always challenge a \$1 bet, because there is no reason why you should place such a bet with a picture card. Hence you expect to lose 2 dollars in 9 of 13 games. Suppose your opponent concedes a \$5 bet with probability p. With a picture card, you win $5p + 10(1 - p) = 10 - 5p$ dollars, in 3 of 13 games. With an Ace, you win $5p - 10(1 - p) = -10+15p$ dollars, in 1 of 13 games. Since $9(-2) + 3(10 - 5p) + (-10 + 15p) = 2$, you expect to win on the average 2 dollars every 13 games.

5. There are ten such pieces, and together they form a $4 \times 4 \times 5$ block as shown in the diagram, drawn in four layers. Parts with identical labels belong to the same piece.



6. Since $OQ > PR$, there exists a point E between O and Q such that $OE = PR$. Then $OP = QR - (OQ - OE) = QR - QE < RE$ by the Triangle Inequality. Consider the triangles EOR and PRO. We have $OE = PR$, $OR = RO$ and $REOP$. Hence $\angle EOR > \angle PRO$ by the Side-Angle-Side Inequality. It follows from the Angle-Side Inequality that $CR > CO$.

7. If (a) is true, then so is (e) which claims that (a) is not. Hence (a) is false. Since (c) claims that (a) is true, (c) is also false. If (d) is true, then one of (a), (b) and (c) is true. Since (a) and (c) are false, (b) must be true, but it claims that (d) is not. Hence (d) is also false, and so is (b). If (f) is true, then all of (a), (b), (c), (d) and (e) are false, but that would make (e) true. Hence (f) is false. Hence (e) is the only one that can be true, and this leads to no contradictions.

8. Clearly, four points are not enough. Suppose we have five. Consider the smallest convex polygon containing them. If it is a pentagon or a quadrilateral, we have a convex quadrilateral as desired. Suppose it is a triangle ABC, with D and E inside. The line DE must intersect two sides of this triangle, say AB and AC, with D closer to AC. Then $BCDE$ is a convex quadrilateral.

9. The amount of solid substance remains unchanged. Since its percentage has doubled, the total weight must have halved to 250 kilograms.

10. The geometric series $1 + x + \cdots + x^n$ sums to $\frac{1}{1-x}$ if $x < 1$. Hence
$M = \left(1+\frac{1}{2}+\cdots\right)\left(1+\frac{1}{3}+\cdots\right)\left(1+\frac{1}{5}+\cdots\right)\cdots\left(1+\frac{1}{p}+\cdots\right)$.
When we multiply out the right side, we will get the reciprocals of all positive integers. For instance, $\frac{1}{60}$ will come from $\left(\frac{1}{2}\right)^2 \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot 1 \cdot 1 \cdots$. Hence $M = 1+\frac{1}{2}+\frac{1}{3}+\cdots$. Now $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{2}$, and so on. If we go far enough, the right side will exceed M. This contradiction shows that there must be infinitely many prime numbers.

11. Let AB be the diameter dividing the domains of the first two ice fishermen. By symmetry, these remain equal in area as long as the newcomer sets up his ice house on AB. If it is set up at A, clearly he will be at a disadvantage. If it is at the centre O, he will have gained the upper hand. Hence there is a point somewhere between A and O at which the third ice fisherman can set up his ice house so that all three domains have equal area.

12. (a) This is a village in Wales. The name means *Church of St. Mary in a hollow of white hazel, near to a rapid whirlpool, to St. Tysilio Church and to a red cave.*

(b) This is a lake in Massachusetts, USA. The name means *I fish on my side and you fish on your side and nobody fishes in the middle*.

(c) This is a place in New Zealand. The name means *The brow of the hill where Tamatea who sailed all round the land played his nose flute to his lady love*.

13. The answer is "no". For instance, $(1 \oplus 1) \oplus 2 = 4 \oplus 2 = 12$ while $1 \oplus (1 \oplus 2) = 1 \oplus 6 = 14$.

14. Consider first the special case where $a_3 = 85$ for $1 \le i \le 19$. Then the desired sum is clearly $19 \cdot 85 = 1700$. If not, let j be the largest index such that $a_j < 85$. Suppose $a_j = k$. If we increase a_j by 1, then all the b's remain unchanged except for b_{k+1}, which decreases by 1. This balances out the gain by a_j, so that the sum remains unchanged. By repeating this increment process, we will eventually arrive at the special case considered earlier. Hence the desired sum is always equal to 1700.

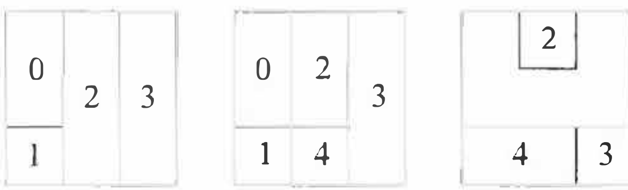
15. We claim that the probability is $\frac{2}{15}$. We prove this by induction on the number n of boxes. For $n = 2$, the result is trivial. Suppose the claim holds for some $n \ge 2$. Consider the next case with $n + 1$ boxes. Suppose the $(n + 1)$st key is in the $(n + 1)$st box. This happens $\frac{1}{n+1}$ of the time, and the probability of success now is 0. Suppose the kth key is there instead for some k, $1 \le k \le n$. This happens $\frac{n}{n+1}$ of the time. If we get to the $(n + 1)$st key, we can open the $(n + 1)$st box and retrieve the kth key. Hence we may pretend that the $(n + 1)$st key and box were not there, and that the kth key is where the $(n + 1)$st key was. By the induction hypothesis, the probability of success now is $\frac{2}{n}$. It follows that the overall probability is $\frac{1}{n+1}(0)+\frac{n}{n+1}\left(\frac{2}{n}\right)=\frac{2}{n+1}$. This completes the inductive argument.

16. If a prisoner thinks only of himself, he would reason as follows: suppose the other guy confesses. Then either I get a light sentence by also confessing, or get a heavy sentence otherwise. If he does not confess, I can get an award by ratting on him. No matter what he does, I am better off confessing. The irony is that both would end up getting a light sentence, whereas they could have gone free if neither confesses.

17. Five campers will be sufficient. The counsellor sends all of them down path A while she explores path B. If the campers return a 5:0 or 4:1 decision, it can be believed and the counsellor will know the true situation about paths A and B. She will have time to explore path C if necessary. Suppose the campers return a 3:2 decision and the campsite is not down path B. If two of them say "No", the counsellor sends one of them down path C while she checks out path A. If she does not find the campsite herself, the report from path C will be reliable. Suppose after the first exploration, three campers say "No". The counsellor sends all of them down path C while she checks out path A. If she does not find the campsite herself, both of those who say "Yes" are liars, and the majority decision from path C will be correct. Four campers are not sufficient because the one who always tells the truth cannot outnumber the liars unless he is alone. This means that the counsellor must be able to identify him after the first exploration. However, one of the liars can confuse the issue by telling the truth up to that point.

18. First, ask the Amazing Sand Counter to mentally note down the number of grains of sand in the bucket. Second, ask him to turn around while you remove a number of grains. Third, ask him to glance at the bucket again. If he has the power claimed, he will be able to tell you how many grains are missing.

19. The solution is shown in the diagram, where the cube is drawn in three layers. Parts with identical labels belong to the same piece.



20. With seven varieties, there are $\frac{7 \cdot 6}{2} = 21$ pairs of them. Each plot accounts for 3 pairs, so that there must be 7 plots. Labelling the varieties 1 to 7, the plots may contain (1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 4, 7) and (3, 5, 6), rspectively. It is routine to verify that all conditions are met.

21. First, take both a and b to be the irrational number $\sqrt{2}$. If a^b is rational, we have what we want. If $\sqrt{2}^{\sqrt{2}}$ is irrational, take it to be a and keep b unchanged. Then $a^b = 2$ is rational.

22. Let the segment be AB. Take a point C not on the two parallel lines. Join C to A and B, cutting the other line at E and D respectively. Join BE and AD, intersecting at P. The point F of intersection of the lines CP and AB is the midpoint of AB. To see this, let Q be the point of intersection of the lines CP and DE. Then triangles PBF and PEQ are similar, as are PAF and ADQ. Hence $\frac{FB}{QE}=\frac{FP}{QP}=\frac{FA}{QD}$. Now the triangles CBF and CDQ are also similar, as are CAF and CEQ. Hence $\frac{FB}{QE}=\frac{FC}{QC}=\frac{FA}{QE}$. Multiplication yields $FB^2 = FA^2$ or $FB = FA$.

23. The total number of possible outcomes of tossing $2n$ coins is 2^{2n}. The number of those with exactly n heads is equal to the binomial coefficient $\frac{(2n)!}{n!n!}$. Hence the probability is $p_n = \frac{(2n)!}{2^{2n}n!n!}$. Now

$$\frac{p_{n+1}}{p_n} = \frac{(2n+2)!}{2^{2n+2}(n+1)!(n+1)!}\cdot\frac{2^{2n}n!n!}{(2n)!} = \frac{2n+1}{2n+2} < 1.$$

Hence p_n decreases as n increases. It follows that the maximum occurs when $n = 1$.

24. We show first that it suffices to consider the case in which all students of each school are of the same sex. Indeed, if some school has both a boy and a girl, then the number of singles matches played by this boy is the same as the number of mixed single matches played by the girl, and vice versa. It follows that sending both the boy and the girl home alters neither of the conditions of the problem. Now, suppose that, as above, each school has either all girls or all boys, and that k schools have an odd number of students. Suppose there are, in all, B boys, G girls, S singles and M mixed singles, with $|B - G|$ and $|S - M|$. Then $M = BG$ and

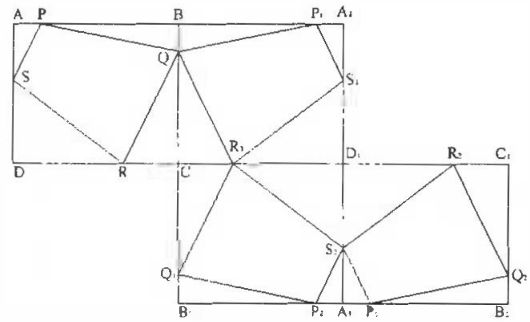
$$-1 \le S - M \le \frac{1}{2}(B(B-1)+G(G-1)) - BG = \frac{1}{2}$$

$$((B-G)^2 - (B+G)) \le \frac{1}{2}(1-(B+G)).$$

It follows that $-3 \le -(B + G)$ or $B + G \le 3$. Since each of the k schools has at least one student, $k \le B + G \le 3$, and there are at most 3 schools with an odd number of students. The upper bound $k = 3$ is attained if two of them have 1 girl and 0 boys each, and a third school has 0 girls and 1 boy.

25. There are nine blocks of four adjacent numbers. Each number appears in four of these blocks. Now four times the sum of the numbers is 360. Hence the average sum of each block is 40, and at least one block is no less than 40.

26. Reflect the rectangle $ABCD$ and its inscribed quadrilateral $PQRS$ three times as shown in the diagram. By the Triangle Inequality, $PQ + QR + RS + SP = PQ + QR_1 + R_1S_2 + S_2P_3 \ge PP_3 = AA_2$, and the desired result follows immediately.



27. If we color the spaces black and white in checkerboard fashion, with A black, then there are 11 black spaces and 9 white ones. Each of the first, second, fourth and fifth pieces covers 2 black and 2 white spaces, leaving 3 black and 2 white spaces. Since the third piece must cover 3 squares of one color and 1 of the other, it covers 3 black and 1 white spaces. It follows that the unit square is on a white space, which can only be B or D. Since it obviously cannot be in B, the answer is D, and it is not hard to see how the other five pieces can fit in around it.

28. Let g_n be the number of ways of paving a straight path, 1 metre wide and n metres long. Clearly, $g_1 = 1$ and $g_2 = 2$. For $n \ge 3$, we can either start with a square paving stone or a rectangular one. In the first case, we are left with an unpaved path of length n-1, and there are g_{n-1} ways of completing the paving. In the second case, we are left with an unpaved path of length n - 2, and there are g_{n-2} ways of completing the paving. Hence $g_n = g_{n-1} + g_{n-2}$ for $n \ge 3$. It follows that $g_3 = 3$, $g_4 = 5$, $g_5 = 8$, $g_6 = 13$, $g_7 = 21$, $g_8 = 34$, $g_9 = 55$, $g_{10} = 89$, $g_{11} = 144$, $g_{12} = 233$, $g_{13} = 377$, $g_{14} = 610$, $g_{15} = 987$ and $g_{16} = 1597$.

29. Assign values to the squares as shown in the diagram. Initially, the single counter occupies a square of value 1. Note that the value of each square is equal to the sum of the values of the squares to the north and east of it. It follows that in any move, the total value of the squares occupied by counters is unchanged, and must be equal to 1 all the time. The total value of all the squares, counting one row at a time, is equal to $2+1+\frac{1}{2}+\frac{1}{4}+\cdots=4$. The total value of the six home squares is $2\frac{3}{4}$. The total value of the squares on the first row, not counting the home squares, is $\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{4}$. By symmetry, this is also the total value of the squares on the first column, not counting the home squares. It follows that the total value of the remaining squares is $\frac{3}{4}$. Suppose after a finite number of moves, the home squares are cleared of all counters. Now there must be exactly one counter on the first row and exactly one on the first column, each occupying a square of value at most $\frac{1}{8}$. It follows that the total value of the occupied squares is strictly less than $\frac{1}{8}+\frac{1}{8}+\frac{3}{4}=1$, and we have a contradiction.

$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

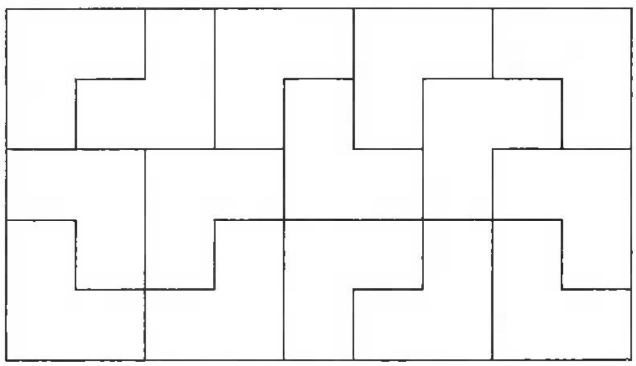
30. We start with 1, 2 and 3, each of which divides their sum 6. If we throw in 6, and each of these four numbers will divide their sum 12. This eventually leads to 1, 2, 3, 6, 12, 24, 48, 96, 192 and 384.

31. First, if we take 997 evenly spaced points on a line, then there are exactly 1991 red points, consisting of the 996 midpoints between consecutive points as well the 995 points other than the 2 at the ends. In general, let M and N be 2 points at the greatest distance apart. Let their midpoint be Q and let the perpendicular bisector of MN be l. Let the other points be P_i, $1 \le i \le 1995$, and let X_i and Y_i be the midpoints of MP_i and NP_i respectively. Note that the X_i are distinct, as are the Y_i. Moreover, none coincides with Q. Now $MX_i \le MQ$ since $MP_i \le MN$. It follows that each X_i is on the same side of l as M. Similarly, each Y_i is on the same side of l as N. Hence no X_i can coincide with any Y_i. Thus we have at least 1991 red points, consisting of the X_i, the Y_i and Q.

32. First note that $\sqrt{a}-\sqrt{b}=\frac{a-b}{\sqrt{a}+\sqrt{b}}$ is also rational. Hence $\sqrt{a}=\frac{\sqrt{a}+\sqrt{b}}{2}+\frac{\sqrt{a}-\sqrt{b}}{2}$ is rational, as is $\sqrt{b}=\frac{\sqrt{a}+\sqrt{b}}{2}-\frac{\sqrt{a}-\sqrt{b}}{2}$.

33. Divide the triangle into four congruent ones by drawing the three segments joining midpoints of the sides. With 9 points, at least 3 will be in the same small triangle. Hence they determine a triangle of area at most $\frac{1}{4}$.

34. A solution is shown in the diagram.



35. Note that the knights can never move to the central square of the chessboard. Number the remaining squares from 1 to 8 in clockwise order, with the white knights on squares 1 and 3, and the black knights on squares 5 and 7. Now they can only move along the cycle (1, 6, 3, 8, 5, 2, 7, 4), and their relative positions on this cycle can never change. Hence the desired task is not possible.

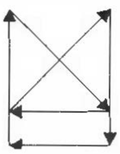
36. After t hours, the first car will be at a distance of 10 - 60t kilometres and the second car, 10 - 30t kilometres from the intersection. Instead of minimizing directly the distance between them, we may minimize the square of this distance, which by Pythagoras' Theorem is equal to $(10-60t)^2+(10-30t)^2=200-1800t+4500t^2=5(4+(6-30t)^2)$. The minimum occurs at $t=\frac{1}{5}$, which is 12 minutes past noon. At that moment, the first car is at 10 - 12 = -2 or 2 kilometres past the

intersection, and the second at 10 - 6 = 4 kilometres before the intersection.

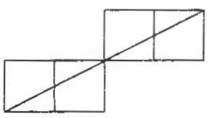
37. It has 32 edges, 24 two-dimensional faces and 8 three-dimensional faces.

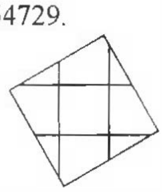
38. The quadratic expression expands into $Ax^2 + Bx + C$, where $A = q_1^2 + q_2^2 + \cdots + q_n^2$, $B = 2(p_1q_1 + p_2q_2 + \cdots + p_nq_n)$ and $C = p_1^2 + p_2^2 + \cdots + p_n^2$. Since it is a sum of squares, it is never negative for any real values of x. It follows that its discriminant $B^2 - 4AC$ must be non-positive. Hence $(\frac{B}{2})^2 \leq AC$, which is equivalent to the desired result.

39. Take an equilateral triangle ABC. Let D be on the perpendicular bisector of BC, with $AD = BC$. Then $BD = CD \neq AD$. This gives rise to two configurations as D can be on either side of BC. A third consists of two equilateral triangles ABC and CDA, and a fourth consists of a square $ABCD$. The fifth is an isosceles trapezoid $ABCD$ with $AB = BC = CD$ and $AD = AC = BD$.



40. The only solution is 381654729.





41. A solution is shown in the diagram.

42. A solution is shown in the diagram.

43. Jock led it, and was in fact the only one who spoke truthfully.

44. We have $12 = y^4 - x^2 = (y^2 - x)(y^2 + x)$. One of the factors on the right side must be even, which implies that both are even. The only possibility is $y^2 - x = 2$ and $y^2 + x = 6$, leading to $x = y = 2$.

45. We may assume that PA ≤ PB. Drop the perpendicular OD from O onto AB. Then AD = BD. Suppose P in inside the circle. By Pythagoras' Theorem, $PA \cdot PB = (AD - PD)(BD + PD) = AD^2 - PD^2 = (OA^2 - OD^2) - (OP^2 - OD^2) = r^2 - OP^2$. If P is outside the circle, we have $PA \cdot PB = (PD - AD)(PD + BD) = OP^2 - r^2$.

46. There are 10 potential lines of division. If the task is possible, each must be blocked by a domino across its path. However, if exactly one domino is in the way, then the line divides the remaining part of the square into two regions each containing an odd number of cells. This is impossible because each is supposed to be tiled with dominoes. Hence each line is blocked by at least two dominoes. Clearly, each domino can block only one line. Since there are only 18 dominoes, at most 9 lines can be blocked. Hence the task is impossible.

47. After taking a paved road and a country road, the lost tourist should take three more country roads and finally another paved road. It is routine to verify that no matter where he starts, the lost tourist will always end up in D.

48. Let the two points be O and A. Draw a circle γ with centre O passing through A, and a circle λ with centre A passing through O. Use the common radius to mark off three successive arcs on λ, starting from O, so that we end up at point B which is at the other end of the diameter from O. Draw a circle with centre B passing through O, cutting γ at the points P and Q. With these two points as centres, draw two circles passing through O, and intersecting each other again at a point M. Then M is the desired midpoint of OA. To see that it is so, note by symmetry that M indeed lies on the line OA. Now POM and BOP are isosceles triangles. Since they have a common base angle, they are similar to each other. Hence $\frac{MO}{OP} = \frac{OP}{OB}$ which leads to $MO = \frac{1}{2}OA$.

49. If the party were held when grandpa was 99, then $x67y2 \equiv x + 10y + 69 \equiv 0 \pmod{99}$. Since x and y are single-digit numbers, we must have $x + 10y = 30$. However, this means that x is a multiple of 10, and $x67y2$ cannot be a five-digit number. It follows that the party was held when grandpa was 98, and $x67y2 \equiv 4x + 10y + 38 \equiv 0 \pmod{98}$. This leads to $2x + 5y = 30$ and we must have $x = 5$. Then $y = 4$ and the number of guests was $56742 \div 98 = 579$.

50. For each question, denote the answers 0, 1 and 2 in non-ascending order of their numbers of respondents. Suppose there were 10 or more students. Then at least 7 of them answered 0 or 1 in Question 4, which would not be useful for distinguishing triples of them. Similarly, at least 5 of these 7 answered 0 or 1 in Question 3, at least 4 of these 5 answered 0 or 1 in Question 2 and at least 3 of these 4 answered 0 or 1 in Question 1.

Students	1	2	3	4	5	6	7	8	9
Question 1	0	0	0	1	1	1	2	2	2
Question 2	0	1	2	0	1	2	0	1	2
Question 3	0	1	2	2	0	1	1	2	0
Question4	0	1	2	1	2	0	2	0	1

These 3 did not answer any of the four questions differently. It follows that the largest possible number of students is 9, and this is indeed possible, as shown in the table above.

51. A triangle inside a parallelogram has at most one half its area, and the same goes for a parallelogram inside a triangle. Let the angles at *P, Q, R* and *S* be α, β, γ and δ respectively. Since $(\alpha + \beta) + (\gamma + \delta) = 360°$, we may assume that $\alpha + \beta \geq 180°$, and similarly that $\alpha + \delta \geq 180°$. If we complete the parallelogram *PQTS*, then *T* is inside *PQRS*. Now *PQS* has at most one half the area of *PQTS* which in turn has at most one half the area of *ABC*.

52. When divided by 4, an even square leaves remainder 0 while an odd square leaves remainder 1. When the sum of the squares of five consecutive integers is divided by 4, the remainder is either $1 + 0 + 1 + 0 + 1 = 3$ or $0 + 1 + 0 + 1 + 0 = 2$. Thus this sum cannot be a square itself.

53. Triangles *PAB* and *PCD* are similar, as are *QCD* and *QEF*. It follows that $\frac{AP}{CP} = \frac{AB}{CD} = \frac{EF}{CD} = \frac{EQ}{CQ}$. Hence *PQ* is parallel to *AE* and *BD*.

54. Let $BC = a$, $CA = b$, $AB = c$ and $\angle ABC = \beta$. Then the area of *ABC* is $\frac{1}{2}ca\sin\beta$ and that of *BFG* is $\frac{1}{2}ca\sin(180°-\beta)$. Hence they have the same area. Similarly, *ADE* and *CHJ* also have the same area as *ABC*. Now the area of *ACJD* is $b^2 = c^2 + a^2 - 2ca\cos\beta$. The total area of *ABFE* and *BCHG* is $c^2 + a^2$ while that of the remaining part of *DEFGHJ* is $c^2 + a^2 - 2ca\cos(\beta) + 2ca\sin\beta$. Equating these two values, we have $\theta = 45°$.

55. Let p be any prime number and for any positive integer *k*, let *H*(*k*) denote the highest powers of *p* which divides *k*. Let $H(l) = a$, $H(m) = b$ and $H(n) = c$. Then $H(l\,m) = ab$, $H(mn) = bc$ and $H(nl) = ca$. It follows that $H(\gcd(l,m,n)) = \min\{a,b,c\}$ and $H(\operatorname{lcm}(l\,m,mn,nl)) = \max\{ab,bc,ca\}$. The desired result follows immediately.

56. Note that

$$\frac{1}{1+x+xy} = \frac{z}{z+zx+1} = \frac{yz}{yz+1+y} \text{ and } \frac{1}{1+z+zx} = \frac{y}{y+yz+1}.$$

It follows that the desired sum is equal to

$$\frac{yz}{yz+1+y} + \frac{1}{1+y+yz} + \frac{y}{y+yz+1} = 1$$

57. We have

$$\begin{aligned} x \odot y &= (x \odot y) \odot (x \odot y) \\ &= ((x \odot y) \odot x) \odot y \\ &= ((y \odot x) \odot x) \odot y \\ &= ((x \odot x) \odot y) \odot y \\ &= (y \odot y) \odot (x \odot x) \\ &= y \odot x. \end{aligned}$$

58. Suppose we have an integer solution. Clearly, $x^4 + y^4 + z^4$ is even. We may assume that *x* is even and *y* and *z* are either both odd or both even. If they are even, then the left side of the equation is divisible by 16, but 24 is not. Hence *y* and *z* are both odd. However, the left side of the equation is congruent modulo 16 to $0 + 1 + 1 - 2 - 2x^2 - 2x^2$. This leads to $-4x^2 \equiv 8 \pmod{16}$ or $x^2 \equiv 2 \pmod 4$, which is impossible.

59. Suppose $(3m + 2)^2 = n^2 + p$ where *m* and *n* are positive integers and *p* is a prime. Then $p = (3m + 2 - n)(3m + 2 + n)$, which means that $3m + 2 - n = 1$ and $3m + 2 + n = p$. Solving this system of equations yields m = $\{p - \frac{1}{2}\}$ or $p = 3(2m + 1)$. Thus *p* cannot be a prime. Hence $(3m + 2)^2$ does not have the desired form for any positive integer *m*.

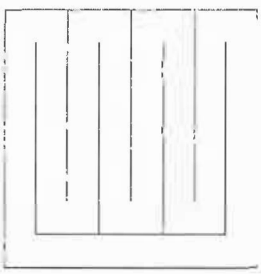
60. For every two scientists, there must be a lock which neither can open. However, each of the other four must have a key, as otherwise some three of them will not be able to open it. Hence we should have one lock for each pair

of scientists and give keys only to the other four. This requires 15 locks and 10 keys for each scientist.

61. Let OA and PB be the respective radii. We may assume that $OA \geq PB$. Drop the perpendicular PC from P onto OA. By Pythagoras' Theorem, $AB^2 = OC^2 = OP^2 - CP^2 = (OA + PB)^2 - (OA - PB)^2 = 4OA \cdot PB$.

62. The dominoes can tile the chessboard by following the path shown in the diagram. The removal of two cells of opposite colors will result in the removal of one domino and the shifting of one group of the remaining dominoes between the two removed cells along the path.

63. The next such integer is 52.

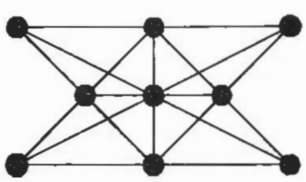


64. Let P be the point inside the equilateral triangle ABC of side 1. Let $2a$, $2b$ and $2c$ be the respective distances of P from BC, CA and AB. Then the area of PBC, PCA and PAB are a, b and c respectively, so that the area of ABC is $a + b + c$. Since its base is 1, its altitude is indeed $2a + 2b + 2c$.

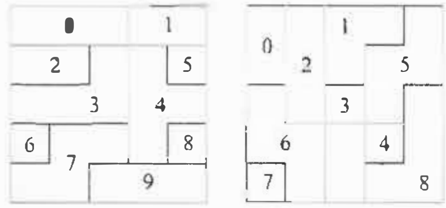
65. Ask the chicken to stand on one foot and the rabbits to put up their front paws. Then there will be $94 \div 2 = 47$ feet, exceeding the number of heads by $47 - 35 = 12$. Each rabbit contributes one foot to this total, so that the number of rabbits is 12, and the number of chickens is $35 - 12 = 23$.

66. If $n = 2k$, a permutation which works is $(a_{k+1}, a_1, a_{k+2}, a_2, \ldots, a_{2k}, a_k)$. If $n = 2k + 1$, just add a_{2k+1} at the end.

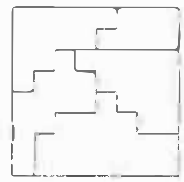
67. A solution is shown in the diagram.



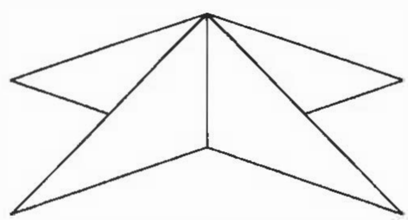
68. The solution is shown in the diagram, where the block is drawn in two layers. Parts with identical labels belong to the same piece.



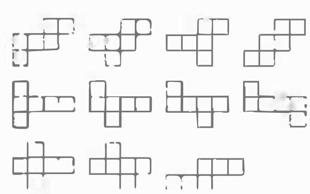
69. A solution is shown in the diagram.



70. Make four creases as follows. The first two are along the diagonals, with the colored side showing. The other two are along the segments joining midpoints of opposite sides, with the plain side showing. Then the paper is folded into the four-winged shape shown in the diagram, with the midpoints of the four sides coming together. Two opposite wings serve as "pockets" while the other two serve as "tongues". Let the colors be red, yellow and blue. Put the red tongues in the yellow pockets, the yellow tongues in the blue pockets and the blue tongues in the red pockets. Paper clips can help during assembly, but they are not needed to hold the structure together once it is completed.



71. The eleven hexominoes are shown in the diagram.



72. The two solutions are shown in the diagram.