Changing Tires

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Abstract

The discussion below has got nothing to do with tires. Also, at times you'll feel like the silent moviegoer who shouts out to the screen heroine "look behind you." Unfortunately I can't hear you. Read on to the end despite my deafness and see why.

Tires on the front of my motor bike last 40 000 km and on the back they last 60 000 km. How far can I go without having to buy a new tire?

Try it and then read on.

I'm sorry to say I don't really own a motorbike. In fact I've never ridden one. My uncle had one for most of his life and was involved in a couple of accidents so I got the hint that they weren't the safest things around. It didn't stop my uncle, though he did add a side car to his machine after my first cousin was born.

Anyway, it's possible that he was confronted by the tire problem at some stage in his life when he wanted to delay spending money on new tires for as long as he could. Because the average of 40 000 and 60 000 is 50 000, it's tempting to believe that by suitably switching tires from front to back, and vice-versa, you might eke out 50 000 km before buying new tires. But how would you manage that? Well, after 25 000 km you could try a switch. Suppose tire A is on the front and tire B on the back. They've now done their 25 000 km stint and we've switched them around. How far can A last on the back? For that matter, how far can B last on the front?

Because B has used up 25 000 km of its 60 000 km life at the back, does it have 35 000 km left up front? Surely not. It won't last as long. How long then, will it last?

Assuming uniform wear, and I don't see how we can avoid that assumption, tire B has spent $\frac{25000}{60000}$ of its life at the rear. So $\frac{35000}{60000}$ of its life is yet to come. It can now spend $\frac{7}{12}$ of its life up front. Since a front tire's life expectancy is 40 000 km, tire B must be able to give us another $\frac{7}{12}\times 40\ 000$ km service. It looks like we'll get another $23\ 333\frac{1}{3}$ km out of it. But that's a problem because $25\ 000 + 23\ 333\frac{1}{3} = 48\ 333\frac{1}{3} < 5\ 000$? In fact, it's a long way short of 50 000. What can we do to get the extra mileage (kilometrage?)? Perhaps we could keep switching the tires back and forth. Maybe we'd get more distance that way.

Back up for a bit. Tire A can only go on the front or back. The same holds for tire B. Doesn't that mean that tire A spends part of its life on the front and part on the back? It doesn't really matter how often you change the tires. The net effect is to keep A on the front for part of the time and then put it on the rear wheel. Forget about all the swapping then. One swap is sufficient.

Oh dear! And by the looks of things it doesn't seem as if we're going to be able to get our full 50 000 km either. I've done a few jottings in the margin and I can't get anywhere near that target. How can I do these experiments systematically enough to produce the maximum I want? Past experience with this sort of thing suggests that perhaps it's time for a bit of algebra.

Suppose A traveled x km on the front and y on the back. Then the total distance d for the old tires is $d = x + y$. And I want to maximize d.

Hmm! One equation with two unknowns x and y. I need another equation if I'm going to get anywhere. What else do I know? I suppose that at the end of the day (or rather the end of the tire) the tire will be worn out. How can I get another equation out of this? Partly worn on the front plus partly worn on the back is all worn! So? The fractional part of A worn on the front is $\frac{x}{40\ 000}$. And the part worn on the back is $\frac{y}{60\ 000}$. When A's done that, it's all gone. That means $\frac{x}{40\ 000}+\frac{y}{60\ 000}=1$ is equal to what?

Is what 1? After all it's one whole life. So this means that we have to maximize d = x + y subject to $\frac{x}{40\ 000}+\frac{y}{60\ 000}=1$ That looks like a linear programming problem. Let me check if there are any other constraints. Yes, clearly $0 < x < 40\ 000$ and $0 < y < 60\ 000$. If you know anything about linear programming you'll know that you only have to test the values of d at P, Q and R in Figure 1. The best value of d is obtained when x = 0 and y = 60 000! Looks like I ride for 60 000 km with the front wheel in the air!

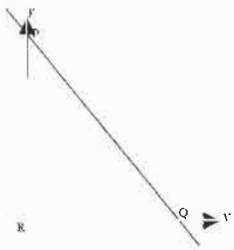

Figure 1

That's screwy. Maybe I forgot to take something else into account? Can't think what. But wait. There's another way to do this. I want to maximize $d = x + y$ and x and y are linked by $\frac{x}{40\ 000} + \frac{y}{60\ 000} = 1$. If I solve the last equation for y, I can substitute into the first equation. This will give d in terms of x and I might be able to use a bit of calculus. Right, $d = 60\ 000 - \frac{3x}{2}$. Hmm. There's no need for calculus. Surely d is biggest when x is smallest. That happens when $x = 0$. Back up on to one wheel!

I'm clearly missing a vital piece of information. But what? What have I not done that I could have done? Wait! So far I've only thought about tire A. Perhaps if I brought B into the action I might make some progress. Well, for B, d is still $x + y$. But for B, x km is spent on the back and y km on the front, so $\frac{x}{60\ 000} + \frac{y}{40\ 000} = 1$. That's the same as for A. I haven't made any progress at all.

Hang on. No, that's crazy. For A I get $\frac{x}{40\ 000} + \frac{y}{60\ 000} = 1$ and for B I get $\frac{x}{60\ 000} + \frac{y}{40\ 000} = 1$. Two equations, two unknowns! Let's solve. You can do that in your head can't you? Well, on a bit of paper then. I get $x = 24\ 000$ and $y = 24\ 000$.

So it looks as if the manufacturers of motor bike tires should put a small red strip in at the 24 000 km mark and then I'll know when to change tires!

But wait. The original question was how far could I go on a pair of tires. The answer is $d = 48\ 000$ km. That's a bit worrying: it's not consistent with what I did a while back. When I thought I could get 50 000 km out of the tires I tried changing them after 25 000 km and managed to get 48 $333\frac{1}{3}$ km worth. What have I done wrong now?

In that case I looked at tire B. There $x = 25\ 000$ and $y = 23\ 333\frac{1}{3}$. Does that fit the equation for B?

$$\frac{25\ 000}{60\ 000} + \frac{23\ 333\frac{1}{3}}{40\ 000} = \frac{5}{12} + \frac{7}{12} = 1.$$

Yes, that's OK. No problems. There's something wrong somewhere though. I'd better check A's equation. There I've got

$$\frac{25\ 000}{40\ 000} + \frac{23\ 333\frac{1}{3}}{60\ 000} = \frac{5}{8} + \frac{7}{18} = \frac{73}{72}$$

which isn't 1! In fact, $\frac{73}{72}$ is bigger than 1! We've got an extra $\frac{1}{72}$ of a life out of tire A. Not bad (but don't tell the tire companies).

OK, so what that all means is that we can't get $48\ 333\frac{1}{3}$ km out of a set of tires. The inconsistency I thought I had removed. It looks as if I can only get 48 000 km out of a set of tires after all.

Given the last experience with $48\ 333\frac{1}{3}$ km though, can we really manage 48000 km? Now

$$\frac{24\ 000}{60\ 000} + \frac{24\ 000}{40\ 000} = \frac{24\ 000}{40\ 000} + \frac{24\ 000}{60\ 000} = 1$$

So we have got everything out of the tires. However, it may be that we can't organize the tires so that they die simultaneously. No, that's stupid. I've just lost a little confidence. Clearly the A and B equations tell us that if we change the tires at 24 000 km, they will both be useless at the 48 000 km mark. That's a relief!

Now wait a minute! Suppose I had three tires. Could I get more than 48 000 km out of them? Is that the right question? Obviously if I use tire A at the front for 40 000 km and then put tire C at the front, I'll get 60 000 km's worth until B goes bald. With three tires I can easily get 60 000 km. If I do that though I really haven't done as well, per tire, as I did with only two tires. It seems to me then, that "with three tires can I do better than an average of 24 000 km per tire?" is the right question to ask. For what it's worth, I do know that I can't do worse than a 20 000 km average.

I guess the way to tackle this one is to use the successful strategy of the two tire case. So if tire A is on the front for x km and on the back for y km and tire B is on the front for z km and the back for u km and tire C is on the front for v km and on the back for w km, I get

$$\frac{x}{40\ 000} + \frac{y}{60\ 000} = 1$$

$$\frac{z}{40\ 000} + \frac{u}{60\ 000} = 1$$

$$\frac{v}{40\ 000}+\frac{w}{60\ 000}=1$$

And that's assuming that I can wear out all three tires too! Let's assume that for now and worry about it later. What a mess!

I might be able to make those equations at least look better if I write a for 40 000 and b for 60 000. It will also save me quite a bit of writing. So

$$\frac{x}{a}+\frac{y}{b}=1,\quad \frac{z}{a}+\frac{u}{b}=1,\quad \frac{v}{a}+\frac{w}{b}=1.$$

I can get them all on one line now!

Actually I'm inclined to write x_A, x_B and x_C for the numbers of kilometres that A, B and C are on the front, respectively, and y_A, y_B and y_C for the numbers of kilometres they're on the back. I can then even reduce the three equations to one:

$$\frac{x_i}{a}+\frac{y_i}{b}=1$$

for i=A,B,C. How about that!

Has that really helped me solve the problem though? I don't think I know what to do with that lot. You see the next step that I would like to make is to say that $x_A = y_B$, which is what I had in the two-tire problem but that's not necessarily going to be the case. Can I link the x's and y's at all? I'll draw a picture (Figure 2).

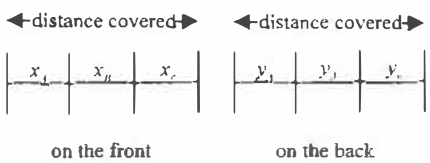

Figure 2

Obviously I don't know which order the tires should go on the front and back. But it's probably not worth changing them too frequently. We might as well leave A on the front for its x_A km, and do the same for B and C. With any luck, we can do the same on the back wheel.

The one thing that the picture is useful for is that it does give us $x_A+x_B+x_C=y_A+y_B+y_C$, though what use this is I don't know. Again, let's call this quantity d. Using the Σ notation I can rewrite this equation as $d=\sum_i x_i=\sum_i y_i$, and we want to maximize d subject to $\frac{x_i}{a}+\frac{y_i}{b}=1$ for i = A, B, C. I don't want to even think about the linear programming approach. If I use calculus, I'll have d as a function of several variables. So the methods that didn't work last time probably won't work now either. And I'm not so sure that solving three equations in six unknowns is going to get me anywhere either! Is there any way I can use the two-tire approach? What good is it to me that $y_i=b-\frac{bx_i}{a}$, without the link I had before between x and y?

Wait though! Using that last equation, I get $\sum_i y_i=\sum_i\left(b-\frac{bx_i}{a}\right)=3b-\frac{b}{a}\sum_i x_i$. But $\sum_i y_i$ and $\sum_i x_i$ are both equal to d. So $d=3b-\frac{bd}{a}$. For what it's worth I can at least solve for d. Then $d=\frac{3ab}{a+b}$. I'm sure you can work that out.

Remembering that $a=40\,000$ and $b=60\,000$, d must be 72 000. The average distance per tire then is just $\frac{d}{3}$. Not 24 000 km again!

Better check that we can actually get $d=72\ 000$. Do the equations tell us how to achieve this grand distance before ending up with three bald tires? I don't think they do. I can't see any way that the x_i and y_i are restricted to be something special. What if we just try $x_i=y_i=24\ 000$? Maybe it'll work out. I'll show this diagrammatically in Figure 3.

	A	B	C
front	←24000→	←24000→	←24000→
back	←24000→	←24000→	←24000→
	B	C	A

Figure 3

So there is a rotation that will give me the 24 000 km average. It's strange though that I don't, and can't, increase the average number of kilometres per tire even if I use an extra tire.

Just reflecting a minute, I see now that the assumption that I could wear out all tires was justified. I wonder though if there are other ways of rotating the tires? And are some ways better for economy or safety? Or have I missed something that forces x_i to equal y_i? And it probably doesn't help me to have four, five or any number of tires. I doubt that I could get more than a 24,000 km tire average. I wonder what my uncle did when he had his side car? If he could get 80 000 km out of a side car tire, I wonder what he averaged per tire? And what should you do with car tires?

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As I said initially, this discussion has really nothing to do with tires. What I was doing would be very similar no matter what the problem in hand because there are some tried and true heuristic techniques that you can use in a lot of situations. If you have them in your armory it will improve your problem solving capability.

One of the first things you'll notice is that **I keep asking questions**. It's almost always impossible to solve a decent problem straight away. Quite often you have to cast around for an approach and you usually have to solve a large number of smaller problems to get where you want to go. I tend to think of problem solving, not like Figure 4(a), but rather like Figure 4(b).

In regular classrooms where certain algorithms are being practiced, Figure 4(a) is often the model used. With more difficult problems many smaller questions are asked and answers sought, a lot of which are not on the final track of a solution at all. Actually in mathematical research many answers are reached unexpectedly. Some answers may not be what the researcher was looking for and do not necessarily answer the original questions. However, they may be very useful and interesting nevertheless. Asking the right questions is more important than answering them.

Q → A

Figure 4(a)

```
          q→a→q→a
          ↑
          a→q→a→q→a→q
          ↑
     Q→a→q      q→a
       ↓        ↑
       q→a→q→a→q→a→q→a→q→A
                ↓
                q→a→q
                    ↓
                    q
```

Figure 4(b)

When starting out with a new problem, it's often a good idea to do a few examples that will give you some **insight** into what's going on. If nothing else, examples will help you understand the problem. Moreover, they are useful yardsticks by which to measure a solution and they may give you some insight to find a solution.

Don't be afraid to **use a diagram** either. Even though the tire problem is essentially algebraic, I got some insight into what was going on by producing Figure 2.

In the search for solutions though, it's a good idea to try to think of situations that you've been in before that seem vaguely similar to the present situation. **Can a previously used technique help?** That's why I thought about linear programming, calculus and solving algebraic equations. They won't all work but maybe one will.

Often you'll get stuck because **you've overlooked a vital piece of information**. In the two-tire problem I was lost until I remembered to use tire B. That had an important contribution to make to the action.

When you finally get an answer, is it **consistent** with the data in the problem or life in general? An answer of 80 000 km average per tire is clearly nonsense. So is an answer that tells you the height of a mountain is 3 cm. If you do get inconsistency, then you need to go back and tidy up. What is the source of the inconsistency? Is it a wrong **assumption**? At the end of the day it's always worth checking to see that any assumption you made to keep things going **can be justified**.

Suprisingly **notation** too is often a key to solving a problem. For a start, you really don't want to use unwieldy numbers like 40 000 and 60 000. Instead use a and b and do the arithmetic when you have to. While changing to x_i's and y_i's may be difficult if you're not used to them, they will often lead to general results. In the discussion, if we replace i = A, B, C by i = $A_1, A_2, \ldots, A_n$ for n tires, the argument which previously led to $\frac{d}{3} = \frac{ab}{a+b}$ will lead to $\frac{d}{n} = \frac{ab}{a+b}$. No matter how many tires you have you can't do better than 24 000 km average! (You can also see that if $a = 50\ 000$ and $b = 70\ 000$ you can immediately produce the n-tire average.)

Toward the end, I talked about four, five and more tires. This is a **generalization**, the solution of which is given in the last paragraph. Generalizations are situations which contain the original problem as a special case. In the last paragraph we found $\frac{d}{n}$ for **any** value of n.

Put $n = 2$ and you get the original problem.

I also talked **extensions**. Extensions are problems similar to the original which are motivated by the original. The side-car problem with three tires is an extension of the two-tire problem. Extensions often lead to interesting results too.

There are clearly more heuristics that are worth learning. Read any of George Polya's books for in-depth discussions.

Bibliography

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