# *Packing Boxes with N-tetracubes*

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## Introduction

With the popularity of the video game *Tetris*, most people are aware of the five connected shapes formed of four unit squares joined edge-to-edge. They are called the I-, L-, N-, O- and T-tetrominoes, after the letter of the alphabet whose shapes they resemble. They form a subclass of the polyominoes, a favorite topic in research and recreational mathematics founded by Solomon Golomb.

Here is a problem from his classic treatise, *Polyominoes*. Is it possible to tile a rectangle with copies of a particular tetromino? Figure 1 shows that the answer is affirmative for four of the tetrominoes but negative for the N-tetromino, which cannot even fill up one side of a rectangle.

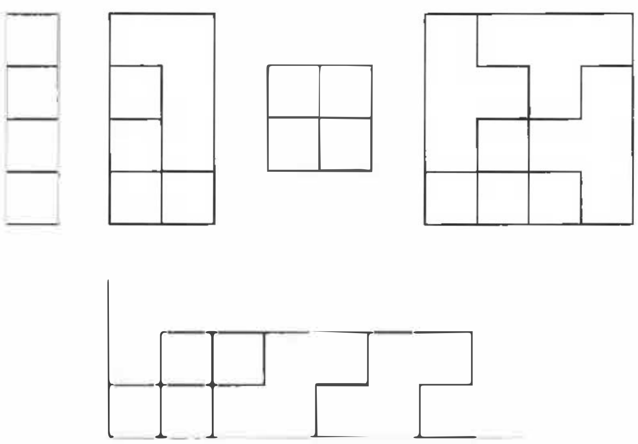

**Figure 1**

Getting off the plane into space, we can join unit cubes face-to-face to form polycubes. By adding unit thickness to the tetrominoes, we get five tetracubes, but there are three others. They are shown in Figure 2, along with the N-tetracube.

Is it possible to pack a rectangular block, or box, with copies of a particular tetracube? The answer is obviously affirmative for the I- L- O- and T-tetracubes, and it is easy to see that two copies of each of the three tetracubes not derived from tetrominoes can pack a $2 \times 2 \times 2$ box. Will the N-tetracube be left out once again? Build as many copies of it as possible and experiment with them.

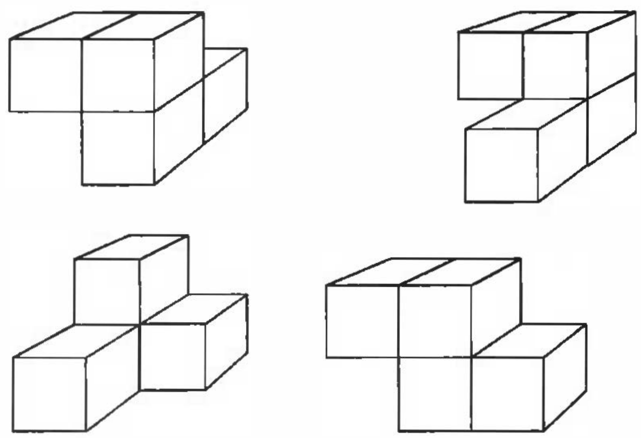

**Figure 2**

If the $k \times m \times n$ box can be packed with the N-tetracube, we call it an *N-box*. Are there any such boxes? Certain types may be dismissed immediately.

**Observation 1**

The $k \times m \times n$ box cannot be an N-box if it satisfies at least one of the following conditions:

(a) one of $k$, $m$ and $n$ is equal to 1;

(b) two of $k$, $m$ and $n$ are equal to 2;

(c) $kmn$ is not divisible by 4.

It follows that the $2 \times 3 \times 4$ box is the smallest box which may be an N-box. Figure 3 shows that this is in fact the case. The box is drawn in two layers, and two dominoes with identical labels form a single N-tetracube.

So there is life in this universe after all! The main problem is to find all N-boxes.

## N-cubes

If $k = m = n$, the $k \times m \times n$ box is called the *k-cube*, and a cube which can be packed by the N-tetracube is called an *N-cube*. We can easily assemble the 12-cube with the $2 \times 3 \times 4$ N-box, which makes it an N-cube. This is a special case of the following result.

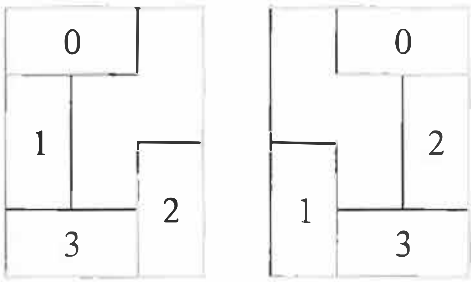


**Figure 3**

**Observation 2**

Suppose the $k \times m \times n$ and $l \times m \times n$ boxes are N-boxes. Let $a$, $b$ and $c$ be any positive integers. Then the following are also N-boxes:

(a) $(k + l) \times m \times n$;

(b) $ak \times bm \times cn$.

The 12-cube is not the smallest N-cube. By Observation 1, the first candidate is $k = 4$. It turns out that this is indeed an N-cube. It can be assembled from the $2 \times 4 \times 4$ N-box, whose construction is shown in Figure 4.

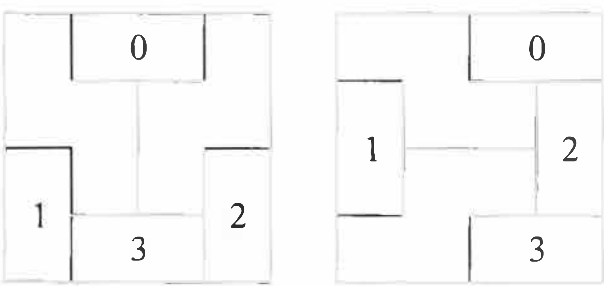


**Figure 4**

The next candidate, the 6-cube, is also an N-cube, but a packing is not that easy to find. In Figure 5, we begin with a packing of a $2 \times 6 \times 6$ box, with a $1 \times 2 \times 4$ box attached to it. To complete a packing of the 6-cube, add a $2 \times 3 \times 4$ N-box on top of the small box, flank it with two $2 \times 4 \times 4$ N-boxes and finally add two more $2 \times 3 \times 4$ N-boxes.

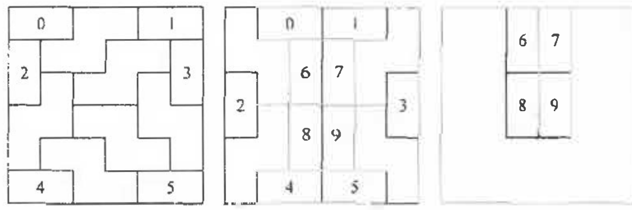


**Figure 5**

Can we pack the 8-cube, the 10-cube, or others? It would appear that as size increases, it is more likely that we would have an N-cube. However, it is time to stop considering one case at a time. We present a recursive construction which expands N-cubes into larger ones by adding certain N-boxes.

**Theorem 1**

The $k$-cube is an N-cube if and only if $k$ is an even integer greater than 2.

**Proof**

That this condition is necessary follows from Observation 1. We now prove that it is sufficient by establishing the fact that if the $k$-cube is an N-cube, then so is the $(k + 4)$-cube. We can then start from either the 4-cube or the 6-cube and assemble all others.

From the $2 \times 3 \times 4$ and $2 \times 4 \times 4$ N-boxes, we can assemble all $4 \times m \times n$ boxes for all even $m,n \geq 4$, via Observation 2. By attaching appropriate N-boxes from this collection, we can enlarge the $k$-cube first to the $(k + 4) \times k \times k$ box, then the $(k + 4) \times (k + 4) \times k$ box and finally the $(k + 4)$-cube. This completes the proof of Theorem 1.

## Further Necessary Conditions

Observation 1 contains some trivial necessary conditions for a box to be an N-box. We now prove two stronger results, one of which supercedes (c) in Observation 1.

**Lemma 1**

The $k \times m \times n$ box is not an N-box if at least two of $k$, $m$ and $n$ are odd.

**Proof**

We may assume that m and n are odd. Place the box so that the horizontal cross-section is an $m \times n$ rectangle. Label the layers $L_1$ to $L_k$ from bottom to top. Color the unit cubes in checkerboard fashion, so that in any two which share a common face, one is black and the other is white. We may assume that the unit cubes at the bottom corners are black. It follows that $L_i$ has one more black unit cube than white if $i$ is odd, and one more white unit cube than black if $i$ is even.

Suppose to the contrary that we have a packing of the box. We will call an N-tetracube *vertical* if it intersects three layers. Note that the intersection of a layer with any N-tetracube which is not vertical consists of two or four unit cubes, with an equal number in black and white. The intersection of a vertical N-tetracube with its middle layer consists of one unit cube of each color.

Since $L_1$ has a surplus of one black unit cube, it must intersect $l_1$ vertical N-tetracubes in white and $l_1 + 1$ vertical N-tetracubes in black, for some non-negative integer $l_1$. These N-tetracubes intersect $L_3$ in $l_1$ black unit cubes and $l_1 + 1$ white ones. Hence the remaining part of $L_3$ has a surplus of two black. They can only be packed with $l_3$ vertical N-tetracubes intersecting $L_3$ in white, and $l_3 + 2$ vertical N-tetracubes in black, for some non-negative integer $l_3$. However, the surplus in black unit cubes in $L_5$ is now three, and this surplus must continue to grow. Thus the $k \times m \times n$ box cannot be packed with the N-tetracube. This completes the proof of Lemma 1.

**Lemma 2**

The $k \times m \times n$ box is not an N-box if kmn is not divisible by 8.

**Proof**

Suppose a $k \times m \times n$ box is an N-box. In view of Lemma 1, we may assume that at least two of $k$, $m$ and $n$ are even. Place the packed box so that the horizontal cross-section is an $m \times n$ rectangle, and label the layers $L_1$ to $L_k$ from bottom to top. Define vertical N-tetracubes as in Lemma 1 and denote by $t_i$ the total number of those that intersect $L_i$, $L_{i+1}$ and $L_{i+2}$, $1 \leq i \leq k-2$.

Since at least one of $m$ and $n$ is even, each layer has an even number of unit cubes. It follows easily that each $t_i$ must be even, so that the total number of vertical N-tetracubes is also even. The same conclusion can be reached if we place the box in either of the other two non-equivalent orientations. Hence the total number of N-tetracubes must be even, and $kmn$ must be divisible by 8. This completes the proof of Lemma 2.

## N-Boxes of Height 2

We now consider $2 \times m \times n$ boxes. By Observation 1, $m \geq 3$ and $n \geq 3$. By Lemma 2, $mn$ is divisible by 4. We may assume that $m$ is even. First let $m = 4$. We already know that the $2 \times 4 \times 3$ and $2 \times 4 \times 4$ boxes are N-boxes. Figure 6 shows that so is the $2 \times 4 \times 5$ box.

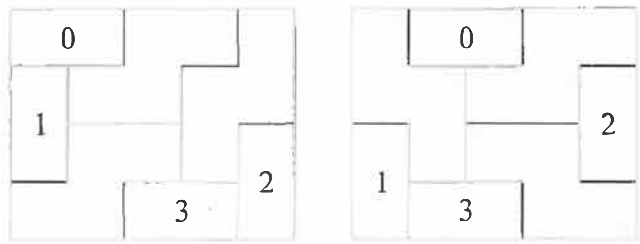


**Figure 6**

If the $2 \times 4 \times n$ box is an N-box, then so is the $2 \times 4 \times (n + 3)$ box by Observation 2. It follows that the $2 \times 4 \times n$ box is an N-box for all $n \geq 3$.

Now let $m = 6$. Then $n$ is even. We already know that the $2 \times 6 \times 4$ box is an N-box. However, the $2 \times 6 \times 6$ box is not. Our proof consists of a long case-analysis, and we omit the details. On the other hand, the $2 \times 6 \times 10$ box is an N-box. In Figure 7, we begin with the packing of a $2 \times 3 \times 6$ box with a $2 \times 2 \times 3$ box attached to it. We then build the mirror image of this solid and complete the packing of the $2 \times 6 \times 10$ box by adding a $2 \times 4 \times 3$ N-box.

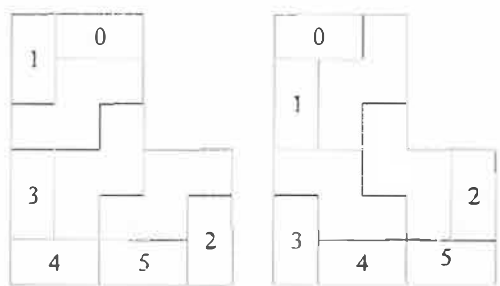


**Figure 7**

If the 2 ´ 6 ´ (n + 4) box by observation 2. It follows that the 2 ´ 6 ´ n box is an N-box for n = 4 and all even n ≥ 8.

**Theorem 2**

The $2 \times m \times n$ box is an N-box if and only if $m \geq 3$, $n \geq 3$ and $mn$ is divisible by 4, except for the $2 \times 6 \times 6$ box.

**Proof**

If $m$ is divisible by 4, the result follows immediately from Observation 2. Let $m = 4l + 2$ for some positive integer $l$. We already know that the $2 \times 10 \times 6$ box is an N-box. If the $2 \times (4l + 2) \times n$ box is an N-box, then so is the $2 \times (4l + 6) \times n$ box by Observation 2. This completes the proof of Theorem 2.

## The Main Result

**Theorem 3**

The $k \times m \times n$ box is an N-box if and only if it satisfies all of the following conditions:

(a) $k \geq 2$;

(b) $m \geq 3$;

(c) at least two of $k$, $m$ and $n$ are even;

(d) $kmn$ is divisible by 8;

(e) $(k, m, n) \neq (2, 6, 6)$.

**Proof**

Necessity has already been established, and we deal with sufficiency. We may assume that $k$, $m$ and $n$ are all at least 3, since we have taken care of N-boxes of height 2. Consider all $3 \times m \times n$ boxes. By (c), both $m$ and $n$ must be even. By (d), one of them is divisible by 4. All such boxes can be assembled from the $3 \times 2 \times 4$ box.

Consider now the $k \times m \times n$ box. We may assume that $m$ and $n$ are even. If $k$ is odd, then one of $m$ and $n$ is divisible by 4. Slice this box into one $3 \times m \times n$ box and a number of $2 \times m \times n$ boxes. Since these are all N-boxes, so is the $k \times m \times n$ box.

Suppose $k$ is even. Slice this box into a number $2 \times m \times n$ boxes, each of which is an N-box unless $m = n = 6$. The $4 \times 6 \times 6$ box may be assembled from the $4 \times 2 \times 3$ box, and we already know that the 6-cube is an N-cube. If the $k \times 6 \times 6$ box is an N-box, then so is the $(k + 4) \times 6 \times 6$ box. This completes the proof of Theorem 3.

## Research Projects

**Problem 1.**

Try to prove that the $2 \times 6 \times 6$ box is not an N-box. It is unlikely that any elegant solution exists.

**Problem 2**

An N-box which cannot be assembled from smaller N-boxes, is called a *prime* N-box. Find all prime N-boxes.

**Problem 3.**

Prove or disprove that an N-box cannot be packed if we replace one of the N-tetracubes by an O-tetracube.

**Problem 4.**

For each of the other seven tetracubes, find all boxes which it can pack.

## Bibliography

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