How to Flip without Flipping

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A red triangular cardboard is lying on a desk. We wish to get a physical copy of its mirror image. The simplest way is to flip the cardboard over. However, we discover that it is plain on the other side, and we want a red copy of the mirror image. We are allowed to use straight cuts to dissect the triangle into pieces for reassembly. The problem is to minimize the number of pieces.

No dissection is necessary if the triangle is isosceles, as this is the only class of triangle for which the problem can be solved in one piece. On the other hand, three pieces are sufficient for any triangle, as illustrated in Figure 1.

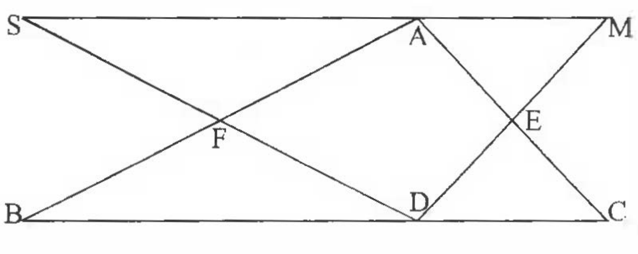

Figure 1

Let ABC be any triangle with BC its longest side. Then the foot D of the perpendicular AD from A lies between B and C. Make the cuts along DE and DF, where E is the midpoint of AC and F the midpoint of AB. Then $AE = DE = CE$ and $AF = DF = BF$. We can rotate CDE about E into AME and BDF about F into ASF. It follows easily that DSM is a mirror image of ABC.

The main part of our investigation is to determine all triangles for which the problem is solvable in two pieces. In other words, only one straight cut is allowed. This class trivially includes all isosceles triangles which we will ignore for now. Then there are two cases, according to the position of the cut.

Case 1

The cut passes through a vertex of the triangle, which is dissected into two isosceles triangles. If we overlay the original triangle on its mirror image so that the triangular pieces with the largest interior angles coincide, we obtain Figure 2.

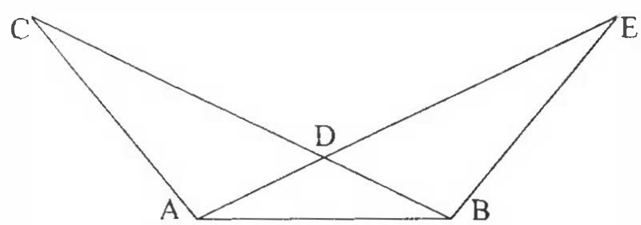

Figure 2

Being the largest interior angle, $\angle ADB$ is greater than $\angle ADC$, so that it must be obtuse. In order for BAD to be an isosceles triangle, we must have ∠BAD = ∠ABD. Denote their common value by θ. Then ∠ADC = 2θ. There are three ways in which CAD may become an isosceles triangle.

Subcase 1(a)

Here ∠ACD = 2θ. Then ∠CAD = 180° - 4θ > 0°. This class consists of all triangles in which two of the angles are in the ratio 1:2, where the smaller angle θ satisfies 0° < θ < 45°.

Subcase 1(b)

Here, ∠CAD = 2θ. Then ∠CAB = 3θ and ∠ ACD = 180° - 4θ > 0°. This class consists of all triangles in which two of the angles are in the ratio 1:3, where the smaller angle θ satisfies 0° < θ < 45°.

Subcase 1(c)

Here, ∠ACD = ∠CAD. Then their common value is 90° - θ so that we have ∠CAB = 90°. This class consists of all right triangles.

Case 2

The cut does not pass through any vertex of the triangle, which is dissected into a kite and an isosceles triangle. If we overlay the original triangle on its mirror image so that the kite pieces coincide, we obtain Figure 3.

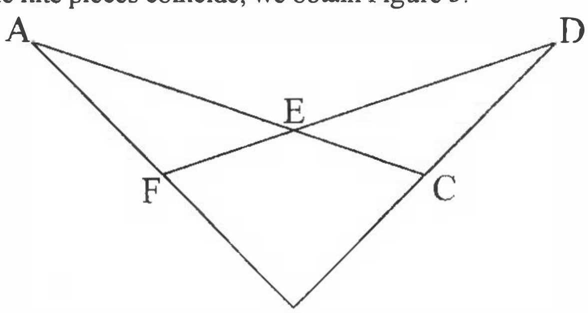

Figure 3

We must have *BC* = *BF* and *CE* = *EF*. Note that we cannot possibly have *AE* = *EF* = *CE* = *DE* as otherwise *ADCF* would be a rectangle, and *AF* will not meet *DC*. Hence there are only two ways in which *AEF* may become an isosceles triangle. Denote the common value of ∠*AEF* = ∠*DEC* by θ.

Subcase 2(a)

Here ∠CAB = θ. Then ∠BCA = 2θ and ∠ ABC = 180° - 3θ > 0°. This class consists of all triangles in which two of the angles are in the ratio 1:2, where the smaller angle θ satisfies 0° < θ < 60°, and contains as a subclass all triangles under Subcase 1(a).

Subcase 2(b)

Here ∠AFE = θ. Then ∠CAB = 180° - 2θ > 0°, ∠BCA = 180° - θ and ∠ABC = 3θ - 180° > 0°. This class consists of all triangles of the form (180° - θ, 180° - 2θ, 3θ - 180°), where 60° < θ < 90°.

We summarize our finding in the following statement.

Theorem

A triangle may be dissected by a straight cut into two pieces which can be reassembled into a mirror image of the original triangle if and only if the triangle satisfies at least one of the following conditions:

- it is isosceles;
- it has a right angle;
- two of its angles are in the ratio 1:2, with the smaller angle θ satisfying 0°< θ < 60°;
- two of its angles are in the ratio 1:3, with the smaller angle θ satisfying 0° < θ < 45°;
- it is of the form (180° - θ, 180° - 2θ, 3θ - 180°), where 60° < θ < 90°.

Supplementary Problem

A non-isosceles triangle is dissected by a straight cut into two pieces which can be assembled, without turning either piece over, into a mirror image of the original triangle, in three different ways. Find all possible values of the angles of such a triangle.