A Tale of Two Cities

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In the summer of 1993, I went from Graz, Austria to Beloretsk, Russia, for the International Mathematics Tournament of the Towns Problem-solving Workshop. For an account of the event, see [1].

I was the top prize winner for my solution of one of the five problems posed. Part of my work has been reported in [2]. Here are some further results. To set the scene, let me restate the problem.

The road network of a certain city consists of a continuous chain of circles. At the point of tangency of two adjacent circles, the roads cross over as shown in Figure 1, which illustrates the case with four circles.

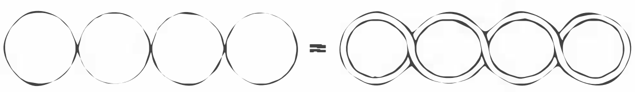

Figure 1

A ring road is constructed around the city, and is integrated with the inner chain at various points. At each integration point, the ring road is crossed over with the inner chain as shown in Figure 2, which illustrates the case with two integration points.

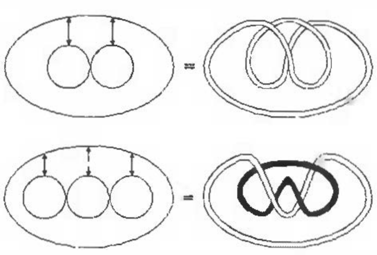

Figure 2

Note that the first integrated network consists of only one component while the second one consists of two mutually inaccessible components. We call integrated networks like the former "regular" and those like the latter "irregular".

In [2], it was proved that for studying regularity, we may assume that all integrated networks have the following properties:

- Each circle in the inner chain has at most one integration point.
- Each circle at either end of the inner chain has an integration point.
- All integration points are on the north side of the ring road.

Integrated networks that have these properties are called normalized integration networks, abbreviated to NINs. Both examples in Figure 2 are NINs. A necessary and sufficient condition for regularity was established in [2] for them. Henceforth, we restrict our attention to NINs.

Suppose two cities with regular NINs decide to merge. Their road networks are combined as follows. The inner chains are attached end-to-end, and the two ring roads are replaced by a common one, with all integration points preserved.

Figure 3 shows two examples. The new NIN is regular in one case and irregular in the other. The problem is to determine when such a merger leads to a regular NIN. Of course, we can apply the characterization in [2] afterwards, but we would like to tell beforehand just from the properties of the individual NINs.

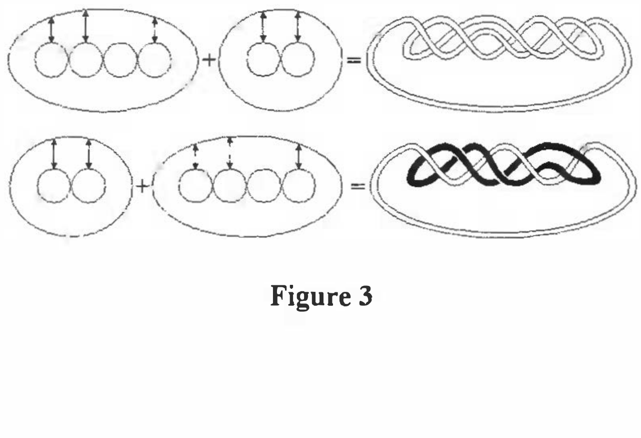

Figure 3

Consider the regular NIN in Figure 4. Each of the two circles is divided by their point of tangency and the integration points into two arcs. The latter divide the ring road into two segments. The arrows indicate the direction of travel on each arc and segment as we go once round the NIN.

We define the orientation of an arc of a circle as follows: an arc is "positive" if it is traversed in the same direction as the segment of the ring road closest to it, and "negative" otherwise. All arcs in Figure 4 are negative.

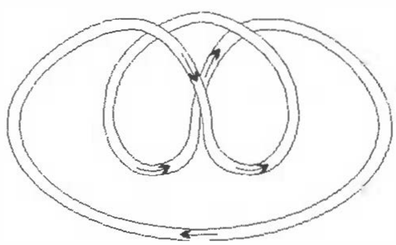

Figure 4

Theorem 1

All arcs on the same circle have the same orientation.

Proof

If the inner chain has only one circle, the whole circle is a single arc, and the result is trivial. Hence we may assume that the inner chain has at least two circles. We first observe that two arcs separated by an integration point have the same orientation. Figure 5 shows the only two non-equivalent configurations. Both arcs in either case have the same orientation.

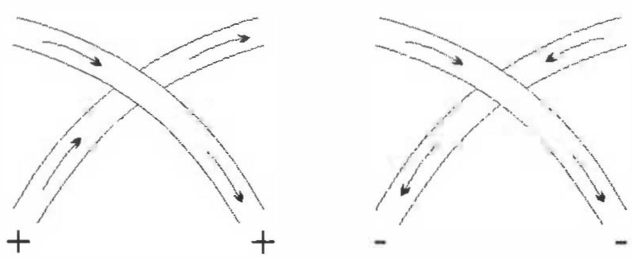

Figure 5

If we number the circles consecutively, we can prove the desired result inductively along the inner chain, starting from the circle at the west end. It consists of two arcs separated by an integration point, and we have already pointed out that they have the same orientation.

Suppose the result holds for a particular circle. Consider the one to the east. Figure 6 shows the four non-equivalent configurations. In each case, the two arcs on the next circle separated by the point of tangency have the same orientation.

If this circle has only two arcs, the desired result is established for it. If it has a third arc, it must be separated from one of the other two by an integration point. From our earlier remark, we know that it also has the same orientation. This completes the proof of Theorem 1.

By virtue of Theorem 1, we can now define the orientation of a circle of the inner chain as that of any of its arcs. The following follows from the observation that the first configuration in Figure 6 is actually impossible.

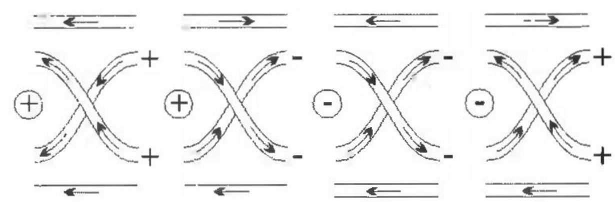

Figure 6

Corollary

Two positive circles cannot be tangent to each other.

Theorem 2

A merger of two regular NINs yields a regular NIN if and only if the orientations of the two circles brought into contact are the same.

Proof

Figure 7 shows a situation where a positive circle at the east end of the inner chain of one regular NIN is brought into contact with a negative circle at the west end of another. Of the four loose ends in the first NIN, the direction of travel forces A and C to be connected to B and D.

Since the NIN is regular, A must be connected to D and C connected to B. Similarly, E must be connected to G and F connected to H in the second NIN. After the merger, one component of the new NIN goes from A via F to H and returns via D. It is inaccessible from the component which goes from C via G to E and returns via B. Hence it is irregular.

Figures 8 and 9 show the cases where circles with the same orientation are brought together. Analogous arguments show that these mergers always yield regular NINs. This completes the proof of Theorem 2.

Note that in Figures 8 and 9, the circles which are brought into contact become negative circles in the new regular NIN. This yields the following result on the inverse operation of merging.

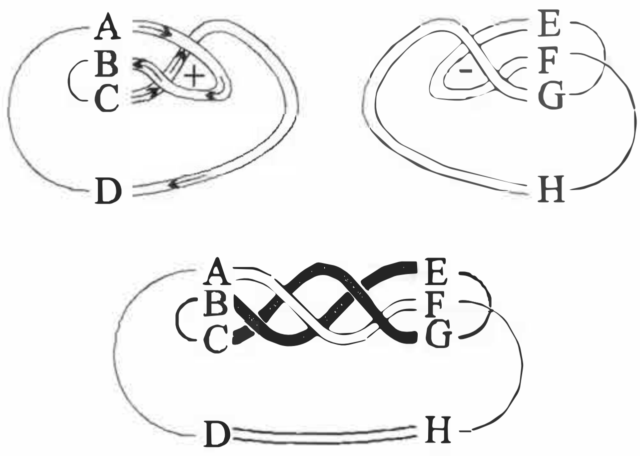

Figure 7

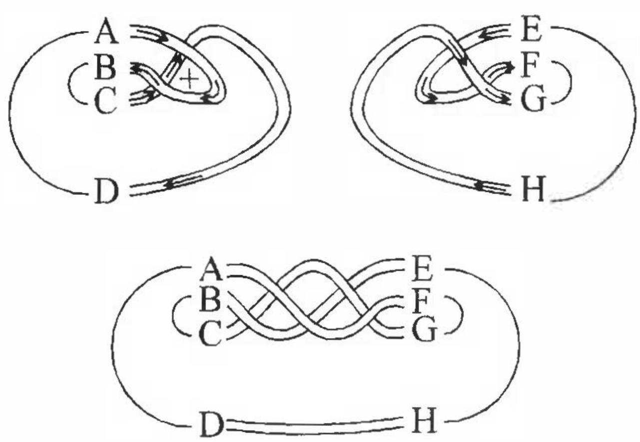

Figure 8

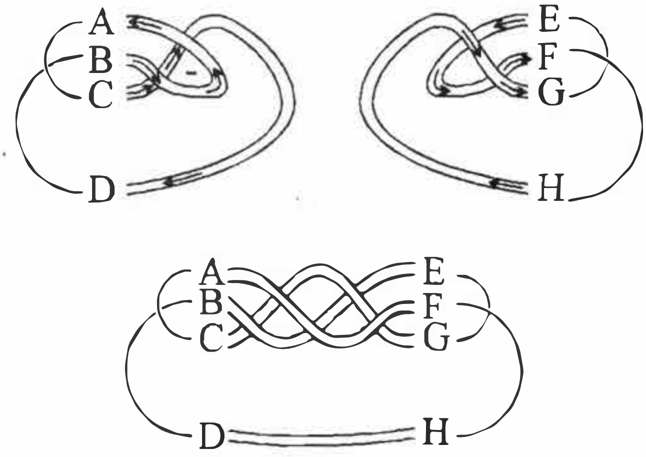

Figure 9

Corollary

A regular NIN may be detached to form two regular NINs if detachment occurs between two negative circles each with an integration point.

Bibliography

Liu, Andy. "A Mathematical Journey." *Crux Mathematicorum*, 20, 1994: 1 - 5.

Liu, Andy. "A Tale of One City." *Quantum, 4* May/June 1994: 50 - 52.

Supplementary Problem

In a regular normalized integrated network, a positive circle can have only negative circles as neighbors, but a negative circle may have neighbors of either kind. Find a necessary and sufficient condition for a negative circle to have one neighbor of each kind.