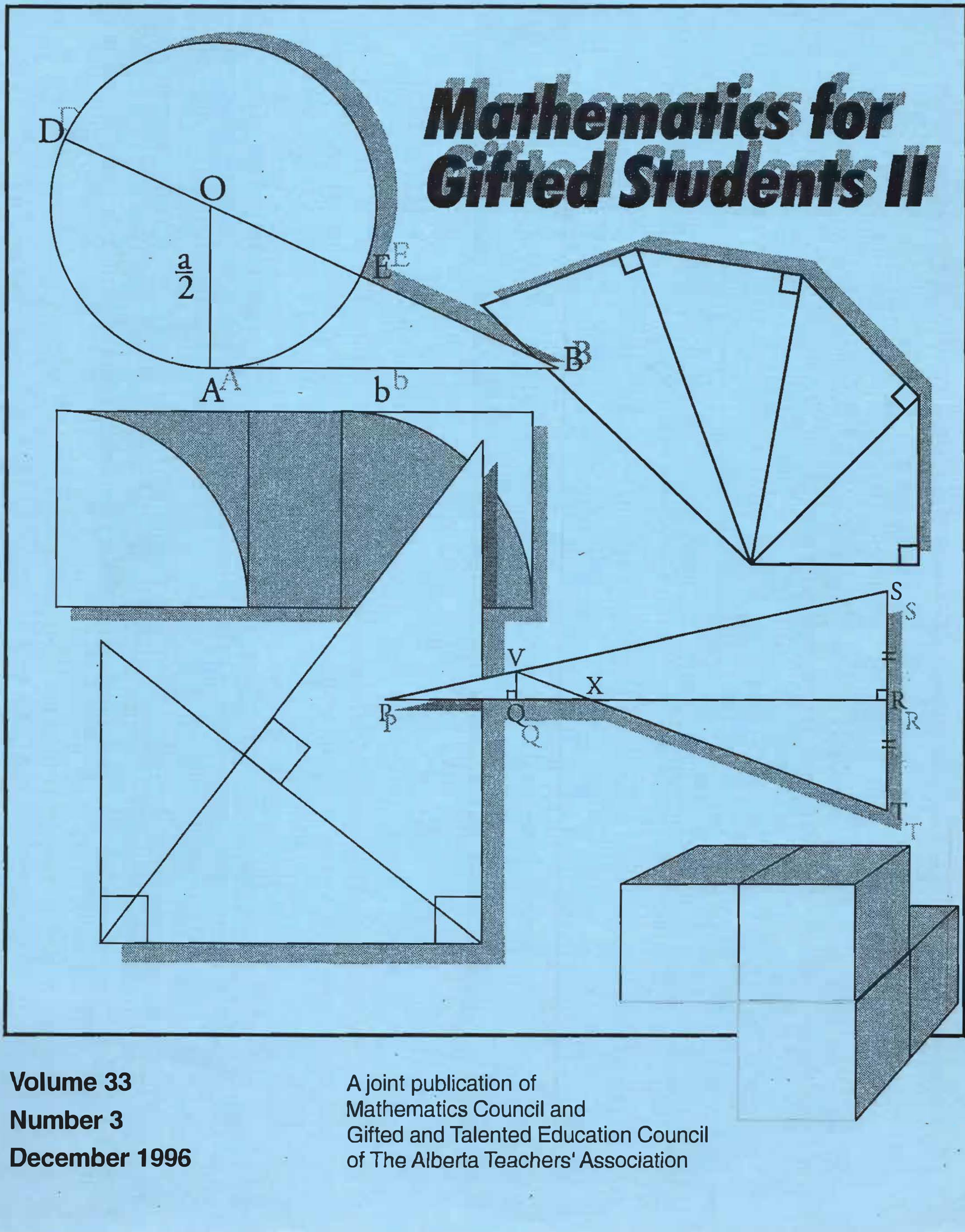
delta-k

Special Edition

Mathematics for Gifted Students II



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The journal seeks to stimulate thinking, to explore new ideas and to offer various viewpoints. It serves to promote MCATA's convictions about mathematics education.

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Pak-Hong Cheung graduated in 1979 with a BSc in Mathematics and Physics from the University of Hong Kong, where he later also received a Certificate and an Advanced Diploma in Education. In 1991 he was awarded the degree of MSocSc in Applied Statistics, also by the University of Hong Kong. He taught in secondary schools for eleven years. In 1990 he was appointed a lecturer in Mathematics Education in the Department of Curriculum Studies of the University of Hong Kong. His interests include problem-solving, use of Macintosh in mathematics education and publishing in the World Wide Web.

Zonghu Qiu is a professor in the Institute of System Sciences, Academia Sinica in Beijing. Before 1980, he worked on operations research and mathematical programming in the Institute of Mathematics. His involvement with mathematics competitions in China dates back to 1978. Since 1980, he has been responsible for organizing and setting problems for mathematics competitions hosted by the Chinese Mathematical Society, which include annual national leagues for middle and high schools, the Chinese Primary School Mathematical Olympiad and the Chinese Mathematical Olympiad. He is also interested in recreational mathematics and has written articles and books in this area.

Derek Holton combines research in graph theory and mathematics education with a host of activities including running school math camps, founding a national mathematics competition, leading the New Zealand team at various IMOs, and being a consultant for a TV game show. His book *Let's Solve Some Math Problems* is published by the Canadian Mathematics Competition. A new book written by Derek in collaboration by Peter Hilton, and Jean Pedersen is called *Mathematical Reflections* and is to be published by Springer-Verlag toward the end of this year.

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Guest Editor's Comments

I am very pleased that *Mathematics for Gifted Students*, which appeared in 1989, was very well received, and grateful that the Alberta Teachers' Association has entrusted me with the editing of this sequel. Of the five articles in **MfGS1**, two were from North America and three from Western Europe. For this edition, I have solicited articles from South America, Asia, Oceania, Eastern Europe and Africa.

Maria de Losada reminds us of the long and glorious tradition of mathematics. With the information revolution in full rage, we seem to be forever going back to the future. However, a journey to the past can be equally rewarding in the general quest for knowledge as well as in the particular context of mathematics competitions.

Pak-Hong Cheung and **Zonghu Qiu** describe the elaborate structure for and meticulous care in the construction of competition problems for elementary school students. The appropriateness of contests for students in this age group has often been debated. The authors highlight positive aspects of competition and also identify some pitfalls to avoid.

Derek Holton takes the reader on a personal tour of problem-solving, with introspective and humorous commentary. **Andris Cibulis** presents a fascinating problem on building blocks which young children may be encouraged to explore, though the complete solution is far from being easy. **John Webb** gives a behind-the-scenes look at South Africa's triumphant entry into the International Mathematical Olympiad.

I would like to thank my friends and colleagues who have brought a global perspective to mathematics education to Alberta. The student projects in Appendix I show that the process is not a one-way street: our young people can also make significant contributions.

Andy Liu

History - A Great Source of Challenging Problems

Maria de Losada

Finding challenging but "elementary" problems for young people can be a challenge in itself. Many creative mathematicians generate new and exciting problems, some specializing from their research, some recreating personal problem-solving experiences, and others simply producing great new ideas for young people.

Problems not requiring advanced ideas but that test problem-solving abilities or require creative thought, are the lifeblood of mathematical challenges and enrichment programs in general. While the sources we have already mentioned are vital, many such problems can be found in the research questions of the past.

For experts dedicated to mathematical challenges, searching historical material for inspiration is an established practice. In the Hungarian Problem Books (English translation published by the MAA), the use of historical references made routinely by József Kürschák is particularly striking. Although the majority of the methods and theorems cited there form part of the mainstream of mathematical thought and are, therefore, quite well-known, it is clear that the history of mathematics was indeed one of his most prized sources.

In what follows we will try to show a variety of ways in which problems from the past can make good material for enrichment and mathematical challenges, be these in the style of multiple choice, short-answer or requiring full solution and proof. Our remarks will show how some of the historical material can be or is being used. However in order to find problems tailor-made for the students' interests and the lines of research that they naturally would wish to follow, teachers must rely on firsthand contact with the history of mathematics.

Some suggestions of sources for multiple-choice type problems

Some early recreational problems that can be found in the Greek Anthology and the Liber Abaci of Fibonacci are still a source of good material for enrichment and popular challenges, although many have filtered into everyday methods and textbooks (the famous "rule of three," for example) and become commonplace.

Well suited to multiple-choice competition problems are those involving a variety of diophantine equations, based on vintage problems such as these:

- Diophantus (II-III Century a.d., Book I, Problem 16). Find three numbers such that when they are added in pairs, the sums are 20, 30 and 40.

Teachers should certainly try to invent their own version of this problem. It has appeared, and continues to appear, in various guises in mathematical challenges. The following is an especially fanciful British version, posed in the First UK School Mathematics Challenge:

- Weighing the baby at the clinic was a problem. The baby would not keep still and caused the scale to wobble. So I held the baby and stood on the scale while the nurse read off 78 kg. Then the nurse held the baby while I read off 69 kg. Finally, I held the nurse while the baby read off 137 kg. What would be the combined weight, in kilograms, of nurse, baby and me?

  (a) 142 (b) 147 (c) 206 (d) 215 (e) 284

- Christoff Rudolff (1526). Find the number of men, women and children in a company of 20 persons, if together they pay 20 coins, each man paying 3, each woman 2 and each child ½.

The following problems, although they come from three different mathematical cultures and traditions nevertheless address the same question.

- Yen Kung (1372). We have an unknown number of coins. If you make 77 strings of them, you are 50 coins short; but if you make 78 strings it is exact. How many coins are there?
- Bhaskara (1150). What number when divided by 6 leaves a remainder of 5, when divided by 5 leaves a remainder of 4, divided by 4 leaves a remainder of 3, and divided by 3 leaves a remainder of 2?

- Fibonacci (1202). Find a multiple of 7 having remainders 1, 2, 3, 4, 5 when divided by 2, 3, 4, 5, 6.

The details of the solutions of some of these problems make them long and messy. Thus a numerical answer included among the various solutions makes checking easy. A slight modification in the question, such as asking for the sum of the two smallest numbers with the given property, requires the development of an essentially complete solution, making the problem especially suitable for multiple-choice or short-answer challenges. Below is the approach taken in the 1985 Australian Mathematics Competition, Junior Division:

- The smallest positive integer which, when divided by 6 gives remainder 1 and when divided by 11 gives remainder 6, is in the range

(a) 115 to 120 (b) 90 to 95 (c) 125 to 130
(d) 60 to 65 (e) 35 to 40

Although somewhat overworked at present, these are still nice little problems for beginners.

Problems suitable for short answers or easy proofs

A class of problems appropriate for a short-answer format are those that concern spending money in intricate ways. The following example can be found in Fibonacci. A merchant doing business in Lucca doubled his money there and then spent 12 denarii. On leaving he went to Florence, where he also doubled his money and then spent 12 denarii. Returning home to Pisa, he there doubled his money and again spent 12 denarii, nothing remaining. How much did he have at the beginning?

This is representative of problems in which the well-known strategy of working backwards is tested and promoted among young scholars and "aficionados" of mathematics.

Many results in geometry from the past constitute excellent moderately difficult problems, that can be adapted to short-answer challenges. The following deals with the harmonic mean $\frac{2ab}{a+b}$ of two quantities a and b.

In his Mathematical Collection, Pappus constructed the harmonic mean of two segments PQ and PR as follows. On the perpendicular to PR at R construct and let the perpendicular to PR at Q meet PS at V. Draw VT and let X be the point of intersection of VT and PR. Then X is the harmonic mean of PR and PQ.

This construction can form the basis of a problem that asks the direct question of expressing PX in terms of PQ and PR, or it can be tailored to other questions such as proving that X is independent of the length of RT.

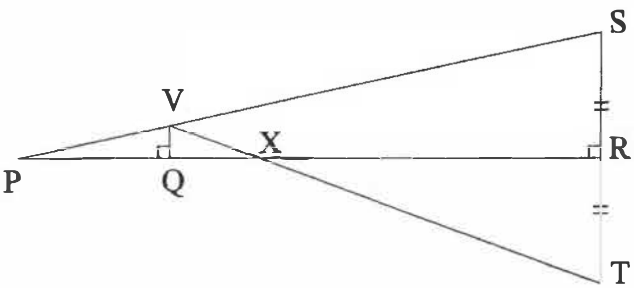

Figure 1

It is known (as told in the *Theatetus*) that the mathematics tutor of Plato, Theodorus of Cyrene, proved the irrationality of $\sqrt{3}, \sqrt{5}, \ldots, \sqrt{17}$. Theodorus had shown how to construct a segment of length $\sqrt{n}$ as the leg of a right triangle with hypotenuse $\frac{n+1}{2}$ and second leg $\frac{n-1}{2}$. And it is but a short step from here to Plato's formulas for Pythagorean triples. Now this can be used to construct a series of right triangles with a common vertex such that the length of the leg opposite the common vertex in each case is 1. The hypotenuse of the nth triangle in this sequence has length $\sqrt{n+1}$.

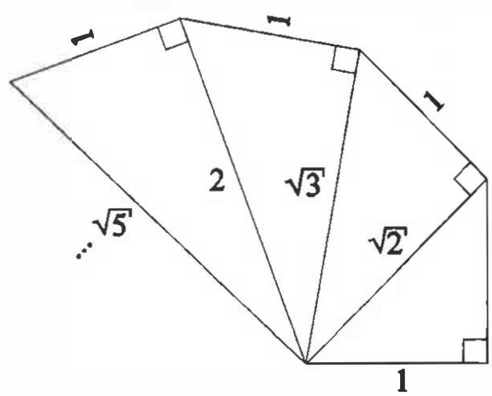

Figure 2

Questions that can be posed relative to the situation described abound. It can even be used to explain why Theodorus stopped his proofs of irrationality at $\sqrt{17}$. We invite our readers to think up two or three.

The classical problem of finding the side a of a cube with twice the volume of a given cube of side b may be reduced to that of finding two mean proportionals c and d between a and b, in other words, $\frac{a}{c}=\frac{c}{d}=\frac{d}{b}$. This is attributed to Hypocrates of Chios. Many of the attempts to produce a ruler and compass construction based on this idea lead to figures that can be used to pose good problems.

Consider a construction attributed to Plato. Let *ABC* and *DBC* be two right triangles lying on the same side of their common side *BC*. Furthermore let hypotenuses *AC* and *BD* meet at right angles at point *P* such that $AP = a$ and $DP = 2a$. It is clear that *BP* and *CP* are two mean proportionals between a and $2a$.

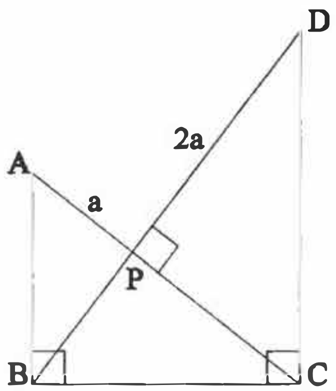

Figure 3

Now, given the construction as described, we can ask students to prove this fact, or we can elaborate on the problem in a variety of ways. For example, we may ask them to find the area of triangle *BPC*.

To solve the quadratic equation $x^2 + ax = b^2$, Descartes used the following construction. Draw a line segment *AB* of length b and a circle of diameter a tangent to *AB* at *A*. Let *O* be the center of the circle. Finally draw BO cutting the circle in points *E* and *D*. It is clear that the length x of the segment *BE* satisfies the given equation.

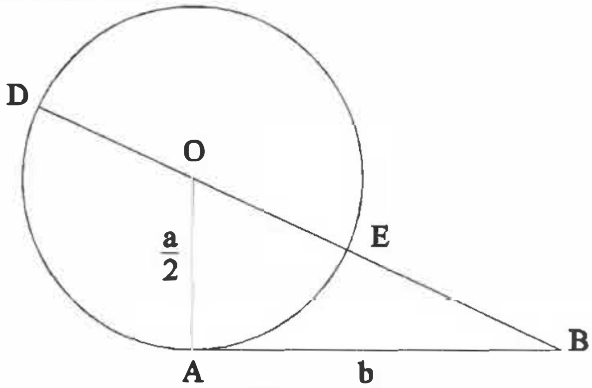

Figure 4

We can ask the student to prove this fact, a very easy task indeed. Or we can start from the geometric construction and ask the student to construct segments of lengths a and b so that the length x of a given segment *BE* is a root of the quadratic equation $x^2 + ax + = b^2$.

The point in all of these problems is to start with a relatively unknown but interesting construction of certain importance at one time in the history of mathematics. We then formulate questions that require the student to explore the special properties of the construction.

A good problem is adaptable

History gives us an enormous store of suitable problems in geometry that can be adapted to almost any challenge format or more generally to an enrichment program; and there is no doubt that Archimedes is a great source for challenges. Consider, for example, Archimedes' result regarding the "broken" chord. If *AB* and *BC* make up any "broken" chord in a circle (where $BC > AB$), and if *M* is the midpoint of the arc *ABC* and *MF* the perpendicular to the longer chord, then *F* is the midpoint of the broken chord. That is, $AB + BF = FC$.

An easy problem related to this result would ask for the measure of angle *BMF*, given the measures of angles *CBM* and *BAM*.

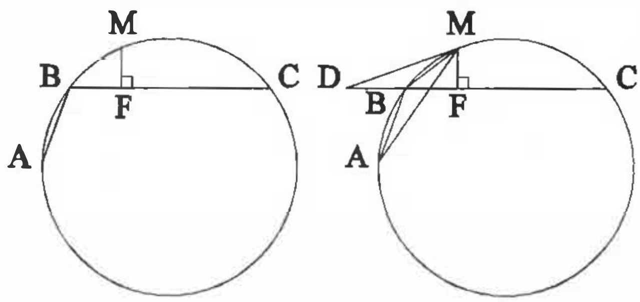

Figure 5

Now Archimedes proved his result by extending chord *CB* to *D* with $BD = BA$. A somewhat harder problem would give the lengths of *AB* and *BF* and ask for the length of *CF*. The observation that is required here is in essence the proof of Archimedes' result. Asking for that proof is also a good option (and has a nice solution using rotations). Finally, a problem can be formulated by stating several tentative properties of the figure and asking a question about which combinations of these are true (I and II only, I and III only, etc.).

Another problem studied by Archimedes concerns the "arbelos". This means the "shoemaker's knife", and refers to the curvilinear region in Figure 6 bounded by the three semicircles. Problems related to this figure, given on the cover of the late Sam Greitzer's journal of the same name, take this form:

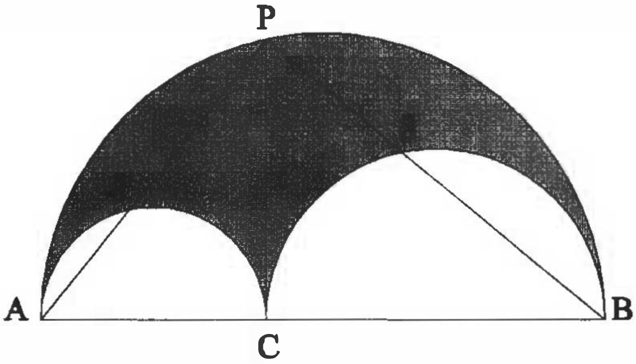

Figure 6

Segment AB is divided in two parts at C. On the same side of AB semicircles with diameters AB, AC and CB are drawn. If PC is perpendicular to AB at C and if R and S are the points of intersection of AP and BP with the respective semicircles, then show that

- RS is a common tangent of the two smaller semi-circles;
- PC and RS bisect one another;
- the total area of the arbelos is equal to the area of the circle with diameter PC;
- the circles inscribed in 'segments' ACP and BCP have equal radii.

Original problems start with Archimedes' results but go beyond them. For example, another problem concerning the arbelos can be found in the International Competition held in Luxembourg in 1980 (a year when there was no IMO) and attended by Luxembourg, Belgium, the Netherlands, Great Britain and the former Yugoslavia. There the problem posed was that of expressing the ratio of the areas of the triangles PRS and PAC in terms of the radii of the two smaller semicircles.

How can a teacher use this material? Obviously we can pose a problem giving numerical values for the radii and ask students to determine the length of PC, the area of triangle PAB, the area of the trapezoid with vertices R, S and the centres of the two smaller semicircles, etc. Or we can ask our students to prove one of the general properties of the arbelos or even to state and prove a property that has not been mentioned among our results. There is much room for interesting and even provocative mathematics here.

As a last Archimedean example, let us consider the following problem taken from the Book of Lemmas which we find appropriate as a problem requiring a full proof from the students.

Let AB be a diameter of a circle and t the tangent to the circle at B. From point P on t draw a second line tangent to the circle at D. Let F be the foot of the perpendicular dropped from D to AB and E the intersection of P and DF. Prove that $DE = EF$.

Simple problems from the past help solve/invent difficult problems for the present

In his book, *Liber Quadratorum*, Fibonacci posed this very simple problem: given the squares of three successive odd numbers, show that the largest square exceeds the middle square by eight more than the middle square exceeds the smallest.

And yet the result can be used to solve in part, or to invent, an olympiad problem at the international level like this one from the 1986 IMO in Poland.

Let d be any positive integer not equal to 2, 5 or 13. Show that it is possible to find two different numbers a and b belonging to the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

We leave it to our readers to discover the link between these two problems.

Methods lost

Other sources of inspiration for Olympiad problems include specialized tools originally developed to solve specialized problems, but never integrated into the mainstream of mathematical thought. Formerly important tools, like continued fractions, finite differences and many geometric methods, which have now been almost forgotten (at least in the basic secondary curriculum) can also serve as sources of inspiration.

The following is an example of a specialized tool. Consider the formula known as the "bloom of Thymaridas" given in a first-century manuscript of Iamblichus and judged to be copied from an ancient source. We have n unknown quantities $x_1, x_2, \ldots, x_n$ to which we add one more unknown quantity x. The following sums are given:

$$\begin{aligned} x + x_1 + x_2 + \cdots + x_n &= S \\ x + x_1 &= a_1 \\ x + x_2 &= a_2 \\ \cdots &= \cdots \\ x + x_n &= a_n \end{aligned}$$

Iamblichus gives a general solution to the problem that is equivalent to the formula

$$x = \frac{(a_1 + a_2 + \ldots + a_n) - S}{n - 1}$$

As van der Waerden tells us, this solution can be found using a well-known method of Diophantus which requires us to set $x = s$ and then observe that the remaining unknowns are $a_1 - s, a_2 - s, \ldots, a_n - s$ so that the sum S is given by

$$(a_1 + \ldots + a_n) - (n - 1)s.$$

Iamblichus uses this solution in a clever way to solve (diophantine) equations of somewhat different appearance, such as the system:

$$\begin{aligned} x + y &= 2(z + u) \\ x + z &= 3(y + u) \\ x + u &= 4(y + z) \end{aligned}$$

which constitutes an interesting problem; one of its solutions starts by putting these into the same form as the bloom of Thymaridas.

The given equations are equivalent to

$$\begin{aligned} x + y + z + u &= 3(u + v) \\ x + y + z + u &= 4(y + u) \\ x + y + z + u &= 5(y + z) \end{aligned}$$

Now the sum of the four numbers must be divisible by 3, 4 and 5. So Iamblichus sets $S = 120$ and arrives at the same equations as given in the bloom of Thymaridas as follows.

$$\begin{aligned} x + y + z + u &= & S &= 120 \\ x + y = 2(z + u) &= & \tfrac{2}{3}S &= 80 \\ x + z = 3(y + u) &= & \tfrac{3}{4}S &= 90 \\ x + u = 4(y + z) &= & \tfrac{4}{5}S &= 96 \end{aligned}$$

At this point the formula of Thymaridas can be applied. (Undoubtedly common elimination procedures will also do the job, but may not be as much fun nor as well focused.) This same problem reappeared in Fibonacci's *Liber Abaci* and was used this year in the Colombian Olympiad's final round for students from sixth to eighth grades, as transcribed below.

Consider this problem also from *Liber Abaci*. Three men, *A*, *B* and *C*, each of whom already has some gold coins, found a purse of gold coins. *A* said: If you give me all of the coins in the purse, I will have twice as many coins as *B* and *C* put together. *B* said: If you give me all of the coins in the purse, I will have three times as many coins as *A* and *C* together. Finally *C* said: If you give me all of the coins in the purse, I will have four times as many coins as *A* and *B* together. What is the least number of coins that can be in the purse?

A final thought

Ideas are our most important, and (fortunately) renewable resources. They can be used over and over, on varying levels of difficulty, and never be fully exhausted.

We hope that these few examples illustrate how ideas of the past can be put to use in the formation of young mathematics students of today and that teachers looking for challenging and intriguing problems will be able to discover in the history of mathematics a vast reserve that awaits their exploration.

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Primary School Mathematics Olympiad in China

Pak-Hong Cheung & Zonghu Qiu

Although mathematics competitions for high school students have occurred for more than a century, mathematics competitions for primary school students are still young. Large-scale mathematics competitions for primary school students in China date back to 1986. The Chinese Mathematical Society introduced the Primary School Mathematics Olympiad (hereafter referred to as the PSMO) in 1991. Since then we have witnessed both achievements and problems associated with these competitions. A question which we must answer is: how can mathematics competitions contribute to mathematics education in primary schools?

Some Principles

This year there are two million participants in the PSMO. Students, teachers and parents are very enthusiastic about it. There are two main reasons behind their enthusiasm:

- China continuously achieved good results in the IMO.
- Many prestigious middle schools welcome winners in the PSMO to study in their schools.

Obviously, these two factors cannot be the sole driving forces for mathematics competitions forever. We believe that the aim of mathematics competitions is to arouse students' interest and encourage them to study mathematics. It is inappropriate to give each winner too many awards or to label the winners as talented or gifted.

- A crucial part of a mathematics competition is to set problems that suit the ability of most participants. To develop genuine interest in mathematics among students, we must devote more effort to the setting of the problems. A good competition paper enables participants to demonstrate their abilities and ingenuity, which in turn influences classroom teaching in mathematics positively. With these objectives in mind, the Popularization Committee of the Chinese Mathematical Society laid down two rules on problem-setting for the PSMO: Knowledge required for answering PSMO problems must be within the syllabuses for primary school mathematics.
- Every PSMO problem must have a simple arithmetic solution.

We should motivate primary school students to think rather than to know more facts in mathematics. We must not include secondary school mathematics in mathematics competitions for primary school students. In particular, the solution of applications problems using equations should be discouraged at this level. Applications problems, if included, must have simple and quick arithmetic solutions. The PSMO emphasizes familiarity with the basic operations in arithmetic and, building upon this, the ability to skillfully solve some problems.

Problem-setting is a creative task and original problems are most welcome. Setting new problems without increasing the difficulty of the competition paper is a formidable task. However, problem setters must work toward the goal of setting interesting problems using words understandable by primary school students.

Levels of Difficulty

At present, the determination of the difficulty of PSMO problems is based on the experience of the setters, so the judgment is not always reliable. Problem setters must overcome their own prejudices and the biases brought about by their problem-solving abilities. In some sense, problem-solving is analogous to taking off a dress—the more buttons, the more difficult it is to take off the dress. A stumbling block in the problem-solving process acts like a button: a problem setter should be aware of the existence of buttons and count the number of buttons when evaluating difficulty of problems.

PSMO problems are classified according to three levels of difficulty: A (difficult), B (medium) and C (easy). Each level is further divided into three sub-levels to allow for greater precision. For example, level A is divided into $A+$, A and $A-$. To quantify the levels, the following weights are assigned to them: *C-* = *1*, *C* = 2, *C+* = 3, *B-* = *4*, *B* = 5, *B+* = 6, *A-* = 7, *A* = 8 or 9 and *A+* = 10 or 11. The sum of the weights of the individual problems represents the difficulty of the whole paper. Extra weights are accorded to the *A* and *A+* problems so as to avoid setting a paper which is far too difficult.

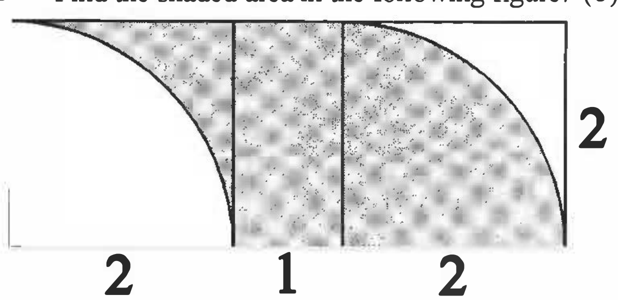
On a scale of 0 to 11, even two experts can differ widely in their assessment of the difficulty of a particular problem. To achieve some consistency, we identify three specific buttons, to continue the earlier analogy, that stand in the way of problem-solvers, and compute the weight of a problem as the sum of the following three components. While the final result also ranges from 0 to 11, experts seldom differ by more than 1 point.

- Is the problem easily understood? Is the statement too long? Are there terms which may be unfamiliar? This aspect of the problem is graded on a scale of 0 to 3.
- Are the computations, if any, very involved? Is the knowledge of specific techniques required? This aspect of the problem is also graded on a level of 0 to 3.
- Is there a natural approach to the problem? Does it require insight? How much synthesis and analysis are involved? How difficult is it to see a solution through from start to finish? This aspect of the problem is graded on a scale of 0 to 5.

We now give one sample problem of each sub-level. These are taken from past PSMOs.

C- Calculate 1991 + 199.1 + 19.91 + 1.991 (2212.001)

C Find the shaded area in the following figure. (6)



C+ There are two numbers A and B. If we move the decimal point in A two places to the left, then we get 8 of B. How many times of B is A? (12.5)

B- Among some sweets, 45% are milk sweets. If we add 16 fruit sweets, then the percentage of milk sweets changes to 25%. How many milk sweets are there? (20)

B A 7-digit number 1993*** is divisible by 2, 3, 4, 5, 6, 7, 8 and 9. Find its last three digits. (320)

B+ Five students A, B, C, D and E scored more than 91 marks each, out of 100, in an examination in which only integral scores were awarded. The average score of A, B and C is 95 while that of B, C and D is 94. A got the highest score and E got the third highest with 96 marks. Find the score of D. (97)

A- Vessel A contains 11 litres of pure alcohol. Vessel B contains 15 litres of water. Part of the alcohol in vessel A is poured into vessel B and mixed with the water. Then, part of the solution in B is poured into vessel A. Now vessels A and B contain 62.5% and 25% of pure alcohol respectively. Find the amount of solution poured from vessel B into vessel A. (6 litres)

A A car travels from town A to town B. If its speed is increased by 20%, then it will arrive at town B one hour earlier. If the car travels at the original speed for 120 km and then its speed is increased by 25%, it will arrive at town B 40 minutes earlier. Find the distance between towns A and B. (270 km)

A+ A circular track has a length of 2 kilometres. A, B and C, starting simultaneously from the same spot and in the same direction, are required to complete two laps each. A walks at 5 kilometres per hour while B and C walk at 4 kilometres per hour. All three can ride a bike at 20 kilometres per hour. There are two bikes. A starts on foot while B and C start on the bikes. Riders may get off anywhere and leave the bike for the others. Devise a scheme such that all three people and the two bikes arrive simultaneously at the destination. Then, find the shortest time required. (19.2 minutes)

There are many mathematics Olympiad schools all over China offering extracurricular instruction for mathematics competitions. Teachers in these schools teach solutions to many problems and students get practice working on problems. Even though these students are the winners in mathematics competitions, their problem solving abilities are difficult to predict.

Consider the following problem.

One bottle of soft drink can be exchanged with five empty bottles. A class of students drank 161 bottles of soft drink, some of which were obtained by exchanging with bottles emptied. Find the least number of bottles of soft drink bought. (129)

This problem has a simple solution and so we originally classified it as $C+$. However, after the competition, we agreed that it should have been at least $B+$. Ideally, a C problem should be solved by more than 50% of the contestants, a B problem by 15% to 50%, and an A problem by less than 15%. However, if the number of contestants increases, these percentages should be lowered.

Format

The First Round of the PSMO is held annually in March and the Final Round in April. There are 12 problems in the one-hour First Round, in which two million students participate each year. The Final Round has 12 answer-only problems to be attempted in 90 minutes. Each year there are 200,000 to 300,000 participants.

With thirty provinces in China, there is a wide range of abilities in mathematics among students across the country. In light of this, there are three versions of papers for the First Round in the PSMO: Paper A (hardest), Paper B (medium) and Paper C (easiest). The participating cities are allowed to choose the version most suitable for their students. Apart from the Han race, which is the largest ethnic group in China, there are 55 minority races. Because of historical factors, the standards of education are lower. To encourage minority students to participate in mathematics competitions and to demonstrate their attainment, a Minorities Paper (MP) has been set since 1993. The following table is an analysis of the levels of difficulty of the Papers in the first four years.

		Level C	Level B	Level A	Total Weight
	Paper C	9	3	0	37
	Paper B	6	6	0	47
	Paper A	3	9	0	53
	Final	3	6	3	59
	Paper C	7	5	0	38
	Paper B	5	7	0	44
	Paper A	4	8	0	47
	Final	4	6	2	57
	Paper MP	9	3	0	30
	Paper B	7	5	0	38
	Paper A	5	7	0	46
	Final MP	6	6	0	39
	Final	4	6	2	52
	Paper MP	9	3	0	33
	Paper B	8	4	0	38
	Paper A	6	5	0	46
	Final (MP)	5	7	0	45
	Final	4	6	2	58

To adjust the standards of the First round and Final Papers so as to popularize them and provide opportunities for a small number of bright students to demonstrate their abilities, a Summer Camp has been held annually since 1993 as the Overall Final Competition. The Overall Final comprises four parts:

- First Contest (One hour) - 10 answer-only problems covering four levels of difficulty ($C+$, $B-$, B, $B+$) with emphasis on basic skills in arithmetic.
- Second Contest (90 minutes) - 6 problems which are more demanding and which require deeper thinking.
- Relay - Three students in a team work out six problems. Student 1 is responsible for Problems 1 and 4, Student 2 for Problems 2 and 5, and Student 3 for Problems 3 and 6. Every problem, except the first one, contains a parameter which takes the value of the answer to the previous one. Five minutes are allowed for each of Problems 1 to 5, and ten minutes for Problem 6. Students may correct the solutions to the preceding problems. The relay contest was welcomed by teachers and students.
- Calculation Contest - Some Chinese teachers and students are more interested in solving difficult problems and they neglect training on basic operations in arithmetic. A consequence is that students frequently make mistakes in calculations involved in secondary school algebra later on. Calculation is not just a skill: it also trains students to think. In this contest, students are required to do 25 problems in one hour. A relatively easy problem is

$$0.3125 \times 457.83 \times 32 = ?$$

A more difficult one is

$$\frac{10}{13} \div 2\frac{19}{22} - 1\frac{2}{5} \times \frac{11}{13} \div 7 + \frac{1}{5} \times \frac{11}{63} = ?$$

With more different forms of competitions, students can develop an interest in mathematics from different perspectives and have more opportunities to demonstrate their abilities. This also discourages students from focusing only on a particular aspect of mathematics.

Prospect

Because classroom teaching usually caters to the majority, more able students cannot be satisfied with the mathematics taught in school. In China, it is essential to organize extracurricular activities in mathematics and competition is a suitable choice. However, excessive training is a burden on teachers and students. There are too many mathematics competitions in China. Participating in several competitions in a semester is too much for students, and may lead to a loss of interest in learning mathematics. Future PSMO problems should be easier and interesting. This is a difficult task requiring the effort of many mathematicians and teachers.

As more and more Asian countries organize competitions for primary school students, it is worth considering the possibility of organizing a regional competition like the APMO. Primary school mathematics is simple but competitions at the primary level are influential in general education because of the large number of students involved.

Changing Tires

Derek Holton

Abstract

The discussion below has got nothing to do with tires. Also, at times you'll feel like the silent moviegoer who shouts out to the screen heroine "look behind you." Unfortunately I can't hear you. Read on to the end despite my deafness and see why.

Tires on the front of my motor bike last 40 000 km and on the back they last 60 000 km. How far can I go without having to buy a new tire?

Try it and then read on.

I'm sorry to say I don't really own a motorbike. In fact I've never ridden one. My uncle had one for most of his life and was involved in a couple of accidents so I got the hint that they weren't the safest things around. It didn't stop my uncle, though he did add a side car to his machine after my first cousin was born.

Anyway, it's possible that he was confronted by the tire problem at some stage in his life when he wanted to delay spending money on new tires for as long as he could. Because the average of 40 000 and 60 000 is 50 000, it's tempting to believe that by suitably switching tires from front to back, and vice-versa, you might eke out 50 000 km before buying new tires. But how would you manage that? Well, after 25 000 km you could try a switch. Suppose tire A is on the front and tire B on the back. They've now done their 25 000 km stint and we've switched them around. How far can A last on the back? For that matter, how far can B last on the front?

Because B has used up 25 000 km of its 60 000 km life at the back, does it have 35 000 km left up front? Surely not. It won't last as long. How long then, will it last?

Assuming uniform wear, and I don't see how we can avoid that assumption, tire B has spent $\frac{25000}{60000}$ of its life at the rear. So $\frac{35000}{60000}$ of its life is yet to come. It can now spend $\frac{7}{12}$ of its life up front. Since a front tire's life expectancy is 40 000 km, tire B must be able to give us another $\frac{7}{12}\times 40\ 000$ km service. It looks like we'll get another $23\ 333\frac{1}{3}$ km out of it. But that's a problem because $25\ 000 + 23\ 333\frac{1}{3} = 48\ 333\frac{1}{3} < 5\ 000$? In fact, it's a long way short of 50 000. What can we do to get the extra mileage (kilometrage?)? Perhaps we could keep switching the tires back and forth. Maybe we'd get more distance that way.

Back up for a bit. Tire A can only go on the front or back. The same holds for tire B. Doesn't that mean that tire A spends part of its life on the front and part on the back? It doesn't really matter how often you change the tires. The net effect is to keep A on the front for part of the time and then put it on the rear wheel. Forget about all the swapping then. One swap is sufficient.

Oh dear! And by the looks of things it doesn't seem as if we're going to be able to get our full 50 000 km either. I've done a few jottings in the margin and I can't get anywhere near that target. How can I do these experiments systematically enough to produce the maximum I want? Past experience with this sort of thing suggests that perhaps it's time for a bit of algebra.

Suppose A traveled x km on the front and y on the back. Then the total distance d for the old tires is $d = x + y$. And I want to maximize d.

Hmm! One equation with two unknowns x and y. I need another equation if I'm going to get anywhere. What else do I know? I suppose that at the end of the day (or rather the end of the tire) the tire will be worn out. How can I get another equation out of this? Partly worn on the front plus partly worn on the back is all worn! So? The fractional part of A worn on the front is $\frac{x}{40\ 000}$. And the part worn on the back is $\frac{y}{60\ 000}$. When A's done that, it's all gone. That means $\frac{x}{40\ 000}+\frac{y}{60\ 000}=1$ is equal to what?

Is what 1? After all it's one whole life. So this means that we have to maximize d = x + y subject to $\frac{x}{40\ 000}+\frac{y}{60\ 000}=1$ That looks like a linear programming problem. Let me check if there are any other constraints. Yes, clearly $0 < x < 40\ 000$ and $0 < y < 60\ 000$. If you know anything about linear programming you'll know that you only have to test the values of d at P, Q and R in Figure 1. The best value of d is obtained when x = 0 and y = 60 000! Looks like I ride for 60 000 km with the front wheel in the air!

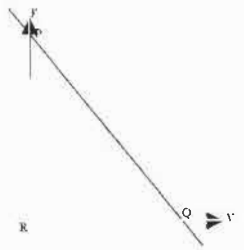


Figure 1

That's screwy. Maybe I forgot to take something else into account? Can't think what. But wait. There's another way to do this. I want to maximize $d = x + y$ and x and y are linked by $\frac{x}{40\,000} + \frac{y}{60\,000} = 1$. If I solve the last equation for y, I can substitute into the first equation. This will give d in terms of x and I might be able to use a bit of calculus. Right, $d = 60\,000 - \frac{3x}{2}$. Hmm. There's no need for calculus. Surely d is biggest when x is smallest. That happens when $x = 0$. Back up on to one wheel!

I'm clearly missing a vital piece of information. But what? What have I not done that I could have done? Wait! So far I've only thought about tire A. Perhaps if I brought B into the action I might make some progress. Well, for B, d is still $x + y$. But for B, x km is spent on the back and y km on the front, so $\frac{x}{60\,000} + \frac{y}{40\,000} = 1$. That's the same as for A. I haven't made any progress at all.

Hang on. No, that's crazy. For A I get $\frac{x}{40\,000} + \frac{y}{60\,000}$ and for B I get $\frac{x}{60\,000} + \frac{y}{40\,000} = 1$. Two equations, two unknowns! Let's solve. You can do that in your head can't you? Well, on a bit of paper then. I get $x = 24\,000$ and $y = 24\,000$.

So it looks as if the manufacturers of motor bike tires should put a small red strip in at the 24 000 km mark and then I'll know when to change tires!

But wait. The original question was how far could I go on a pair of tires. The answer is $d = 48\,000$ km. That's a bit worrying: it's not consistent with what I did a while back. When I thought I could get 50 000 km out of the tires I tried changing them after 25 000 km and managed to get 48 $333\frac{1}{3}$ km worth. What have I done wrong now?

In that case I looked at tire B. There $x = 25\,000$ and $y = 23\,333\frac{1}{3}$. Does that fit the equation for B?

$$\frac{25\,000}{60\,000} + \frac{23\,333\frac{1}{3}}{40\,000} = \frac{5}{12} + \frac{7}{12} = 1.$$

Yes, that's OK. No problems. There's something wrong somewhere though. I'd better check A's equation. There I've got

$$\frac{25\,000}{40\,000} + \frac{23\,333\frac{1}{3}}{60\,000} = \frac{5}{8} + \frac{7}{18} = \frac{73}{72}$$

which isn't 1! In fact, $\frac{73}{72}$ is bigger than 1! We've got an extra $\frac{1}{72}$ of a life out of tire A. Not bad (but don't tell the tire companies).

OK, so what that all means is that we can't get $48\,333\frac{1}{3}$ km out of a set of tires. The inconsistency I thought I had removed. It looks as if I can only get 48 000 km out of a set of tires after all.

Given the last experience with $48\,333\frac{1}{3}$ km though, can we really manage 48000 km? Now

$$\frac{24\,000}{60\,000} + \frac{24\,000}{40\,000} = \frac{24\,000}{40\,000} + \frac{24\,000}{60\,000} = 1$$

So we have got everything out of the tires. However, it may be that we can't organize the tires so that they die simultaneously. No, that's stupid. I've just lost a little confidence. Clearly the A and B equations tell us that if we change the tires at 24 000 km, they will both be useless at the 48 000 km mark. That's a relief!

Now wait a minute! Suppose I had three tires. Could I get more than 48 000 km out of them? Is that the right question? Obviously if I use tire A at the front for 40 000 km and then put tire C at the front, I'll get 60 000 km's worth until B goes bald. With three tires I can easily get 60 000 km. If I do that though I really haven't done as well, per tire, as I did with only two tires. It seems to me then, that "with three tires can I do better than an average of 24 000 km per tire?" is the right question to ask. For what it's worth, I do know that I can't do worse than a 20 000 km average.

I guess the way to tackle this one is to use the successful strategy of the two tire case. So if tire A is on the front for x km and on the back for y km and tire B is on the front for z km and the back for u km and tire C is on the front for v km and on the back for w km, I get

$$\frac{x}{40\,000} + \frac{y}{60\,000} = 1$$

$$\frac{z}{40\,000} + \frac{u}{60\,000} = 1$$

$$\frac{v}{40\,000} + \frac{w}{60\,000} = 1$$

And that's assuming that I can wear out all three tires too! Let's assume that for now and worry about it later. What a mess!

I might be able to make those equations at least look better if I write a for 40 000 and b for 60 000. It will also save me quite a bit of writing. So

$$\frac{x}{a}+\frac{y}{b}=1, \quad \frac{z}{a}+\frac{u}{b}=1, \quad \frac{v}{a}+\frac{w}{b}=1.$$

I can get them all on one line now!

Actually I'm inclined to write x_A, x_B and x_C for the numbers of kilometres that A, B and C are on the front, respectively, and y_A, y_B and y_C for the numbers of kilometres they're on the back. I can then even reduce the three equations to one:

$$\frac{x_i}{a}+\frac{y_i}{b}=1$$

for i=A,B,C. How about that!

Has that really helped me solve the problem though? I don't think I know what to do with that lot. You see the next step that I would like to make is to say that $x_A = y_B$, which is what I had in the two-tire problem but that's not necessarily going to be the case. Can I link the x's and y's at all? I'll draw a picture (Figure 2).

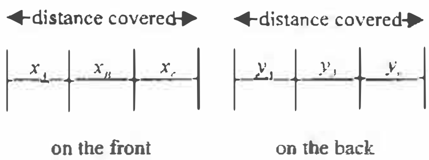


Figure 2

Obviously I don't know which order the tires should go on the front and back. But it's probably not worth changing them too frequently. We might as well leave A on the front for its x_A km, and do the same for B and C. With any luck, we can do the same on the back wheel.

The one thing that the picture is useful for is that it does give us $x_A + x_B + x_C = y_A + y_B + y_C$, though what use this is I don't know. Again, let's call this quantity d. Using the Σ notation I can rewrite this equation as $d = \sum_i x_i = \sum_i y_i$, and we want to maximize d subject to $\frac{x_i}{a}+\frac{y_i}{b}=1$ for i = A, B, C.

I don't want to even think about the linear programming approach. If I use calculus, I'll have d as a function of several variables. So the methods that didn't work last time probably won't work now either. And I'm not so sure that solving three equations in six unknowns is going to get me anywhere either! Is there any way I can use the two-tire approach? What good is it to me that $y_i = b - \frac{bx_i}{a}$, without the link I had before between x and y?

Wait though! Using that last equation, I get $\sum_i y_i = \sum_i \left(b - \frac{bx_i}{a}\right) = 3b - \frac{b}{a}\sum_i x_i$. But $\sum_i y_i$ and $\sum_i x_i$ are both equal to d. So $d = 3b - \frac{bd}{a}$. For what it's worth I can at least solve for d. Then $d = \frac{3ab}{a+b}$. I'm sure you can work that out.

Remembering that $a = 40\,000$ and $b = 60\,000$, d must be 72 000. The average distance per tire then is just $\frac{d}{3}$. Not 24 000 km again!

Better check that we can actually get $d = 72\,000$. Do the equations tell us how to achieve this grand distance before ending up with three bald tires? I don't think they do. I can't see any way that the x_i and y_i are restricted to be something special. What if we just try $x_i = y_i = 24\,000$? Maybe it'll work out. I'll show this diagrammatically in Figure 3.

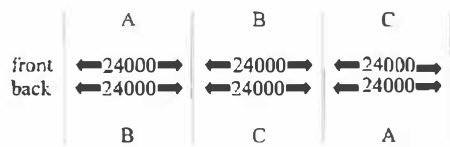


Figure 3

So there is a rotation that will give me the 24 000 km average. It's strange though that I don't, and can't, increase the average number of kilometres per tire even if I use an extra tire.

Just reflecting a minute, I see now that the assumption that I could wear out all tires was justified. I wonder though if there are other ways of rotating the tires? And are some ways better for economy or safety? Or have I missed something that forces x_i to equal y_i? And it probably doesn't help me to have four, five or any number of tires. I doubt that I could get more than a 24,000 km tire average. I wonder what my uncle did when he had his side car? If he could get 80 000 km out of a side car tire, I wonder what he averaged per tire? And what should you do with car tires?

* * * * * * * * * *

As I said initially, this discussion has really nothing to do with tires. What I was doing would be very similar no matter what the problem in hand because there are some tried and true heuristic techniques that you can use in a lot of situations. If you have them in your armory it will improve your problem solving capability.

One of the first things you'll notice is that **I keep asking questions**. It's almost always impossible to solve a decent problem straight away. Quite often you have to cast around for an approach and you usually have to solve a large number of smaller problems to get where you want to go. I tend to think of problem solving, not like Figure 4(a), but rather like Figure 4(b).

In regular classrooms where certain algorithms are being practiced, Figure 4(a) is often the model used. With more difficult problems many smaller questions are asked and answers sought, a lot of which are not on the final track of a solution at all. Actually in mathematical research many answers are reached unexpectedly. Some answers may not be what the researcher was looking for and do not necessarily answer the original questions. However, they may be very useful and interesting nevertheless. Asking the right questions is more important than answering them.

$$\mathbf{Q \to A}$$

Figure 4(a)

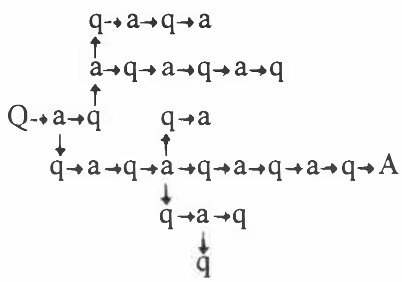


Figure 4(b)

When starting out with a new problem, it's often a good idea to do a few examples that will give you some **insight** into what's going on. If nothing else, examples will help you understand the problem. Moreover, they are useful yardsticks by which to measure a solution and they may give you some insight to find a solution.

Don't be afraid to **use a diagram** either. Even though the tire problem is essentially algebraic, I got some insight into what was going on by producing Figure 2.

In the search for solutions though, it's a good idea to try to think of situations that you've been in before that seem vaguely similar to the present situation. **Can a previously used technique help?** That's why I thought about linear programming, calculus and solving algebraic equations. They won't all work but maybe one will.

Often you'll get stuck because **you've overlooked a vital piece of information**. In the two-tire problem I was lost until I remembered to use tire B. That had an important contribution to make to the action.

When you finally get an answer, is it **consistent** with the data in the problem or life in general? An answer of 80 000 km average per tire is clearly nonsense. So is an answer that tells you the height of a mountain is 3 cm. If you do get inconsistency, then you need to go back and tidy up. What is the source of the inconsistency? Is it a wrong **assumption**? At the end of the day it's always worth checking to see that any assumption you made to keep things going **can be justified**.

Suprisingly **notation** too is often a key to solving a problem. For a start, you really don't want to use unwieldy numbers like 40 000 and 60 000. Instead use a and b and do the arithmetic when you have to. While changing to x_i's and y_i's may be difficult if you're not used to them, they will often lead to general results. In the discussion, if we replace i = A, B, C by i = $A_1, A_2, \ldots, A_n$ for n tires, the argument which previously led to $\frac{d}{3} = \frac{ab}{a+b}$ will lead to $\frac{d}{n} = \frac{ab}{a+b}$. No matter how many tires you have you can't do better than 24 000 km average! (You can also see that if a = 50 000 and b = 70 000 you can immediately produce the n-tire average.)

Toward the end, I talked about four, five and more tires. This is a **generalization**, the solution of which is given in the last paragraph. Generalizations are situations which contain the original problem as a special case. In the last paragraph we found $\frac{d}{n}$ for **any** value of n.

Put $n = 2$ and you get the original problem.

I also talked **extensions**. Extensions are problems similar to the original which are motivated by the original. The side-car problem with three tires is an extension of the two-tire problem. Extensions often lead to interesting results too.

There are clearly more heuristics that are worth learning. Read any of George Polya's books for in-depth discussions.

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Packing Boxes with N-tetracubes

Andris Cibulis

Introduction

With the popularity of the video game *Tetris*, most people are aware of the five connected shapes formed of four unit squares joined edge-to-edge. They are called the I-, L-, N-, O- and T-tetrominoes, after the letter of the alphabet whose shapes they resemble. They form a subclass of the polyominoes, a favorite topic in research and recreational mathematics founded by Solomon Golomb.

Here is a problem from his classic treatise, *Polyominoes*. Is it possible to tile a rectangle with copies of a particular tetromino? Figure 1 shows that the answer is affirmative for four of the tetrominoes but negative for the N-tetromino, which cannot even fill up one side of a rectangle.

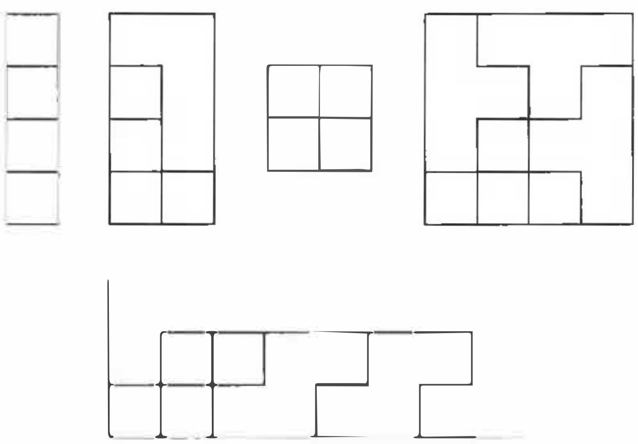

Figure 1

Getting off the plane into space, we can join unit cubes face-to-face to form polycubes. By adding unit thickness to the tetrominoes, we get five tetracubes, but there are three others. They are shown in Figure 2, along with the N-tetracube.

Is it possible to pack a rectangular block, or box, with copies of a particular tetracube? The answer is obviously affirmative for the I- L- O- and T-tetracubes, and it is easy to see that two copies of each of the three tetracubes not derived from tetrominoes can pack a $2 \times 2 \times 2$ box. Will the N-tetracube be left out once again? Build as many copies of it as possible and experiment with them.

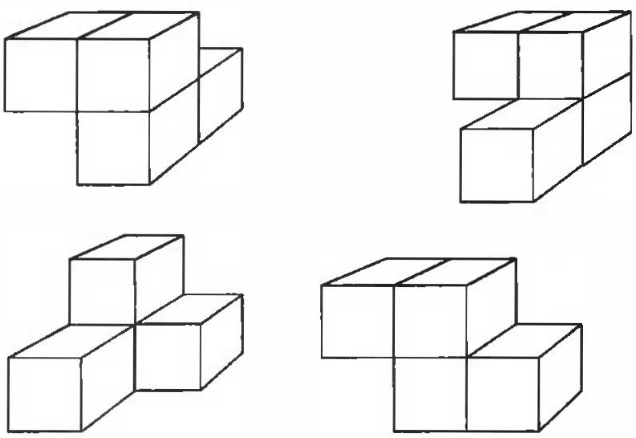

Figure 2

If the $k \times m \times n$ box can be packed with the N-tetracube, we call it an *N-box*. Are there any such boxes? Certain types may be dismissed immediately.

Observation 1

The $k \times m \times n$ box cannot be an N-box if it satisfies at least one of the following conditions:

(a) one of k, m and n is equal to 1;

(b) two of k, m and n are equal to 2;

(c) kmn is not divisible by 4.

It follows that the $2 \times 3 \times 4$ box is the smallest box which may be an N-box. Figure 3 shows that this is in fact the case. The box is drawn in two layers, and two dominoes with identical labels form a single N-tetracube.

So there is life in this universe after all! The main problem is to find all N-boxes.

N-cubes

If $k = m = n$, the $k \times m \times n$ box is called the *k-cube*, and a cube which can be packed by the N-tetracube is called an *N-cube*. We can easily assemble the 12-cube with the $2 \times 3 \times 4$ N-box, which makes it an N-cube. This is a special case of the following result.

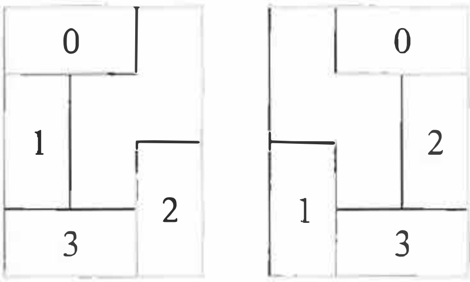

Figure 3

Observation 2

Suppose the $k \times m \times n$ and $l \times m \times n$ boxes are N-boxes. Let a, b and c be any positive integers. Then the following are also N-boxes:

(a) $(k + l) \times m \times n$;

(b) $ak \times bm \times cn$.

The 12-cube is not the smallest N-cube. By Observation 1, the first candidate is $k = 4$. It turns out that this is indeed an N-cube. It can be assembled from the $2 \times 4 \times 4$ N-box, whose construction is shown in Figure 4.

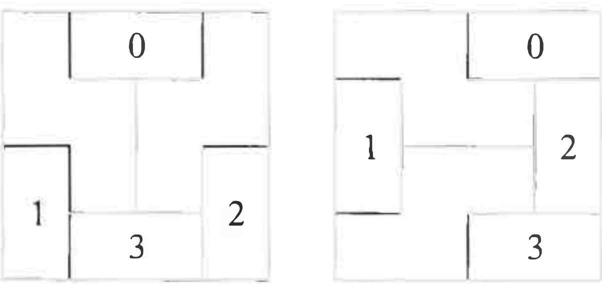

Figure 4

The next candidate, the 6-cube, is also an N-cube, but a packing is not that easy to find. In Figure 5, we begin with a packing of a $2 \times 6 \times 6$ box, with a $1 \times 2 \times 4$ box attached to it. To complete a packing of the 6-cube, add a $2 \times 3 \times 4$ N-box on top of the small box, flank it with two $2 \times 4 \times 4$ N-boxes and finally add two more $2 \times 3 \times 4$ N-boxes.

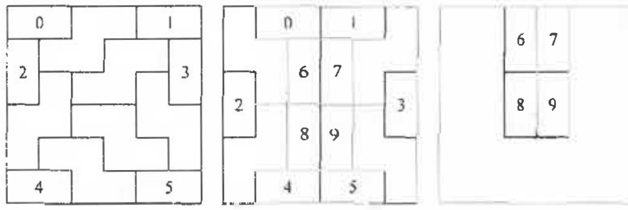

Figure 5

Can we pack the 8-cube, the 10-cube, or others? It would appear that as size increases, it is more likely that we would have an N-cube. However, it is time to stop considering one case at a time. We present a recursive construction which expands N-cubes into larger ones by adding certain N-boxes.

Theorem 1

The k-cube is an N-cube if and only if k is an even integer greater than 2.

Proof

That this condition is necessary follows from Observation 1. We now prove that it is sufficient by establishing the fact that if the k-cube is an N-cube, then so is the $(k + 4)$-cube. We can then start from either the 4-cube or the 6-cube and assemble all others.

From the $2 \times 3 \times 4$ and $2 \times 4 \times 4$ N-boxes, we can assemble all $4 \times m \times n$ boxes for all even $m,n \geq 4$, via Observation 2. By attaching appropriate N-boxes from this collection, we can enlarge the k-cube first to the $(k + 4) \times k \times k$ box, then the $(k + 4) \times (k + 4) \times k$ box and finally the $(k + 4)$-cube. This completes the proof of Theorem 1.

Further Necessary Conditions

Observation 1 contains some trivial necessary conditions for a box to be an N-box. We now prove two stronger results, one of which supercedes (c) in Observation 1.

Lemma 1

The $k \times m \times n$ box is not an N-box if at least two of k, m and n are odd.

Proof

We may assume that m and n are odd. Place the box so that the horizontal cross-section is an $m \times n$ rectangle. Label the layers L_1 to L_k from bottom to top. Color the unit cubes in checkerboard fashion, so that in any two which share a common face, one is black and the other is white. We may assume that the unit cubes at the bottom corners are black. It follows that L_i has one more black unit cube than white if i is odd, and one more white unit cube than black if i is even.

Suppose to the contrary that we have a packing of the box. We will call an N-tetracube *vertical* if it intersects three layers. Note that the intersection of a layer with any N-tetracube which is not vertical consists of two or four unit cubes, with an equal number in black and white. The intersection of a vertical N-tetracube with its middle layer consists of one unit cube of each color.

Since L_1 has a surplus of one black unit cube, it must intersect l_1 vertical N-tetracubes in white and $l_1 + 1$ vertical N-tetracubes in black, for some non-negative integer l_1. These N-tetracubes intersect L_3 in l_1 black unit cubes and $l_1 + 1$ white ones. Hence the remaining part of L_3 has a surplus of two black. They can only be packed with l_3 vertical N-tetracubes intersecting L_3 in white, and $l_3 + 2$ vertical N-tetracubes in black, for some non-negative integer l_3. However, the surplus in black unit cubes in L_5 is now three, and this surplus must continue to grow. Thus the $k \times m \times n$ box cannot be packed with the N-tetracube. This completes the proof of Lemma 1.

Lemma 2

The $k \times m \times n$ box is not an N-box if kmn is not divisible by 8.

Proof

Suppose a $k \times m \times n$ box is an N-box. In view of Lemma 1, we may assume that at least two of k, m and n are even. Place the packed box so that the horizontal cross-section is an $m \times n$ rectangle, and label the layers L_1 to L_k from bottom to top. Define vertical N-tetracubes as in Lemma 1 and denote by t_i the total number of those that intersect L_i, L_{i+1} and L_{i+2}, $1 \leq i \leq k\text{-}2$.

Since at least one of m and n is even, each layer has an even number of unit cubes. It follows easily that each t_i must be even, so that the total number of vertical N-tetracubes is also even. The same conclusion can be reached if we place the box in either of the other two non-equivalent orientations. Hence the total number of N-tetracubes must be even, and kmn must be divisible by 8. This completes the proof of Lemma 2.

N-Boxes of Height 2

We now consider $2 \times m \times n$ boxes. By Observation 1, m ≥ 3 and n ≥ 3. By Lemma 2, mn is divisible by 4. We may assume that m is even. First let $m = 4$. We already know that the $2 \times 4 \times 3$ and $2 \times 4 \times 4$ boxes are N-boxes. Figure 6 shows that so is the $2 \times 4 \times 5$ box.

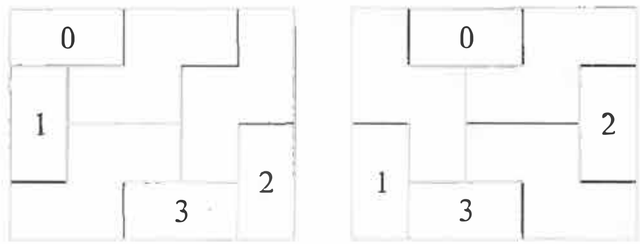

Figure 6

If the $2 \times 4 \times n$ box is an N-box, then so is the $2 \times 4 \times (n + 3)$ box by Observation 2. It follows that the $2 \times 4 \times n$ box is an N-box for all $n \geq 3$.

Now let $m = 6$. Then n is even. We already know that the $2 \times 6 \times 4$ box is an N-box. However, the $2 \times 6 \times 6$ box is not. Our proof consists of a long case-analysis, and we omit the details. On the other hand, the $2 \times 6 \times 10$ box is an N-box. In Figure 7, we begin with the packing of a $2 \times 3 \times 6$ box with a $2 \times 2 \times 3$ box attached to it. We then build the mirror image of this solid and complete the packing of the $2 \times 6 \times 10$ box by adding a $2 \times 4 \times 3$ N-box.

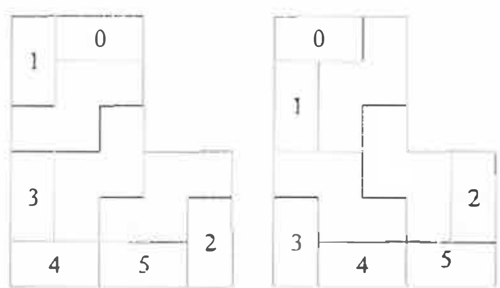

Figure 7

If the 2 ´ 6 ´ (n + 4) box by observation 2. It follows that the 2 ´ 6 ´ n box is an N-box for n = 4 and all even n ≥ 8.

Theorem 2

The $2 \times m \times n$ box is an N-box if and only if $m \geq 3$, $n \geq 3$ and mn is divisible by 4, except for the $2 \times 6 \times 6$ box.

Proof

If m is divisible by 4, the result follows immediately from Observation 2. Let $m = 4l + 2$ for some positive integer l. We already know that the $2 \times 10 \times 6$ box is an N-box. If the $2 \times (4l + 2) \times n$ box is an N-box, then so is the $2 \times (4l + 6) \times n$ box by Observation 2. This completes the proof of Theorem 2.

The Main Result

Theorem 3

The $k \times m \times n$ box is an N-box if and only if it satisfies all of the following conditions:

(a) $k \geq 2$;

(b) $m \geq 3$;

(c) at least two of k, m and n are even;

(d) kmn is divisible by 8;

(e) $(k, m, n) \neq (2, 6, 6)$.

Proof

Necessity has already been established, and we deal with sufficiency. We may assume that k, m and n are all at least 3, since we have taken care of N-boxes of height 2. Consider all $3 \times m \times n$ boxes. By (c), both m and n must be even. By (d), one of them is divisible by 4. All such boxes can be assembled from the $3 \times 2 \times 4$ box.

Consider now the $k \times m \times n$ box. We may assume that m and n are even. If k is odd, then one of m and n is divisible by 4. Slice this box into one $3 \times m \times n$ box and a number of $2 \times m \times n$ boxes. Since these are all N-boxes, so is the $k \times m \times n$ box.

Suppose k is even. Slice this box into a number $2 \times m \times n$ boxes, each of which is an N-box unless $m = n = 6$. The $4 \times 6 \times 6$ box may be assembled from the $4 \times 2 \times 3$ box, and we already know that the 6-cube is an N-cube. If the $k \times 6 \times 6$ box is an N-box, then so is the $(k + 4) \times 6 \times 6$ box. This completes the proof of Theorem 3.

Research Projects

Problem 1.

Try to prove that the $2 \times 6 \times 6$ box is not an N-box. It is unlikely that any elegant solution exists.

Problem 2

An N-box which cannot be assembled from smaller N-boxes, is called a *prime* N-box. Find all prime N-boxes.

Problem 3.

Prove or disprove that an N-box cannot be packed if we replace one of the N-tetracubes by an O-tetracube.

Problem 4.

For each of the other seven tetracubes, find all boxes which it can pack.

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The South African Mathematical Talent Search

John Webb

In April 1991 South Africa received its first invitation to take part in the 32nd International Mathematical Olympiad (IMO) in Sweden, which I attended as an Observer. This opened the way to South Africa's full participation in the IMO, and since 1992 I, as Team Leader, along with Graeme West (University of the Witwatersrand) as Deputy Leader, have taken a team of six students to each IMO.

For many years South Africa had the infrastructure for running a successful IMO program. The quarterly mathematics magazine for high schools, *Mathematical Digest*, features regular problem-solving competitions and has a free circulation to all high schools in South Africa and neighboring countries. With a circulation of about 4,000, *Mathematical Digest* has just celebrated its 25th birthday.

The South African Mathematics Olympiad celebrated its 30th birthday in 1995. For many years it had fewer than 5,000 contestants in its first round, with 100 writing the second round. It is at present expanding its first round participation, with Junior and Senior Rounds, and total participation is now over 20,000.

In addition, there are regional mathematical competitions in some parts of the country. The University of Cape Town Mathematics Competition began in 1977, and today attracts over 4,000 participants who come to the UCT campus one evening in April to write competition papers at five different levels. While it is not a big competition in international terms, it claims (with confidence, but no real justification) to be the largest mathematics competition in the world written in one place at one time.

Despite this background, the identification and preparation of IMO teams presented difficulties which the South African Mathematics Olympiad was not in a position to solve adequately. One problem was the very low participation by black students. Another was the timing of the Olympiad: with its final round written in September, it cannot be used as a final selection test for a team to go to an IMO in July.

When South Africa was admitted to the IMO, the South African Mathematics Society took on the task of selecting and training the teams. A nationwide Talent Search was launched in the form of a correspondence course in problem-solving.

The Talent Search is publicized through teachers' and pupils' magazines such as *Mathematical Digest*. Posters are sent to schools, and personal invitations to take part are sent to students who have distinguished themselves in a regional competition or the national Olympiad. Entry is open to all, and is free.

The structure of the Talent Search is simple. Students are sent a round of problems to solve. They mail in their solutions which are marked and returned with model solutions, suggestions for further reading, a short article on some aspect of problem-solving, and the next round of problems. The Talent Search is self-paced. However, normally the top students get through ten rounds of problems in a year.

The Talent Search enrollment builds up during the year to about 200 students. However, many of the participants soon find the problems too difficult; they are then diverted into a more accessible series of problems appropriate to their abilities.

The Talent Search follows the Southern hemisphere academic year which runs from January to December. At the end of the year, certificates of achievement are sent to all participants and the top students are invited to attend a "Mathematical Camp", held at the University of Stellenbosch at the beginning of the summer vacation. During the six days of the camp the students write a series of tests and attend lectures on solving Olympiad problems. There is also time for sightseeing excursions.

In January the new Talent Search begins, but the survivors of the Stellenbosch Camp continue with a program of problem-solving by correspondence, culminating in an IMO Selection Camp held at Rhodes University in April. The team of six, with a reserve, is selected and goes into heavy training, once again by correspondence.

A small but useful series of publications has been built up. After the IMO in Hong Kong, a book entitled *South Africa and the 35th IMO* was published. This 136-page book, which I co-wrote with Graeme West, contains the problems used in the 1993/94 Talent Search and the camps, plus all 144 problems from the 1994 IMO problems. Full solutions of all the problems are also included.

In addition, five titles have been published in the series *South African Mathematical Society Olympiad Training Notes*:

- *The Pigeon-hole Principle*, by Valentin Goranko;
- *Topics in Number Theory*, by Valentin Goranko;
- *Inequalities for the Olympiad Enthusiast*, by Graeme West;
- *Graph Theory for the Olympiad Enthusiast*, by Graeme West;
- *Functional Equations for the Olympiad Enthusiast*, by Graeme West.

For the first two years, South Africa's performance at the IMO was not impressive, but in Hong Kong the team won three Bronze Medals and an Honorable Mention. The next year, at the IMO in Canada, every member of the team won an award: two Bronze Medals and four Honorable Mentions.

The opportunity to compete in an International Mathematical Olympiad has turned out to be an important incentive in encouraging promising young mathematicians in South Africa to develop their talents. South Africa may not be in the First League of the IMO, but we are challenging for a position at the top of the Second.

Appendix I: Student Projects

We are proud to present five projects by university and high school students. Four are by members of "Saturday Mathematics Activities, Recreations and Tutorials," the enrichment program offered by the Department of Mathematical Sciences of the University of Alberta. The fifth comes from Austria.

Listed below are further publications by members of the SMART program.

T. Boag, C. Boberg and **L. Hughes**
"On Archimedean Solids," *Mathematics Teachers 72* 1979: 371-376.

M. Rabenstein
"An Example of an Error-Correction Code," *Mathematics Magazine 58* 1985: 225-226.

H. Chan, S. Laffin and **D. van Vliet**
"Knight Tours," *Mathematics and Informatics Quarterly 2* 1992: 135-150.

C. Li and **A. Liu**
"The coaches' dilemma," *Mathematics and Informatics Quarterly 2* 1992: 155-157.

G. Denham and **A. Liu**
"Circuits checking circuits," *Mathematics in Education*, ed. by Th. M. Rassias, Univ. LaVerne 1992: 145-150.

J. Chan, P. Laffin and **D. Li**
"Martin Gardner's Royal Problem," *Quantum 4* Sept/Oct 1993: 45-46.

G. Denham, M. Leu and **A. Liu**
"All 4-stars are Skolem-graceful," *Ars Combinatoria 36* 1993: 183-191.

D. Robbins, S. Sivapalan and **M. Wong**
"Dissecting Triangles into Similar Triangles," *Crux Mathematicorum, no.22* 1996: 97-100

An Imaginary Postal Service

Steven Laffin

student, École J. H. Picard, Edmonton

Mathematics is fun. Unfortunately, some mathematics classes are not. Steven imagines himself in such a class with Todd. All the children are bored by the mechanical drills. They pass notes to one another so often that Todd opens a post office. He issues his first stamp, the penny black.

1

Business is booming. So Steven opens a rival post office across the classroom. He issues a set of stamps and a rate chart.

1	2

Funny Notes	1
Silly Notes	2
Naughty Notes	1+2

Now this is a very imaginative idea. There are exactly three connected blocks in this set of stamps. Each has a different value, and all the values are consecutive starting from 1 cent. It is a perfect design.

Todd points out that his penny black is a perfect design too, though not a particularly interesting one. He then comes up with the following.

1	3	2

NOTES	half-page	full-page
Funny	1	3
Silly	2	3+2
Naughty	1+3	1+3+2

Not to be outdone, Steven looks at the 1 by 4 set of stamps to see if it can be turned into a perfect design. He counts 10 connected blocks in the 1 by 4 set of stamps, 4 singles, 3 pairs, 2 triples and the whole set. Therefore, if it can be made into a perfect deign, the values of the blocks must be 1, 2 and so on, up to 10 cents.

Clearly, the values of the individual stamps must be 1, 2, 3 and 4 cents in some order. To get a 9-cent block, the 1 must be at one end. To get an 8-cent block, the 2 must be at the other end. To get a 6-cent block, the 4 must now be next to the 2, with the 3 filling in the gap between the 1 and the 4. Unfortunately, there are no 5-cent blocks.

So no one can make the 1 by 4 set into a perfect design. Steven now tries to see if he can get consecutively up to 9 cents. He starts with his first perfect design, where a 1 is next to a 2. This also gives him a 3-cent block. Next, he puts a 4 on the other side of the 1. This gives him a 5 and a 7-cent block. Finally, by putting a 6 on the other side of the 2, he has the missing 6, 8 and 9-cent blocks. There is also a bonus 13-cent block, even though it is not part of the consecutive values .

4	1	2	6

While not perfect, this is as good a design as anyone can get out of the 1 by 4 set. Still, Steven is annoyed that he has not succeeded in finding a perfect design with 4 stamps.

Suddenly, Steven has an inspiration. There is after all a way to achieve his goal. Instead of a 1 by 4 set of stamps, Steven uses a 2 by 2 set. By bending his latest design into this shape, he turns it into a perfect one. He is so excited that he issues the set right away, along with an expanded rate chart.

1	2
4	6

NOTES	half-page	full-page	oversize
Funny	1 - - -	1 2 - -	- - - 6
Silly	- 2 - -	1 - 4 -	- 2 - 6
Naughty	- - 4 -	1 2 4 -	1 2 - 6

PARCELS	Uninsured	Insured
Inedible	- - 4 6	- 2 4 6
Edible	1 - 4 6	1 2 4 6

Todd is very impressed with this. He goes back to the drawing board, and it does not take him long to modify his perfect 1 by 3 set into the following perfect design.

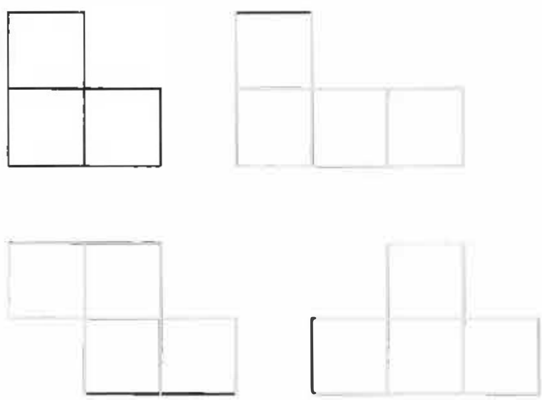
1	**3**
7	**2**

It does exactly what Steven's can do, though in different ways. Since he would be using essentially the same rate chart, he merges his post office with Steven's. The two boys then combine their brain power to look for other perfect designs. They find four other sets containing not more than 4 stamps.

"The first one is really the same as the 1 by 3 set," says Todd, "and I have already made that into a perfect design."

"The next two are really the same as the 1 by 4 set," says Steven, "and I know that it cannot be made into a perfect design."

He shows Todd the work he has done on that, including his near perfect design.



"The last set looks different from the 1 by 4 set as well as from the 2 by 2 set," says Todd.

"It is," says Steven, "because it has 4 singles, 3 pairs, 3 triples and the whole set, so there are 11 connected blocks."

"If it can be made into a perfect design," says Todd, "the values of the individual stamps must be different and add up to 11 cents. They must be 1, 2, 3 and 5 cents."

"The 1 and the 2 cannot be in the middle, or we will be missing either a 10 or a 9-cent block."

"The 3 must be in the middle. Otherwise, there will not be a 4-cent block."

"Now we do not have a 7-cent block," says Steven. "So this set cannot be made into a perfect design."

The two boys then work independently to see if they can get consecutively up to 10 cents. Clearly, they must use 1, 2, 3 and 4. Each comes up with a near perfect design.

"I get mine by adding a 4 on top of my perfect design for the 1 by 3 set," says Todd.

"Mine has the 4 in the middle instead. We always seem to come up with different designs," says Steven, remembering their perfect designs for the 2 by 2 set.

"I will have to find a near perfect design for the 1 by 4 set different from yours," says Todd. "Aha! If I clone the 3 in my perfect design for the 1 by 3 set, I will have what I want."

1	**3**	**3**	**2**

"It is interesting that your stamps add up to under 10 cents, while mine add up to over 10 cents. That is why you can afford a duplicate."

The two boys are silent for a while.

"I have yet another near perfect design for the 1 by 4 set," says Todd. "The stamps add up to 11 cents."

2	**5**	**1**	**3**

"So have I," says Steven, "except that mine add up to 9 cents."

1	**1**	**4**	**3**

"Are there any others?" asks Todd.

"What about the other sets of stamps we have looked at?" asks Steven. "Have we found all the perfect or near perfect designs for them?"

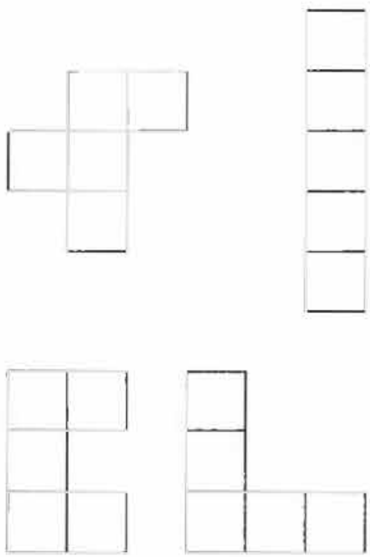
Just then, the bell rings, and the imaginary class dissolves into thin air. Steven hits his alarm-clock and gets ready for his real class.

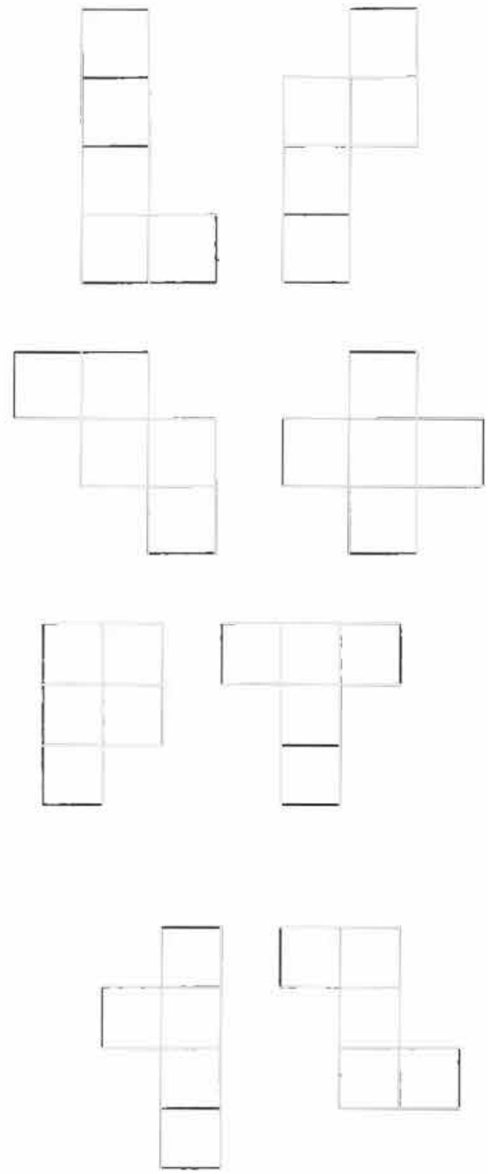
Remark:

A version of this problem was posed by Steven Laffin and Andy Liu as Problem 93-9 in the S.I.A.M. Review, under the title *All-purpose Stamp Design*.

Supplementary Problem

Exactly one of the following twelve sets of 5 stamps can be made into a perfect design. Which one and how?





Dissecting Rectangular Strips Into Dominoes

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The First Problem

A domino is defined to be a 1×2 or 2×1 rectangle. The first one is said to be horizontal and the second vertical. In our Mathematics Club, we learned to count the number g_n of different ways of dissecting a $3 \times 2n$ strip into dominoes. The sequence $\{g_n\}$ satisfies the recurrence relation

$$g_n = 4g_{n-1} - g_{n-2},$$

for all $n \geq 2$, with initial conditions $g_0 = 1$ and, as shown in Figure 1, $g_1 = 3$.

Figure 1

The generating function $G(x)$ for the sequence is defined to be the formal power series $g_0 + g_1x + g_2x^2 + \cdots$. It is easy to deduce from (1) that

$$G(x) = \frac{1-x}{1-4x+x^2}.$$

In examining the dissections of the $3 \times 2n$ strip, we observed that they fall into two kinds. A dissection of the first kind can be divided by a vertical line into two substrips without splitting any dominoes. Such a line is called a *fault line*. A dissection of the second kind, called a *fault-free dissection*, has no fault lines. Figure 2 shows an example of each of the 3×4 strip.

Let f_n be the number of fault-free dissections of the

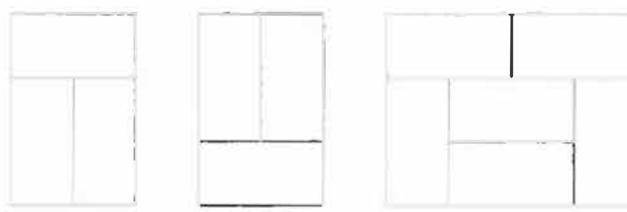

Figure 2

$3 \times 2n$ strip. From Figure 1, $f_1 = 3$. For all $n \geq 2$, a fault-free dissection cannot start with three horizontal dominoes. It must start off as shown in Figure 3, and continue by adding horizontal dominoes except for a final vertical one.

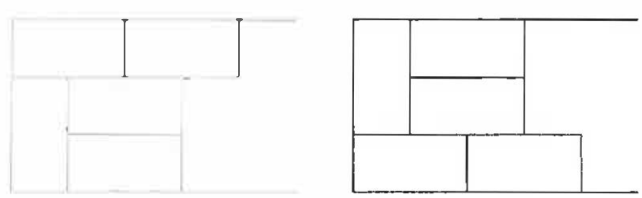

Figure 3

It follows that $f_n = 2$ for all $n \geq 2$, and the sequence satisfies a trivial recurrence relation $f_n = f_{n-1}$ for all $n \geq 3$, with initial conditions $f_0 = 1$, $f_1 = 3$ and $f_2 = 2$. Let $F(x) = f_0 + f_1x + f_2x^2 + \cdots$ be the generating function for the sequence. Then

$$F(x) = -1 + x + 2(1 + x + x^2 + \cdots) = -1 + x + \frac{2}{1-x}.$$

This simplifies to

$$F(x) = \frac{1 + 2x - x^2}{1 - x}.$$

Having solved the simpler problem of counting fault-free dissections of the $3 \times 2n$ strip, we make use of our result to find an alternative solution to the general problem of finding all dissections of this strip. They can be classified according to where the first fault line is. This is taken to be the right end of the strip if the dissection is fault-free. Then the strip is divided into a $3 \times 2k$ substrip on the left and a $3 \times 2(n-k)$ substrip on the right, where $1 \le k \le n$.

Since the first substrip is dissected without any fault lines, it can be done in f_k ways. The second substrip can be dissected in g_{n-k} ways, since we do not care whether there are any more fault lines.

Hence $g_n = f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0$.

From the values of

f_n, $g_n = 3g_{n-1} + 2g_{n-2} + 2g_{n-3} + \cdots + 2g_0$.

If we subtract from this $g_{n-1} = 3g_{n-2} + 2g_{n-3} + \cdots + 2g_0$, we have $g_n - g_{n-1} = 3g_{n-1} - g_{n-2}$, which is equivalent to (1).

We now derive (2) in another way. It follows from (4) that for all $n \ge 1$, $2g_n = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0$.

Multiplying $F(x)$ and $G(x)$ yields

$$\begin{aligned} F(x)G(x) &= (f_0 + f_1 x + f_2 x^2 + \cdots)(g_0 + g_1 x + g_2 x^2 + \cdots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0)x + (f_0 g_2 + f_1 g_1 + f_2 g_0)x^2 \\ &\quad + \cdots. \end{aligned}$$

In view of (5), this becomes

$$F(x)G(x) = g_0 + 2g_1 x + 2g_2 x^2 + \cdots = 2G(x) - 1$$

so that

$$G(x) = \frac{1}{2 - F(x)}$$

Substituting (3) into (6) yields (2).

The Second Problem

Let g_n be the number of ways of dissecting a $4 \times n$ strip into dominoes. Then $g_0 = 1$, $g_1 = 1$ and, as shown in Figure 4, $g_2 = 5$. It is not hard to verify that $g_3 = 11$. We wish to determine the infinite sequence $\{g_n\}$ via recurrence relations and generating functions.

Let f_n be the number of fault-free dissections of the $4 \times n$ strip. We have $f_0 = 0$, $f_1 = 1$ and from Figure 4, $f_2 = 4$. For odd $n \ge 3$, the only fault-free dissections are the extensions of the second and third ones in Figure 4, with horizontal dominoes except for a final vertical one. Hence

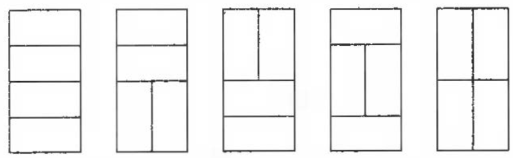

Figure 4

$f_n = 3$. For even $n \ge 4$, $f_n = 3$ since we can also include similar extensions of the fourth dissection in Figure 4.

As in the solution of the First Problem, we have

$$g_n = f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0.$$

This leads to

$$g_n = g_{n-1} + 5g_{n-2} + g_{n-3} - g_{n-4}$$

for all $n \ge 4$, with initial conditions $g_0 = 1$, $g_1 = 1$, $g_2 = 5$ and $g_3 = 11$. Also,

$$\begin{aligned} F(x) &= 1 + x + 4x^2 + 2x^3 + 3x^4 + 2x^5 + 3x^6 + \cdots \\ &= -2 - x + 2x^2 + 3(1 + x^2 + x^4 + x^6 + \cdots) + \\ &\qquad 2x(1 + x^2 + x^4 + \cdots) \\ &= -2 - x + x^2 + \frac{3 + 2x}{1 + x} \\ &= \frac{1 + x + 3x^2 + x^3 - x^4}{1 - x^2}. \end{aligned}$$

Substituting (8) into (6), which is still valid here, we have

$$G(x) = \frac{1 - x^2}{1 - x - 5x^2 - x^3 + x^4}.$$

We now give an alternative solution to the Second Problem, along the line of the solution to the First Problem we learned at the Mathematics Club. We classify the dissections of the $4 \times n$ strip into five types according to how they start. These correspond to those in Figure 4 if we ignore the vertical dominoes in the second column. Call these Types A, B, C, D and E, and let their numbers be a_n, b_n, c_n, d_n and e_n, respectively.

By symmetry, we have $b_n = c_n$ so that

$$g_n = a_n + 2b_n + d_n + e_n$$

for all $n \ge 1$. In a Type A dissection, we are left with a

$4 \times n - 2$ strip, which can be dissected in g_{n-2} ways. Hence

$$a_n = g_{n-2}$$

for all $n \geq 3$, with $a_1 = 0$ and $a_2 = 1$.

In a Type B dissection, if we complete the second column with a vertical domino, the remaining $4 \times n - 2$ strip can be dissected in g_{n-2} ways. The only alternative is to fill the second column with two horizontal dominoes. The remaining part can be dissected in b_{n-1} ways, so that $b_n = g_{n-2} + b_{n-1}$ for all $n \geq 3$, with $b_1 = 0$ and $b_2 = 1$.

In a Type D dissection, the situation is similar except that if we fill the second column with two horizontal dominoes, we must then also fill the third column with two more horizontal dominoes. The remaining part can be dissected in d_{n-2} ways, so that $d_n = g_{n-2} + d_{n-2}$

for all $n \geq 3$, with $d_1 = 0$ and $d_2 = 1$.

Finally, in a Type E dissection, after filling the first column with two vertical dominoes, we are left with a $4 \times n - 1$ strip which can be dissected in g_{n-1} ways. Hence $e_n = g_{n-1}$

for all $n \geq 2$, with $e_1 = 1$.

Now (7) follows from (10), (11), (12), (13) and (14)

$$\begin{aligned} g_n &= a_n + 2b_n + d_n + e_n \\ &= g_{n-2} + (2g_{n-2} + 2b_{n-1}) + g_{n-2} + d_{n-2} + g_{n-1} \\ &= g_{n-1} + 4g_{n-2} + 2(g_{n-3} + b_{n-2}) + d_{n-2} \\ &= g_{n-1} + 4g_{n-2} + 2g_{n-3} + 2b_{n-2} + d_{n-2} \\ &= g_{n-1} + 4g_{n-2} + 2g_{n-3} + g_{n-2} - a_{n-2} - e_{n-2} \\ &= g_{n-1} + 5g_{n-2} + g_{n-3} - g_{n-4}. \end{aligned}$$

Using (7), $(1 - x - 5x^2 - x^3 + x^4)G(x)$ simplifies to $1 - x^2$.

Hence (9) also follows.

Supplementary Problem

Let f_n denote the number of fault-free dissections of the $4 \times n$ strip. Find a recurrence relation for the sequence $\{f_n\}$ with initial conditions.

A Space Interlude

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Kenneth Ng

Philip Stein

students, Ross Sheppard Composite High School, Edmonton

Space Station Intelligentsia received a call for help on the hyperradio from Spaceship Academia. Captain Philip said, "We are on the way home after a successful mission of promoting higher learning in distant star systems. We are surrounded by a Kleingon Fleet. We are unarmed. Please send a relief force."

"Unfortunately, there are no other Spaceships on base at the moment," said Commander Gilbert. "Can you hold out?"

"Affirmative," said Captain Philip, "but we cannot disengage. It would help if you could get a Space Cannon to us."

"No problem. We will send one over by a Space Pod."

"Hang on a minute! Oh, no! The Kleingon Fleet has just been reinforced by a Space Tetropus. It can grab one Space Pod at a time."

"I will send two Space Pods, each carrying a Space Cannon," said Commander Gilbert.

"Do not do that! Repeat! Do not do that!" Captain Philip said urgently. "If a Space Cannon falls into the hands of the Kleingons, we are history. It is too powerful even for us."

"I will get back to you as soon as possible."

Commander Gilbert consulted Lieutenant Kenneth, the scientific advisor. He said, "We can break up a Space Cannon into two component parts and send them separately. This way, the Kleingons can only get half of it, which is of absolutely no use to them."

"Unfortunately, Spaceship Academia will not get too much out of the other half. However, your idea is an excellent one. If we break up two Space Cannons into two component parts in identical fashion and send them by four Space Pods, the Kleingons will still be out of luck, while Space Academia will have enough parts to reassemble a complete one."

The two officers were very pleased with their plan. However, when they tried to put it in operation, they found that there were only three Space Pods on base.

Lieutenant Kenneth thought for a while and said, "We can still do it, but we have to break up two Space Cannons into three component parts in identical fashion. Let us call them A, B and C. The first Space Pod will carry A and B, the second B and C, and the third C and A. We cannot lose both copies of any part, so that Spaceship Academia can still get a complete Space Cannon, while the Kleingons can only get two-thirds of it."

"It would be best if we do not break up the Space Cannons into too many component parts. Couldn't we still do it with only two?"

"No. Since we have four copies and three Space Pods, one of them must carry two. These must be different as there is no point in any Space Pod carrying two identical parts. If the Space Tetropus grabs this one, the Kleingons will get a complete Space Cannon."

"I guess you are right," said Commander Gilbert. "It is lucky that we have three Space Pods. Had there been only two, we could not have done anything."

"Yes, each of Spaceship Academia and the Kleingons will get one. Either both have a chance of getting a complete Space Cannon, or neither has, which is definitely not good for us."

"Let us stop theorizing and put our plan to work. We cannot count on Spaceship Academia holding out forever against the Kleingons."

This was done, and soon words came over the hyperradio that all was well. Before long, Spaceship Academia was docking at Space Station Intelligentsia. Commander Gilbert and Lieutenant Kenneth welcomed Captain Philip's safe return.

"That was a close call," reported Captain Philip. "The Kleingons were about to replace the Space Tetropus with a Space Octopus, which can grab two Space Pods at a time."

"This is serious," said Commander Gilbert. "Let us go to work at once and figure out a solution, rather than wait until we have to face the situation."

"To begin with," said Lieutenant Kenneth, "we have to break up three Space Cannons. This way, we cannot lose every copy of any component part. On the other hand, we do not need to break up more than three, as that will only make things easier for the Kleingons."

"Also, each Space Cannon must be broken up into at least three component parts," Captain Philip said. "If there are only two, the Space Octopus can just nab one Space Pod carrying each part, and the Kleingons will have a complete Space Cannon. If we break it up into exactly three component parts, we will need nine Space Pods so that each one will carry one part. Nothing less will do."

"We seldom have that many Space Pods on base," Commander Gilbert pointed out. "What is the smallest number of Space Pods that can carry out a successful convoy?"

"It has to be five or more. If we send only four, each side will get two, and that is bad news. This is the same argument that we use to explain why two Space Pods are not enough for getting around a Space Tetropus."

"Are five Space Pods enough though?" Commander Gilbert pressed the point.

"This is tough," said Captain Philip. "Let us consider all possible scenarios. If we number the Space Pods 1, 2, 3, 4 and 5, the Space Octopus may nab 1 and 2, 1 and 3, 1 and 4, 1 and 5, 2 and 3, 2 and 4, 2 and 5, 3 and 4, 3 and 5, or 4 and 5. So for any of these ten pairs, there must be at least one component part neither of which is carrying."

"Going back to what I said earlier," chimed in Lieutenant Kenneth, "we must have three copies of each part. Therefore, if 1 and 2 are missing part A, then 3, 4 and 5 must have it."

"This means that we must break up each Space Cannon into ten component parts, so that each of the ten pairs will be missing a different part. This will work. Let us draw a chart to show what each Space Pod should be carrying. We will call the component parts A, B, C, D, E, F, G, H, I and J."

Captured Space Pods	1 2	1 3	1 4	1 5	2 3	2 4	2 5	3 4	3 5	4 5	S P
Parts carried by each of the Space Pods					E	F	G	H	I	J	1
		B	C	D				H	I	J	2
	A		C	D		F	G			J	3
	A	B		D	E		G		I		4
	A	B	C		E	F		H			5

"Wow!" the three officers looked at one another and smiled.

Acknowledgments

This project is based on "The Couriers Problem" in Dennis Shasha's *The Puzzling Adventures of Dr. Ecco*.

Supplementary Problem

What is the minimum number of parts into which each Space Cannon must be divided in order to get around a Space Octopus, if

1. eight Space Pods are available;
2. seven Space Pods are available;
3. six Space Pods are available?

How to Flip without Flipping

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A red triangular cardboard is lying on a desk. We wish to get a physical copy of its mirror image. The simplest way is to flip the cardboard over. However, we discover that it is plain on the other side, and we want a red copy of the mirror image. We are allowed to use straight cuts to dissect the triangle into pieces for reassembly. The problem is to minimize the number of pieces.

No dissection is necessary if the triangle is isosceles, as this is the only class of triangle for which the problem can be solved in one piece. On the other hand, three pieces are sufficient for any triangle, as illustrated in Figure 1.

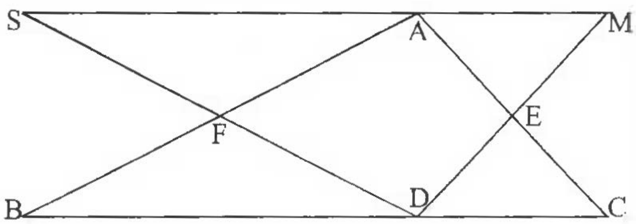

Figure 1

Let ABC be any triangle with BC its longest side. Then the foot D of the perpendicular AD from A lies between B and C. Make the cuts along DE and DF, where E is the midpoint of AC and F the midpoint of AB. Then $AE = DE = CE$ and $AF = DF = BF$. We can rotate CDE about E into AME and BDF about F into ASF. It follows easily that DSM is a mirror image of ABC.

The main part of our investigation is to determine all triangles for which the problem is solvable in two pieces. In other words, only one straight cut is allowed. This class trivially includes all isosceles triangles which we will ignore for now. Then there are two cases, according to the position of the cut.

Case 1

The cut passes through a vertex of the triangle, which is dissected into two isosceles triangles. If we overlay the original triangle on its mirror image so that the triangular pieces with the largest interior angles coincide, we obtain Figure 2.

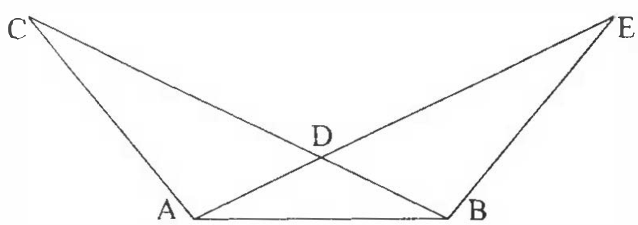

Figure 2

Being the largest interior angle, $\angle ADB$ is greater than $\angle ADC$, so that it must be obtuse. In order for BAD to be an isosceles triangle, we must have $\angle BAD = \angle ABD$. Denote their common value by θ. Then $\angle ADC = 2\theta$. There are three ways in which CAD may become an isosceles triangle.

Subcase 1(a)

Here $\angle ACD = 2\theta$. Then $\angle CAD = 180° - 4\theta > 0°$. This class consists of all triangles in which two of the angles are in the ratio 1:2, where the smaller angle θ satisfies $0° < \theta < 45°$.

Subcase 1(b)

Here, $\angle CAD = 2\theta$. Then $\angle CAB = 3\theta$ and $\angle ACD = 180° - 4\theta > 0°$. This class consists of all triangles in which two of the angles are in the ratio 1:3, where the smaller angle θ satisfies $0° < \theta < 45°$.

Subcase 1(c)

Here, $\angle ACD = \angle CAD$. Then their common value is $90° - \theta$ so that we have $\angle CAB = 90°$. This class consists of all right triangles.

Case 2

The cut does not pass through any vertex of the triangle, which is dissected into a kite and an isosceles triangle. If we overlay the original triangle on its mirror image so that the kite pieces coincide, we obtain Figure 3.

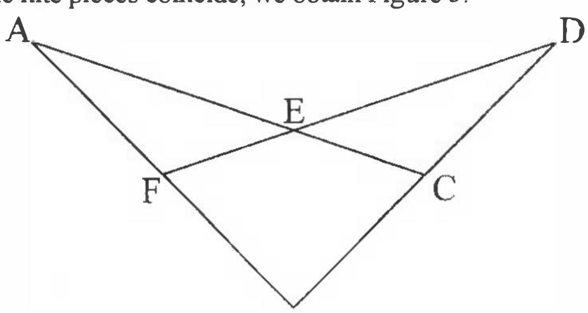

Figure 3

We must have $BC = BF$ and $CE = EF$. Note that we cannot possibly have $AE = EF = CE = DE$ as otherwise $ADCF$ would be a rectangle, and AF will not meet DC. Hence there are only two ways in which AEF may become an isosceles triangle. Denote the common value of $\angle AEF = \angle DEC$ by θ.

Subcase 2(a)

Here $\angle CAB = \theta$. Then $\angle BCA = 2\theta$ and $\angle ABC = 180° - 3\theta > 0°$. This class consists of all triangles in which two of the angles are in the ratio 1:2, where the smaller angle θ satisfies $0° < \theta < 60°$, and contains as a subclass all triangles under Subcase 1(a).

Subcase 2(b)

Here $\angle AFE = \theta$. Then $\angle CAB = 180° - 2\theta > 0°$, $\angle BCA = 180° - \theta$ and $\angle ABC = 3\theta - 180° > 0°$. This class consists of all triangles of the form $(180° - \theta, 180° - 2\theta, 3\theta - 180°)$, where $60° < \theta < 90°$.

We summarize our finding in the following statement.

Theorem

A triangle may be dissected by a straight cut into two pieces which can be reassembled into a mirror image of the original triangle if and only if the triangle satisfies at least one of the following conditions:

- it is isosceles;
- it has a right angle;
- two of its angles are in the ratio 1:2, with the smaller angle θ satisfying $0° < \theta < 60°$;
- two of its angles are in the ratio 1:3, with the smaller angle θ satisfying $0° < \theta < 45°$;
- it is of the form $(180° - \theta, 180° - 2\theta, 3\theta - 180°)$, where $60° < \theta < 90°$.

Supplementary Problem

A non-isosceles triangle is dissected by a straight cut into two pieces which can be assembled, without turning either piece over, into a mirror image of the original triangle, in three different ways. Find all possible values of the angles of such a triangle.

A Tale of Two Cities

Clemens Heuberger

university student, Austria

In the summer of 1993, I went from Graz, Austria to Beloretsk, Russia, for the International Mathematics Tournament of the Towns Problem-solving Workshop. For an account of the event, see [1].

I was the top prize winner for my solution of one of the five problems posed. Part of my work has been reported in [2]. Here are some further results. To set the scene, let me restate the problem.

The road network of a certain city consists of a continuous chain of circles. At the point of tangency of two adjacent circles, the roads cross over as shown in Figure 1, which illustrates the case with four circles.

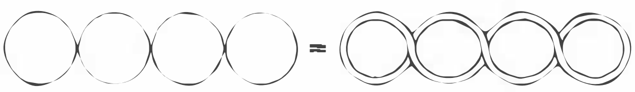

Figure 1

A ring road is constructed around the city, and is integrated with the inner chain at various points. At each integration point, the ring road is crossed over with the inner chain as shown in Figure 2, which illustrates the case with two integration points.

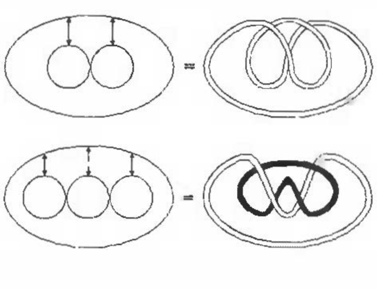

Figure 2

Note that the first integrated network consists of only one component while the second one consists of two mutually inaccessible components. We call integrated networks like the former "regular" and those like the latter "irregular".

In [2], it was proved that for studying regularity, we may assume that all integrated networks have the following properties:

- Each circle in the inner chain has at most one integration point.
- Each circle at either end of the inner chain has an integration point.
- All integration points are on the north side of the ring road.

Integrated networks that have these properties are called normalized integration networks, abbreviated to NINs. Both examples in Figure 2 are NINs. A necessary and sufficient condition for regularity was established in [2] for them. Henceforth, we restrict our attention to NINs.

Suppose two cities with regular NINs decide to merge. Their road networks are combined as follows. The inner chains are attached end-to-end, and the two ring roads are replaced by a common one, with all integration points preserved.

Figure 3 shows two examples. The new NIN is regular in one case and irregular in the other. The problem is to determine when such a merger leads to a regular NIN. Of course, we can apply the characterization in [2] afterwards, but we would like to tell beforehand just from the properties of the individual NINs.

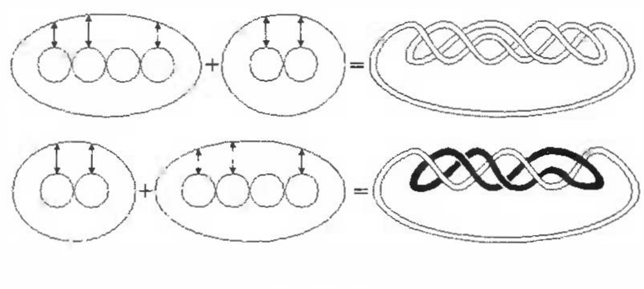

Figure 3

Consider the regular NIN in Figure 4. Each of the two circles is divided by their point of tangency and the integration points into two arcs. The latter divide the ring road into two segments. The arrows indicate the direction of travel on each arc and segment as we go once round the NIN.

We define the orientation of an arc of a circle as follows: an arc is "positive" if it is traversed in the same direction as the segment of the ring road closest to it, and "negative" otherwise. All arcs in Figure 4 are negative.

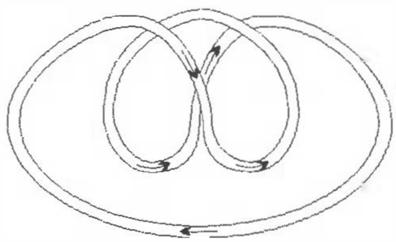

Figure 4

Theorem 1

All arcs on the same circle have the same orientation.

Proof

If the inner chain has only one circle, the whole circle is a single arc, and the result is trivial. Hence we may assume that the inner chain has at least two circles. We first observe that two arcs separated by an integration point have the same orientation. Figure 5 shows the only two non-equivalent configurations. Both arcs in either case have the same orientation.

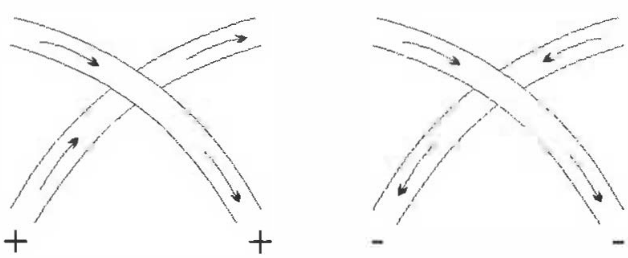

Figure 5

If we number the circles consecutively, we can prove the desired result inductively along the inner chain, starting from the circle at the west end. It consists of two arcs separated by an integration point, and we have already pointed out that they have the same orientation.

Suppose the result holds for a particular circle. Consider the one to the east. Figure 6 shows the four non-equivalent configurations. In each case, the two arcs on the next circle separated by the point of tangency have the same orientation.

If this circle has only two arcs, the desired result is established for it. If it has a third arc, it must be separated from one of the other two by an integration point. From our earlier remark, we know that it also has the same orientation. This completes the proof of Theorem 1.

By virtue of Theorem 1, we can now define the orientation of a circle of the inner chain as that of any of its arcs. The following follows from the observation that the first configuration in Figure 6 is actually impossible.

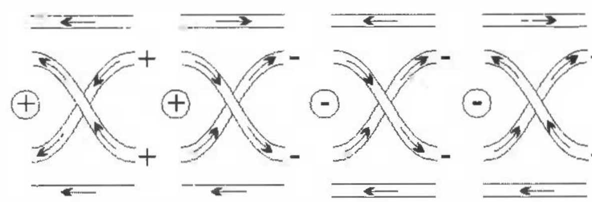

Figure 6

Corollary

Two positive circles cannot be tangent to each other.

Theorem 2

A merger of two regular NINs yields a regular NIN if and only if the orientations of the two circles brought into contact are the same.

Proof

Figure 7 shows a situation where a positive circle at the east end of the inner chain of one regular NIN is brought into contact with a negative circle at the west end of another. Of the four loose ends in the first NIN, the direction of travel forces A and C to be connected to B and D.

Since the NIN is regular, A must be connected to D and C connected to B. Similarly, E must be connected to G and F connected to H in the second NIN. After the merger, one component of the new NIN goes from A via F to H and returns via D. It is inaccessible from the component which goes from C via G to E and returns via B. Hence it is irregular.

Figures 8 and 9 show the cases where circles with the same orientation are brought together. Analogous arguments show that these mergers always yield regular NINs. This completes the proof of Theorem 2.

Note that in Figures 8 and 9, the circles which are brought into contact become negative circles in the new regular NIN. This yields the following result on the inverse operation of merging.

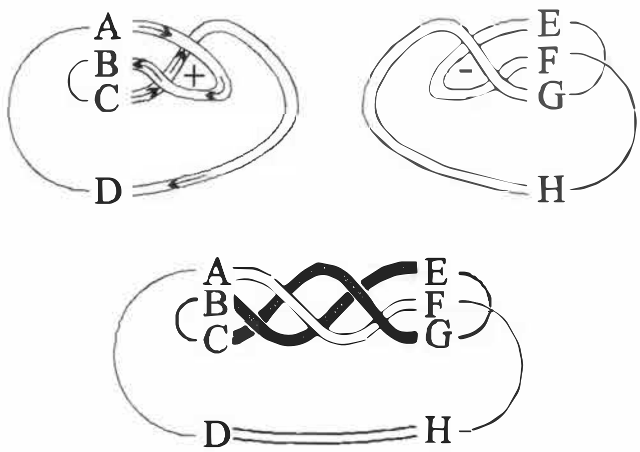

Figure 7

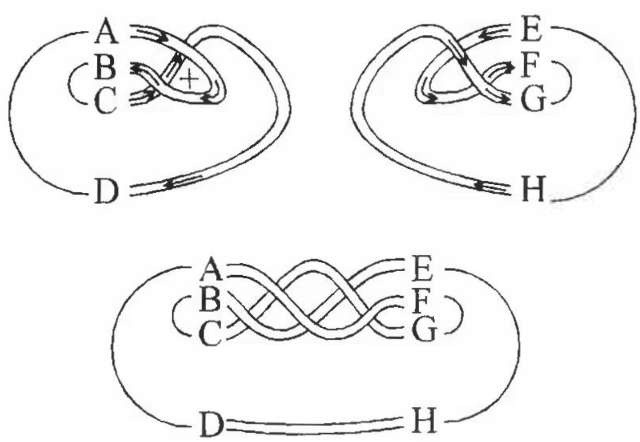

Figure 8

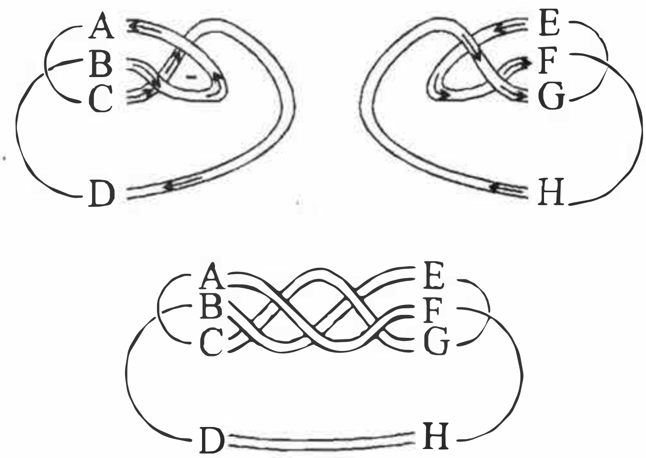

Figure 9

Corollary

A regular NIN may be detached to form two regular NINs if detachment occurs between two negative circles each with an integration point.

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Supplementary Problem

In a regular normalized integrated network, a positive circle can have only negative circles as neighbors, but a negative circle may have neighbors of either kind. Find a necessary and sufficient condition for a negative circle to have one neighbor of each kind.

Appendix II: High School Mathematics Competitions in Alberta

A. Recent History

Professor **Alvin Baragar**, Chair of the Alberta High School Mathematics Competition Board, has retired from the University of Alberta. His successor is his former colleague, Professor **Ted Lewis**, whose hotline for quick consultation is 492-4565. Prof **Claude Laflamme** of the University of Calgary has also joined the Board. As it turned out, Dover Publications Incorporated of New York sponsored our contest for only one year, but the Board is nevertheless grateful for the support.

At the moment, the Canadian Mathematical Society is contemplating overhauling the structure of the Canadian Mathematical Olympiad. As a result, the Alberta High School Mathematics Competition (A.H.S.M.C.) may no longer serve as a qualifier for the national contest. However, it will continue to be the flagship of mathematics competitions in the province.

The newsletter on problem-solving *Postulate* has temporarily ceased publication. In its place, the A.H.S.M.C. Board has been publishing the *Alberta High School Mathematics Competition News*, which contains various announcements from the Board as well as a selection of past contests.

In 1990, Edmonton became the first Canadian city to participate in the **International Mathematics Tournament of the Towns**. This is an inter-city competition based in Moscow, with the Central Organizing Committee under the leadership of Professor **Nikolay Konstantinov** and Professor **Nikolay Vasiliev**. In recent years, some Calgary students also participated through the Local Organizing Committee in Edmonton. Currently, Toronto is the only other Canadian city taking part, with the workload undertaken by Professor **Eugene Kantorovitch**.

The Central Organizing Committee offers no prizes except diplomas in Russian for students who achieve a certain standard. Quite a number of Edmonton area students have won multiple diplomas. They include **Calvin Li** of Archbishop MacDonald, **Jason Colwell** and **Matthew Wong** of Old Scona, **Steven Laffin** of J.H. Picard and **Byung-Kyu Chun** of Harry Ainlay. The Local Organizing Committee offers certificates as well as modest prizes in the form of mathematics books and journals.

In 1993, **Daniel van Vliet** of Salisbury Composite High School and **Matthew Wong** attended the Tournament's Problem-solving Workshop in Beloretsk, Russia. **Clemens Heuberger**, the author of one of the student projects in Appendix I, was also there and won a prize.

In 1995, Canada hosted the International Mathematical Olympiad for the first time. The Problem Selection Committee consisted largely of members of the A.H.S.M.C. Board. National team member **Byung-Kyu Chun** won a bronze medal.

B. Sample Multiple-choice Questions of the A.H.S.M.C., 1967-1983

Multiple-choice questions were first introduced into the Alberta High School Mathematics Competition in 1967. There were 25 of them in 1967-70, and 20 in 1971-83. These were to be attempted, along with a number of essay-type problems, in a single paper. Since 1983-84, the two parts of the competition are written in separate sittings. The following questions have been typeset by **Hubert Chan** of Archbishop MacDonald High School.

1967

1. When the base of a triangle is increased 10%, and the altitude to this base is decreased 10%, the change in area is

   (a) 1% increase
   (b) $\frac{1}{2}$ % increase
   (c) 0%
   (d) $\frac{1}{2}$ % decrease
   (e) 1% decrease

2. If $\frac{4^x}{2^{x+y}}=8$ and $\frac{9^{x+y}}{3^{5y}}=243$, then xy is

   (a) $\frac{12}{5}$
   (b) -4
   (c) 4
   (d) 12
   (e) 6

3. The value of $\frac{(4-\sqrt{5})(2+\sqrt{5})}{7+\sqrt{5}}$ is

   (a) $\frac{8}{7}-\sqrt{5}$
   (b) $\frac{4-\sqrt{5}}{11}$
   (c) $\frac{8+4\sqrt{5}}{11}$
   (d) $\frac{5}{\sqrt{5}-1}$
   (e) $\frac{1+\sqrt{5}}{4}$

4. The graph of the equation $x^2 - 4y^2 = 0$ is

   (a) a parabola
   (b) a point
   (c) an ellipse
   (d) a pair of straight lines
   (e) none of the above

5. If $x - y < x$ and $x + y < y$ then

   (a) $y < x$
   (b) $0 < x < y$
   (c) $x < y < 0$
   (d) $x < 0, y < 0$
   (e) $x < 0 < y$

6. The number of values of x satisfying the equation $\frac{2x^2-10x}{x^2-5x}=x-3$ is

   (a) zero
   (b) one
   (c) two
   (d) three
   (e) more than three

7. The radius of the circle whose equation is $x^2 + y^2 - 16x - 10y + 64 = 0$ is

   (a) 4
   (b) 5
   (c) 6
   (d) 8
   (e) 10

8. One root of the equation $\left(\frac{1}{1+x}\right)^3+\left(\frac{1}{1-x}\right)^3+\frac{1}{2}=0$ is $i=\sqrt{-1}$. Then the number of real roots is

   (a) zero
   (b) one
   (c) two
   (d) three
   (e) four

9. The sides of a triangle are 8, 13 and 15 inches. The number of square inches in its area is

   (a) 52
   (b) $20\sqrt{2}$
   (c) 60
   (d) $30\sqrt{3}$
   (e) none of the above

10. If for all x we have $1 = ax^2 + (bx + c)(x + 1)$, then

    (a) $c + a + 2b = 0$
    (b) $a + b + 2c = 0$
    (c) $b + c + 2a = 0$
    (d) $ab = c^2$
    (e) $bc = a^2$

11. When $x^3 + k^2x^2 - 2kx - 6 = 0$ is divided by $x + 2$, the remainder is 10. Then k must be

    (a) 2
    (b) -2

(c) 2 or -3
(d) 2 or -1
(e) none of the above

12. A student wrote that the product of $a + i$ and $b - i$ was $a + b + i$ where $i = \sqrt{-1}$. If this answer was correct, then the minimum value of ab is

(a) 2
(b) 1
(c) 0
(d) -1
(e) -2

13. The converse of the statement "If $a = 0$, then $ab = 0$" is

(a) If $a \neq 0$, then $ab \neq 0$.
(b) If $a \neq 0$, then $ab = 0$.
(c) If $a = 0$, then $ab \neq 0$.
(d) If $ab = 0$, then $a = 0$.
(e) If $ab = 0$, then $a = 0$ or $b = 0$.

14. Let ABC be a triangle with $\angle A < \angle C < 90° < \angle B$. Consider the external angle-bisectors at A and B, each measured from the vertex to the opposite side (extended). If each of these line segments is equal to AB, then A is

(a) 6°
(b) 9°
(c) 12°
(d) 15°
(e) none of the above

15. The sum and the product of two numbers are each equal to $s + \frac{1}{s} + 2$ where $s > 1$. Then the difference between the squares of the reciprocals of the numbers is

(a) 1
(b) 2
(c) $\left(\frac{s-1}{s+1}\right)^2$
(d) $\frac{s-1}{s+1}$
(e) at least 1

16. The distance that a body falls from rest varies as the square of the time of falling. If it falls from rest at a distance of 256 feet in 4 seconds, then during the tenth second it falls a distance (in feet) of

(a) 288
(b) 304
(c) 320
(d) 336
(e) 384

17. The roots of the equation $x^3 + 3px^2 + q^2x + r^3 = 0$ are in arithmetic progression. Then we must have

(a) $pq^2 = 2p^3 + r^3$
(b) $p = 0$
(c) $q = 3pr^3$
(d) $3p + r^3 = 2q^2$
(e) $3pr = q^2$

18. In calm weather an aircraft can fly from one city to another 200 miles north of the first and back in exactly 2 hours. In a steady north wind the round trip takes 5 minutes longer. The speed of the wind (in miles per hour) is

(a) 8
(b) 16
(c) 32
(d) 35
(e) 40

19. The length of the common chord of two intersecting circles is 16 feet. If the radii are 10 feet and 17 feet, then the distance (in feet) between the centres is

(a) 27
(b) 21
(c) $\sqrt{389}$
(d) 15
(e) none of the above

20. The number of positive integers less than 500 that are divisible by neither 3 nor by 5 is

(a) 269
(b) 267
(c) 265
(d) 234
(e) 201

21. The system of equations $x + (k - 2)y = 1$ and $(k + 2)x - 3y = 1$ can be solved for x and y in terms of k, provided that

(a) $k \neq 1$
(b) $k \neq 0$
(c) $k \neq -1$

(d) $k \neq 1, k \neq -1$

(e) none of the above

22. The smoke trail of a steamship sailing due east at 30 knots is in a direction 60° west of north. It overtakes a freighter sailing east a 10 knots, whose smoke trail is in a direction 30° west of north. The wind must be blowing from the direction

(a) 30° east of north

(b) 135° west of north

(c) due north

(d) due south

(e) 150° west of north.

23. When the last digit of a certain six-digit number N is transferred to the first position (the other digits moving one place to the right), the number is exactly one-third of N.
The sum of the six digits is

(a) 28

(b) 27

(c) 26

(d) 25

(e) 24

24. In the diagram, CA = CF and ∠B = ∠C.
Then we must have

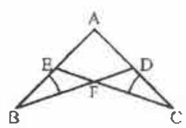
(a) AE = EF

(b) AE = AD

(c) AD = $\frac{1}{2}$CF

(d) AE = EB

(e) AE = DF



25. The guests at a party play as follows: each player in turn names a real number (not 0 or 1), the rules being (i) no number is to be repeated; (ii) either the sum or the product of every pair of successive numbers must be 1.
The greatest positive number is

(a) 2

(b) 4

(c) 6

(d) 8

(e) limitless

1968

1. The equation $x^2 + x + 2 = 0$ has

(a) two positive roots

(b) two negative roots

(c) one positive root and one negative root

(d) no real roots

(e) none of the above

2. The equation $(a + b)x = 3$ in x will have no solution when

(a) $a = b$

(b) $a = -b$

(c) $a + b = 3$

(d) $a + b = -3$

(e) under any circumstance

3. The most general parallelogram which has equal diagonals is a

(a) rhombus

(b) square

(c) rectangle

(d) trapezium

(e) none of the above

4. What is the value of $5^{\log_5 6}$?

(a) 1

(b) 5

(c) 6

(d) $\log_6 5$

(e) none of the above

5. Which of the following constructions is impossible, using only an unmarked ruler and compass?

(a) trisecting a given angle

(b) trisecting a given line

(c) bisecting a given line

(d) bisecting a given angle

(e) none of the above

6. Given that $\log_{10}2 = x$ and $\log_{10}3 = y$, then $\log_{10}15$?

(a) $1 + x + y$

(b) $1 - x - y$

(c) $1 + x - y$

(d) $1 - x + y$

(e) none of the above

7. Let S be the set of points, (x, y), in the plane satisfying both $x^2 + y^2 \leq 1$ and $x^2 + y^2 \geq r^2$. A value of r such that S is the empty set is

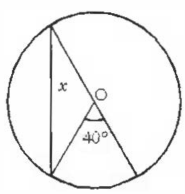
(a) 1
(b) -1
(c) $\frac{1}{2}$
(d) $-\frac{1}{2}$
(e) none of the above

8. If S, T, and V are any sets, then $[(S\cap T)\cup(S\cap V)]$ is the same set as
   (a) S
   (b) $T\cup V$
   (c) $T\cap V$
   (d) $S\cap(T\cup V)$
   (e) none of the above

9. What is the greatest number of possible points of intersection of three lines in the plane, of different slopes?
   (a) 2
   (b) 4
   (c) 6
   (d) 8
   (e) none of the above values

10. What is the greatest number of possible points of intersection of three circles, in the plane, of different radii?
   (a) 2
   (b) 4
   (c) 6
   (d) 8
   (e) none of the above values

11. If O is the centre of the given circle, what is the value of the angle, x?
   (a) 30°
   (b) 40°
   (c) 60°
   (d) 80°
   (e) none of the above



12. A metal disc has one face marked "1" and the other face marked "2". A second metal disc has one face marked "2" and the other marked "3". Assume that, when tossed, the two faces of a disc are equally likely to turn up. If both discs are tossed, what is the probability that "4" is the sum of the numbers turning up?
   (a) $\frac{1}{4}$
   (b) $\frac{1}{2}$
   (c) $\frac{3}{4}$
   (d) $\frac{1}{3}$
   (e) none of these values

13. Which of the following statements about $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ is true?
   (a) it is irrational
   (b) it is rational
   (c) it is imaginary
   (d) it is positive
   (e) none of the above

14. What is the longest rod that can be put in a rectangular box of dimensions 6'×3'×2'?
   (a) 6 ft
   (b) $3\sqrt{5}$ ft
   (c) $2\sqrt{10}$ ft
   (d) 7 ft
   (e) none of the above

15. If $i =$, then i^6 is
   (a) 1
   (b) -1
   (c) i
   (d) $-i$
   (e) none of the above

16. If $i = \sqrt{-1}$, then $i\,\frac{1+i}{1-i}$ is
   (a) 1
   (b) -1
   (c) i
   (d) $-i$
   (e) none of the above

17. The solution set of the inequality $x^2 - x - 2$ is the interval
   (a) $-2 \le x \le 1$
   (b) $-2 < x < 1$
   (c) $-2 \le x < 1$
   (d) $-2 < x \le 1$

(e) none of the above

18. If $a > 0, b > 0, a > b$, which of the following is false?

(a) $\frac{1}{a} > \frac{1}{b}$
(b) $a^2 > b^2$
(c) $a^3 > b^3$
(d) $\sqrt{a} > \sqrt{b}$
(e) none of the above

19. The solution set in the plane of the equation $y^2 = xy$ is

(a) a parabola
(b) a rectangular hyperbola
(c) a pair of straight lines
(d) a circle
(e) none of the above

20. The distance between the two points represented by the complex numbers $1 - 2i$ and $2i - 2$ is

(a) 5
(b) $-1 + 4i$
(c) $\sqrt{17}$
(d) $3 - 4i$
(e) none of the above

21. At the end of a party, everyone shakes hands with everyone else. Altogether there are 28 handshakes. How many people are there at the party?

(a) 8
(b) 14
(c) 20
(d) 56
(e) none of the above

22. How many odd multiples of "3" are there between 100 and 200?

(a) 15
(b) 17
(c) 33
(d) 49
(e) none

23. Let $aabb$ be a four digit number in base r, with $b = 0$. This number is divisible by

(a) r only
(b) $r + 1$ only
(c) $r - 1$ only
(d) more than one of these
(e) none of the above

24. If $\{a_1, a_2, a_3, \cdots\}$ is an infinite sequence of positive numbers with $a_k \geq 1$ for $k = 1, 2, \cdots$, and we define $U_n = a_1 \cdot a_2 \cdots a_n$, then which of the following statements about I, II, and III below is true?
I: $U_n = U_{n+1}$ II : $U_n < U_{n+1}$ III: $U_n > U_{n+1}$

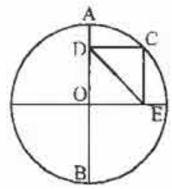
(a) I only
(b) I and II only
(c) I and III only
(d) III only
(e) none of the above

25. A rectangular floor 24' × 40' is covered by squares of sides 1' each. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

(a) 40
(b) 56
(c) 63
(d) 64
(e) none of the above

1969

1. O is the centre of the circle. CDOE is a rectangle. DE is 5 and CE is 3. The diameter of the circle is



(a) $4\sqrt{2}$
(b) 8
(c) 10
(d) $10\sqrt{5}$
(e) cannot be determined

2. A man spends $\frac{1}{3}$ of his money and loses $\frac{2}{3}$ of the remainder. He then has \$12. How much money had he at first?

(a) \$56
(b) \$27
(c) \$108
(d) \$112
(e) none of the above

3. For all real numbers a and b

(a) $a^2 + b^2 \geq 2ab$

(b) $a^2 + b^2 > 2ab$
(c) $a^2 + b^2 < 2ab$
(d) $a^2 + b^2 \leq 2ab$
(e) none of the above

4. The total number of subsets that can be formed from a set containing six elements is

(a) 4
(b) 8
(c) 16
(d) 32
(e) none of the above

5. A gambler visited three gambling houses. At the first he doubled his money, and then spent \$30, at the second he tripled his remaining money and then spent \$54, and at the third he quadrupled his remaining money and then spent \$72, and he then had \$48 left. How much money did he start with?

(a) \$29
(b) \$30
(c) \$31
(d) \$32
(e) \$33

6. Let r be the result of doubling both the base and the exponent of a^b, $b \neq 0$. If r equals the product of a^b by x^b, then x equals

(a) a
(b) $2a$
(c) $4a$
(d) 2
(e) 4

7. The symbol $|a|$ means a if a is a positive number of zero, and $-a$ if a is a negative number. For all real values of x the expression $\sqrt{x^4 + x^2}$ is equal to

(a) x^3
(b) $x^2 + x$
(c) $|x^2 + x|$
(d) $x\sqrt{1+x^2}$
(e) $|x|\sqrt{1+x^2}$

8. In the base ten number system, the number 526_{10} means $5 \cdot 10^2 + 2 \cdot 10 + 6$.If in the base r number system the equation $1000_r - 440_r = 340_r$ holds, then r is

(a) 2
(b) 5
(c) 7
(d) 8
(e) 12

9. While three watchmen were guarding an orchard, a thief slipped in and stole some apples. On his way out he met three watchmen one after another. To each he gave one half of the apples he had at the time and plus an additional two. Thus he managed to escape with one apple. How many apples did he steal originally?

(a) 16
(b) 22
(c) 32
(d) 76
(e) none of the above

10. If one man can dig a hole in one hour, and a second man can dig a hole in one and one-half hours, how many minutes must they work together to dig a hole?

(a) 16
(b) 36
(c) 46
(d) 56
(e) none of the above

11. If the radius of a circle is increased by one unit, the ratio of the new circumference to the new diameter is

(a) $\pi + 2$
(b) $\frac{2\pi+1}{2}$
(c) π
(d) $\frac{2\pi-1}{2}$
(e) $\pi - 2$

12. A square and an equilateral triangle have equal perimeters. The area of the triangle is square inches. Expressed in inches the diagonal of the square is

(a) $\frac{9}{2}$
(b) $2\sqrt{5}$

(c) $4\sqrt{2}$

(d) $\frac{9\sqrt{2}}{2}$

(e) none of the above

13. A hungry hunter came upon two shepherds. One shepherd had three loaves of bread; the other shepherd had five loaves of bread all of the same size. The loaves were divided equally among the three men and the hunter paid 8 cents for his share. How should the shepherds divide the money?

(a) 1 and 7

(b) 2 and 6

(c) 3 and 5

(d) 4 and 4

(e) none of the above

14. The average of a set of 50 numbers is 38. If two numbers of the set, namely 45 and 55 are discarded, the average of the remaining set of numbers is

(a) 38.5

(b) 37.5

(c) 37

(d) 36.5

(e) 36

15. A circle is inscribed in an equilateral triangle, and a square is inscribed in the circle. The ratio of the area of the triangle to the area of the square is

(a) $\sqrt{3}:1$

(b) $\sqrt{3}:\sqrt{2}$

(c) $3\sqrt{3}:2$

(d) $3:\sqrt{2}$

(e) $3:2\sqrt{2}$

16. Every day at noon a ship leaves New York for Lisbon and at the same instant a ship leaves Lisbon for New York. Each trip lasts exactly 8 days. How many ships from Lisbon will each ship from New York meet?

(a) 11

(b) 13

(c) 15

(d) 17

(e) none of the above

17. Points P and Q are both in the line segment AB and on the same side of its midpoint. P divides AB in the ratio 2:3, and Q divides AB in the ratio 3:4. If PQ = 2, then the length of AB is

(a) 60

(b) 70

(c) 75

(d) 80

(e) 85

18. Suppose we have two equiangular polygons P_1 and P_2 with different numbers of sides. Each angle of P_1 is x degrees and each angle of P_2 is kx degrees, where k is an integer greater than 1. The number of possibilities for the pair (x, k) is

(a) infinite

(b) finite but more than two

(c) two

(d) one

(e) zero

19. Given that the following have the same perimeters, which has the largest area?

(a) square

(b) equilateral triangle

(c) circle

(d) regular pentagon

(e) two or more of these are the same

20. The angles at A, B, C, D, E of a pentagon ABCDE are in the ratio 5:3:8:5:6. The largest of these angles has the value in degrees

(a) 90°

(b) 110°

(c) 130°

(d) 150°

(e) none of the above

21. Solve for n: $\binom{n}{8}=\binom{n}{24}$

(a) $n = 8$

(b) $n = 16$

(c) $n = 24$

(d) $n = 32$

(e) none of the above is a solution

22. The number of solutions of $2^{2x} - 3^{2y} = 55$ in which x and y are integers i s
    (a) zero
    (b) one
    (c) two
    (d) three
    (e) greater than three, but finite
23. In racing over a given distance d at uniform speed, A can beat B by 20 yards, B can beat C by 10 yards, and A can beat C by 28 yards. Then d, in yards, equals
    (a) cannot be detennined from information provided
    (b) 58
    (c) 100
    (d) 116
    (e) 120
24. If $x_{k+1} = x_k + \frac{1}{2}$ for $k = 1, 2, \cdots, n-1$ and $x_1 = 1$, find $x_1 + x_2 + \cdots + x_n$
    (a) $\frac{n+1}{2}$
    (b) $\frac{n+3}{2}$
    (c) $\frac{n^2-1}{2}$
    (d) $\frac{n^2+n}{4}$
    (e) $\frac{n^2+3n}{4}$
25. Three disks labelled 1 to 3 are put in a bag. Three other disks labelled 1 to 3 are put in a second bag. A disk is drawn from each bag and stacked in a pile on a table. This is repeated two more times. What is the probability that at least one of the stacks will contain disks with the same number?
    (a) $\frac{1}{6}$
    (b) $\frac{1}{3}$
    (c) $\frac{1}{2}$
    (d) $\frac{2}{3}$
    (e) none of the above

1970

1. The number $10a + b$, where a and b are digits, is divisible by nine if
   (a) $a + b = 7$
   (b) $a + b = 8$
   (c) $a + b = 9$
   (d) $a + b = 10$
   (e) none of the above
2. Given $a \neq b$ and if $ax+b^2 = a^2-bx$, then $x = ?$
   (a) $a + b$
   (b) $a - b$
   (c) $b - a$
   (d) $a^2 + b^2$
   (e) none of the above
3. For the function $f(x) = x^{50} - 2a^{47}x^3 + a^{50}$ the following is a factor.
   (a) $(x - a)$
   (b) $(x - a^2)$
   (c) $(x + a)$
   (d) $(x + a^2)$
   (e) none of the above are factors
4. Which of the following inequalities are true for all real positive x?
   (a) $x + \frac{1}{x} < 2$
   (b) $x + \frac{1}{x} \leq 2$
   (c) $x + \frac{1}{x} > 2$
   (d) $x + \frac{1}{x} \geq 2$
   (e) none of the above are true
5. A steamer was able to go twenty miles per hour downstream and fifteen miles per hour upstream. If on a return trip, the steamer took five hours longer coming up than going down, the total distance travelled by the steamer is
   (a) 500 miles
   (b) 600 miles
   (c) 700 miles
   (d) 800 miles
   (e) none of the above

6. Suppose that n is a positive integer. Then $\frac{n^2+(n+2)^2}{2}$ is

   (a) sometimes an integer
   (b) always a perfect square
   (c) sometimes a perfect square
   (d) never a perfect square
   (e) none of the above

7. The solution set of the inequality $x^2(x-1)^2 \le 0$ consists of

   (a) an interval
   (b) two intervals
   (c) an interval and a point
   (d) an interval and two points
   (e) none of the above

8. The expression $\sqrt[8]{x^8} + \sqrt[7]{7^7}$ is always equal to

   (a) x
   (b) $2x$
   (c) $2x^2$
   (d) 0
   (e) none of the above

9. Given the binary operation * between two integers m and n such that $m * n = m^2 + n^2$, which of the following do not hold?

   (a) commutative law
   (b) associative law
   (c) $m * n$ is an integer
   (d) $m * n \ge 0$
   (e) all of the above hold

10. Given the four quadrants labelled as follows:

II	I
III	IV

   the solution set of the simultaneous inequalities $x^2 + y < 0$, $x^2 + y^2 > 4$ lies entirely in quadrants

   (a) I and II
   (b) II and III
   (c) III and IV
   (d) IV and I
   (e) none of the above

11. Two cyclists race on a circular track. The first can ride around the track in six seconds, and the second in four seconds. If they start off at the same point, the second cyclist can overtake the first in

   (a) 12 seconds
   (b) 14 seconds
   (c) 16 seconds
   (d) 18 seconds
   (e) none of the above

12. An equilateral triangle is inscribed upside down in a larger equilateral triangle. The ratio of the area of the smaller to the area of the larger is

   (a) $\frac{1}{12}$
   (b) $\frac{1}{6}$
   (c) $\frac{1}{4}$
   (d) $\frac{1}{3}$
   (e) none of the above

13. If $f(n) = n^2$, where n is an integer, and $n \neq k$, then $\frac{f(f(n)) - f(f(k))}{f(n) - f(k)} =$

   (a) $n^2 + k^2$
   (b) $n^2 - k^2$
   (c) $\frac{n^2+k^2}{n^2-k^2}$
   (d) $\frac{n^2-k^2}{n^2+k^2}$
   (e) none of the above

14. Which of the following inequalities hold for all real x and y?

   (a) $\sqrt{x^2+y^2} < x + y$
   (b) $\sqrt{x^2+y^2} \le x + y$
   (c) $\sqrt{x^2+y^2} < |x| + |y|$
   (d) $\sqrt{x^2+y^2} \le |x| + |y|$
   (e) none of the above

15. The equation of the line through the origin and perpendicular to the line $y = 3x + 1$ is

   (a) $y = 3x$
   (b) $y = -3x$
   (c) $y = \frac{x}{3}$

(d) $y = -\frac{x}{3}$

(e) none of the above

16. Given that $\log_a b = c$, then $\log_{a^2} b = ?$

(a) c^2

(b) $\frac{c}{2}$

(c) $2c$

(d) $\sqrt{c}$

(e) none of the above

17. A polynomial which passes through the points (3,0), (4,2) and (0,6) is

(a) $y = x - 6$

(b) $y = x^4 + x^3 + 3x - 5$

(c) $y = x^2 - 5x + 6$

(d) $y = x^2 - 9$

(e) none of the above

18. Let $f(x) = 3x^2 + Kx + 1$, where K is a real constant. If r and s are the roots of $f(x)$, which of the following is impossible?

(a) $r = s$

(b) $rs = 1$

(c) $r + s = 1$

(d) r and s are both positive

(e) all of the above are possible

19. If A is the area of an equilateral triangle of side length S, then the area of an equilateral triangle of side length $2S$ is

(a) $2A$

(b) $4A$

(c) A^2

(d) $2A^2$

(e) none of the above

20. The following figures all have area one square unit. Which has the smallest perimeter or circumference?

(a) circle

(b) square

(c) equilateral triangle

(d) regular pentagon

(e) regular hexagon

21. The value of K so that the polynomial will be divisible by $x - 3$ is

(a) 1

(b) -1

(c) 2

(d) -2

(e) none of the above

22. The domain of the function $f(x) = \frac{\sqrt{x+1}}{x}$ is

(a) a single point

(b) an infinite interval

(c) an infinite interval and a single point

(d) an infinite interval with a single point deleted

(e) none of the above

23. Let $a_1, a_2, a_3, \cdots$ be a sequence of numbers with the property that the sum of the first n of them is $\frac{n}{2}(n+1)$. Then

(a) $a_k = k$

(b) 1

(c) $k + 1$

(d) $k - 1$

(e) none of the above

24. If a, b, c are positive integers such that $a + b = 2c$, then $2^a \cdot 2^b = ?$

(a) 2^c

(b)

(c) 4^c

(d)

(e) none of the above

25. A man drives from Edmonton to Calgary at a speed of 30 miles per hour (MPH). At what speed must he drive from Calgary to Edmonton so that the average speed for the whole trip is 40 MPH?

(a) 45 MPH

(b) 50 MPH

(c) 55 MPH

(d) 60 MPH

(e) none of the above

1971

1. Given that x is inversely proportional to y, y is inversely proportional to z and z is inversely proportional to v, the relation between x and v is (k is a constant)

(a) $x = kv$
(b) $x = \frac{k}{v}$
(c) $x = kv^2$
(d) $x = \frac{k}{v^2}$
(e) none of the above

2. Given the function $f(x) = 2^{x-1}$, $f(-1) =$

(a) $\frac{1}{4}$
(b) 1
(c) -1
(d) $-\frac{1}{4}$
(e) none of the above

3. A triangle with sides of length 12, 13 and 5

(a) is a right triangle
(b) is an acute triangle
(c) is an obtuse triangle
(d) does not exist
(e) none of the above

4. If $p = \frac{x+1}{x}$ where $\frac{1}{100} \leq x \leq 100$, for what value of x does p have its smallest value?

(a) 101
(b) $\frac{1}{100}$
(c) $\frac{101}{100}$
(d) 100
(e) none of the above

5. For what values of M and N is the equation $M^{\log N} = N^{\log M}$ true? Both logarithms are base 10.

(a) no values of M and N
(b) all values of M and N
(c) all negative values of M and N
(d) all positive values of M and N
(e) none of the above

6. A triangle with sides of length 5, 16 and 8

(a) is a right triangle
(b) is an obtuse triangle
(c) does not exist
(d) is an acute triangle
(e) none of the above

7. At what time between four and five o'clock is the minute hand exactly two minutes ahead of the hour hand?

(a) 4:21
(b) 4:22
(c) 4:23
(d) 4:24
(e) none of the above

8. Let n be an integer. Define $f(n)$ to be the number of positive integers not exceeding n. Then $f(n) + f(-n) =$

(a) 0
(b) $-n$
(c) n
(d) $2n$
(e) none of the above

9. Let $x = 1 - 2^t$ and $y = 1 + 2^{-t}$. Which of the following is true for all t?

(a) $y = \frac{1}{x-1}$
(b) $y = (x - 2)\cdot\frac{1}{x-1}$
(c) $y = (2 - x)\cdot\frac{1}{x-1}$
(d) $y = x\cdot\frac{1}{x-1}$
(e) none of the above

10. A perfect number is a positive integer such that it is equal to the sum of all positive integers smaller than it which divide evenly into it. Which of the following is perfect?

(a) 4
(b) 6
(c) 8
(d) 10
(e) none of the above

11. Let S be the set of points defined by the inequalities $0 \leq y \leq 1$, and $0 \leq x \leq 1$ and $y \leq x + \frac{1}{2}$. The area of the region determined by S is

(a) $\frac{1}{8}$
(b) $\frac{3}{8}$
(c) $\frac{5}{8}$
(d) $\frac{7}{8}$
(e) none of the above

12. A gear of radius 1 revolves around a fixed gear of radius 2. During one complete revolution, the smaller gear will rotate

    (a) 360°
    (b) 540°
    (c) 720°
    (d) 900°
    (e) none of the above

13. For any integer n, the expression $n(n + 1)(n + 2)$ cannot assume the value

    (a) 0
    (b) 731
    (c) 1320
    (d) 7980
    (e) none of the above

14. The equation of the perpendicular bisector of the line segment with end points (1,5) and (-3,2) is

    (a) $8x - 6y + 29 = 0$
    (b) $8x + 6y + 29 = 0$
    (c) $8x + 6y - 29 = 0$
    (d) $8x - 6y - 29 = 0$
    (e) none of the above

15. Given the quadratic equation $ax^2 + 2bx + 3c$, the absolute value of the difference of the roots is

    (a) $\frac{\sqrt{b^2 - 4ac}}{2a}$
    (b) $2 \cdot \frac{\sqrt{b^2 - 4ac}}{2a}$
    (c) $4 \cdot \frac{\sqrt{b^2 - 4ac}}{2a}$
    (d) $\frac{\sqrt{b^2 - 3ac}}{2a}$
    (e) none of the above

16. A triangle ABC with $\angle A = 30°$ is inscribed in a circle. The radius of the circle is

    (a) BC
    (b) $\frac{1}{2}(AB + AC)$
    (c) $\frac{1}{2}BC$
    (d) AB
    (e) none of the above

17. If $y^2 - 1 + \log_{10}x = 0$, the values of x for which y takes on real values are

    (a) $x \le 10$
    (b) $0 < x \le 10$
    (c) $0 < x$
    (d) all x
    (e) no x

18. The number 123456789012345678901234567890 is not divisible by

    (a) 2
    (b) 3
    (c) 5
    (d) 9
    (e) none of the above

19. Let ABC be a triangle such that $AB = 5$ and $AC = 7$. Let AH be the altitude from A to BC. Then if $AH = 1$, $BC =$

    (a) $\sqrt{24}$
    (b) $2\sqrt{24}$
    (c) $\left(1+\sqrt{2}\right)\sqrt{24}$
    (d) $\sqrt{2}\sqrt{24}$
    (e) none of the above

20. Let x and y be positive numbers. Let $a = \frac{1}{2}(x+y)$, $b = \sqrt{xy}$, and $c = \sqrt{\frac{1}{2}\left(x^2+y^2\right)}$.

    Which of the following is always true?

    (a) $a \le b \le c$
    (b) $c \le b \le a$
    (c) $b \le c \le a$
    (d) $b \le a \le c$
    (e) none of the above

1972

1. A quadrilateral ABCD is inscribed in a circle of radius 5. If the centre of the circle lies on AB, and the length of AD is 7, then diagonal BD has length

    (a) 2
    (b) 3
    (c) $2\sqrt{6}$
    (d) $\sqrt{51}$
    (e) none of the above

2. A three digit number abc is chosen. The difference between bca and abc is calculated

and found to lie between 400 and 500. The number equals

(a) 400
(b) 404
(c) 429
(d) 495
(e) none of the above

3. A pentagon has angles 110°, 143°, 87°, and 52°. Its remaining angle equals

(a) 148°
(b) 110°
(c) 178°
(d) 90°
(e) none of the above

4. In 9 years, John will be $\frac{3}{5}$ as old as his father was ten years ago. In 10 years, John will be half as old as his father was one year ago.
John's age is

(a) 8
(b) 18
(c) 12
(d) 16
(e) none of the above

5. The expression i^{2073}equals

(a) i
(b) $-i$
(c) 1
(d) -1
(e) none of the above

6. Let a and b be the roots of $x^2 - 7x + 3 = 0$. Then $a^3 + b^3$ equals

(a) 91
(b) -18
(c) 324
(d) 360
(e) none of the above

7. The long hand of a clock points exactly at a minute and the short hand points exactly two minutes ahead of the long hand. The time is

(a) 3:17
(b) 4:26
(c) 7:36
(d) 11:58
(e) none of the above

8. The expression $\cos^4\theta - \sin^4\theta$ equals

(a) $2\cos^2\theta - 1$
(b) $\cos\theta$
(c) $2\sin^2\theta - 1$
(d) $\sin\theta$
(e) none of the above

9. What is the next number in the series 1, 1, 2, 6, 15?

(a) 9
(b) 31
(c) 16
(d) 28
(e) 42

10. Which number of the following divides $9^5 + 33^5 + 39^{10} - 4^{10}$?

(a) 4
(b) 7
(c) 3
(d) 2
(e) none of the above

11. ΔABC and ΔDEF have sides AB, BC, DE, and EF of length S. $\angle ABC$ is 2α and $\angle FDE$ is α. Area of ΔABC/Area of ΔDEF is

(a) $\frac{2}{3}$
(b) $\frac{1}{2}$
(c) 2
(d) 1
(e) none of the above

12. *The expression* $f(x) = \log_2\left(-\frac{1}{x^2}\right)$. Then $f(-\frac{1}{4})$ equals

(a) 0
(b) 4
(c) -4
(d) -3
(e) none of the above

13. The sum of the first twenty odd integers is

(a) 420
(b) 800

(c) 400
(d) 190
(e) none of the above

14. The expression $\left|\frac{2+5i}{3-4i}\right|$ equals

(a) 7
(b) -7
(c) $\frac{5}{\sqrt{29}}$
(d) $\left(\frac{5}{\sqrt{29}}\right)^{-1}$
(e) none of the above

15. Let p be an integer. $2x^2 + px + 3 = 0$ has real roots for

(a) all p
(b) no p
(c) $p \le 24$
(d) $p \ge 24$
(e) none of the above

16. If $g(t) = -g(-t)$ for some function g then which of the following are true?

i) $g(t) \ge 0$ for $t \ge 0$
ii) $g(t) + g(-t) = 0$
iii) $g(0) = 0$
iv) $g(t) = t$
v) $(g(t))^2 + 2g(-t) + 1 \ge 0$

(a) all
(b) i) ii) iii)
(c) ii) iii) v)
(d) i) iv) v)
(e) none

17. The expression $i^{4421} + i^{3663}$ equals

(a) 2
(b) -2
(c) $2i$
(d) $-2i$
(d) none of the above

18. A triangle with sides of length 5, 7 and 9 is

(a) impossible
(b) obtuse
(c) acute
(d) right
(e) none of the above

19. A quadrilateral with sides 16, 38, 7, 12 is

(a) impossible
(b) convex
(c) concave
(d) a trapezoid
(e) none of the above

20. Which of the following are true if x and y are any real numbers?

(a) $x^2 + 9y^2 \le 6xy$
(b) $x^2 + 9y^2 \ge x$
(c) $x^2 + 9y^2 \ge 1$
(d) $x^2 + 9y^2 \ge 9xy$
(e) none of the above

1973

1. A triangle ABC is inscribed in a circle of radius three. Given that BC = 2 and AC = 6, then AB equals

(a) 5
(b) 6
(c) $4\sqrt{2}$
(d) $\sqrt{2} + 2\sqrt{3}$
(e) none of the above

2. The line $-x + 3y = 9$ meets the parabola $y^2 = 4x$ in

(a) no points
(b) one point
(c) two points
(d) four points
(e) none of the above

3. Of the following numbers, select the largest.

(a) $2 + 3\sqrt{3}$
(b) $3 + 2\sqrt{2}$
(c) 4
(d) $\sqrt{39}$
(e) $2\sqrt{10}$

4. Let u and v be the roots of $x^2 - 5x + 3 = 0$. Then $u^2 + v^2$ equals

(a) 9
(b) 15
(c) 19
(d) 25

(d) 25
(e) none of the above

5. If a quadrilateral is circumscribed about a circle, then

   (a) the sum of two diagonally opposite angles is 180°
   (b) it must contain a right angle
   (c) it must have two equal sides
   (d) the sum of two opposite sides is half the perimeter of the quadrilateral
   (e) none of the above

6. If * is commutative and associative, and if a * b = c, $c * a = a$, then $b * a * c * a * b$ equals

   (a) a
   (b) b
   (c) c
   (d) $b * b$
   (e) none of the above

7. What is the last digit of 728^{4921}?

   (a) 2
   (b) 4
   (c) 6
   (d) 8
   (e) none of the above

8. What natural number is $\left\{\left[\left(\sqrt{3}\right)^{\sqrt{3}}\right]^{\sqrt{3}}\right\}^{\sqrt{3}}$?

   (a) 3
   (b) 9
   (c) 27
   (d) 1
   (e) none of the above

9. The Edmonton-Calgary Airbus can fly the 189 miles in 40 minutes on a calm day. One day with a headwind, the time was 45 minutes. What was the speed of the wind in miles per hour on the day with the headwind?

   (a) 40.5
   (b) 38.2
   (c) 31.5
   (d) 60.5
   (e) none of the above

10. AB is a diameter of a circle with centre C, and D is another point on the circumference of the circle, such that ∠BCD = 72°. What is ∠BAD?

    (a) 18°
    (b) 24°
    (c) 30°
    (d) 36°
    (e) none of the above

11. A quadrilateral with sides 5, 3, 5, 7 (taken in order) must be a

    (a) parallelogram
    (b) non-isosceles trapezoid
    (c) isosceles trapezoid
    (d) rhombus
    (e) none of the above

12. A pentagon MNPQR has MN = 2, NP = 7, PQ = 4, QR = 5, RM = 1. The sum or the lengths of the diagonals MP + MQ + NQ + NR + PR cannot possibly equal

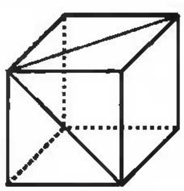
    (a) 53
    (b) 33
    (c) 18
    (d) 16
    (e) none of the above

13. 999,999,999,999 is divisible by

    (a) 23
    (b) 77
    (c) 101
    (d) 162
    (e) none of the above

14. Consider two diagonals of a regular cube, as sketched. They meet at an angle of

    (a) 90°
    (b) 60°
    (c) 45°
    (d) 72°
    (e) none of the above



15. If $f(x) = 2^x x^2$, then $f(1)$ =

    (a) 0
    (b) 1
    (c) 2
    (d) 4
    (e) none of the above

16. The athletic banquet at Dudgeon High School costs \$450.00, and the committee decided that the cost would be shared equally by all those attending. 75 of those eligible to attend did not, and as a result the cost to each attendee was 50¢ higher than it would otherwise have been. How many were eligible to attend?

    (a) 200
    (b) 225
    (c) 300
    (d) 325
    (e) none of the above

17. Find the sum of the first 100 odd numbers (i.e., $1+3+5+\cdots+199$).

    (a) 10,200
    (b) 10,201
    (c) 10,001
    (d) 10,000
    (e) none of the above

18. A given quadrilateral can be inscribed in a circle if

    (a) the sum of its angle is 360°
    (b) the sum of any two opposite angles is 180°
    (c) it has at least two right angles
    (d) the sum of opposite sides is half the perimeter
    (e) none of the above is sufficient

19. Find the next number in the sequence -3, 1, 5, 9, 31, 53, 75, 97, 101, 501

    (a) 301
    (b) 700
    (c) 505
    (d) -3
    (e) none of the above

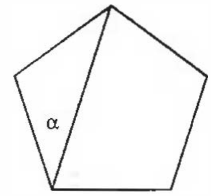
20. The altitude of a regular tetrahedron of edge length 1 is

    (a) $\frac{\sqrt{3}}{2}$
    (b) $\frac{\sqrt{6}}{3}$
    (c) $\frac{\sqrt{3}}{3}$
    (d) $\frac{2\sqrt{3}}{3}$
    (e) none of the above

1974

1. Given a regular pentagon (see sketch), the angle α is

    (a) 18°
    (b) 27°
    (c) 36°
    (d) 45°
    (e) 54°



2. Find the smallest natural number n for which $1+2+3+\cdots+n > 5000$.

    (a) 10
    (b) 99
    (c) 100
    (d) 101
    (e) 1,000

3. Let $\frac{1}{2}\left(\sqrt{5}+\sqrt{7}\right), b=\sqrt{6}$, and $c=\sqrt[4]{35}$. Then

    (a) $a<b<c$
    (b) $b<a<c$
    (c) $c<a<b$
    (d) $c<b<a$
    (e) none of the above

4. John and Susan are both younger than 5 years old. Three times John's age equals twice the age Susan will be five years from now. Susan's age is

    (a) 0
    (b) 1
    (c) 2
    (d) 3
    (e) 4

5. Consider the curve C: $y=\sin x$, $-\infty<x<\infty$. The line $y=\frac{1}{2}$ intersects C

    (a) once
    (b) twice
    (c) never
    (d) infinitely often
    (e) five times

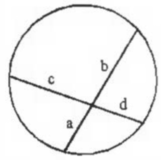
6. A right-angled cross having segments a, b, c, d (see sketch) is inscribed in a circle of radius 2. The largest possible value of a + b + c + d is

    (a) $2\sqrt{2}$
    (b) 8

(c) $4\sqrt{2}$

(d) $6\sqrt{2}$

(e) none of the above



7. The length of the segment joining any top vertex of a cube of side 1 to the midpoint of the bottom side is

   (a) $\sqrt{3}$

   (b) $\sqrt[3]{2}$

   (c) $\sqrt{23}$

   (d) $\sqrt{8}$

   (e) $\frac{1}{2}\sqrt{6}$

8. The graph of the equation $x^2 + \frac{y^2}{9} = 1$ is

   (a) the empty set
   (b) (0,0)
   (c) two straight lines
   (d) an ellipse
   (e) a hyperbola

9. If $0 < x + y < 3$ and $1 < x - y < 2$, then

   (a) $1 < x < 5$
   (b) $|x| < 1$
   (c) $x > 1$
   (d) $\frac{1}{2} < x < \frac{5}{2}$
   (e) none of the above

10. The expression $4^{\log_4 3}$ equals

    (a) 3
    (b) 4
    (c) 64
    (d) $4\frac{1}{3}$
    (e) none of the above

11. A regular polygon of 300 sides, F_1, is inscribed in a circle, as is another regular polygon, F_2 with 600 sides. The perimeters P_1, P_2 of F_1, F_2 respectively, satisfy

    (a) $P_1 = P_2$
    (b) $P_1 < P_2$
    (c) $P_1 > P_2$
    (d) $P_1 + P_2 = P_1 - P_2$
    (e) none of the above

12. A right triangle with area 12 and hypotenuse 5 has a perimeter of

    (a) 49
    (b) 7
    (c) 12
    (d) 37
    (e) none of the above

13. If $\tan \theta = ½$, then $\sin 2\theta =$

    (a) $\frac{1}{\sqrt{2}}$

    (b) $\frac{1}{\sqrt{5}}$

    (c) $\frac{4}{5}$

    (d) $\frac{2}{\sqrt{5}}$

    (e) none of the above

14. The function achieves a minimum at the value

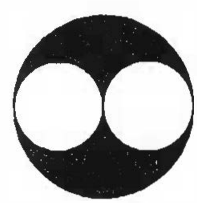
    (a) $x = 0$
    (b) $x = \sqrt{2}$
    (c) $x = 2$
    (d) $x = 2\sqrt{2}$
    (e) does not attain a minimum

15. If b and c are <u>odd integers</u>, which of the following could be the roots of the equation $x^2 + bx + c = 0$?

    (a) 5, 7
    (b) 4, 7
    (c) $3 + 2i$, $3 + 4i$
    (d) $5 + \sqrt{7}, 5 - \sqrt{7}$
    (e) none of the above

16. Two circles of radius 5 are inscribed in a circle of radius 10 (see sketch). If A_1 is the shaded area and A_2 the unshaded are, then

    (a) $A_1 > A_2$
    (b) $A_1 = A_2$
    (c) $A_1 < A_2$
    (d) $A_1 + A_2 = A_1 - A_2$
    (e) $A_1 = \pi A_2$

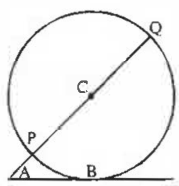


17. Suppose we have a cloth divided into 4 horizontal stripes, and suppose we wish to create different flags by coloring the stripes. If we can use the colors red, blue, white, green and yellow, how

many different flags can we make? (Adjacent stripes must be different colors.)

(a) 80
(b) 210
(c) 320
(d) 400
(e) none of the above

18. A point A is chosen outside a circle with centre C. The tangent from A meets the circle at B, while AC meets the circle at P and Q. Given $\overline{AB} = 10$ and $\overline{AP} = 2$, the radius of the circle must equal



(a) 12
(b) $12\sqrt{2}$
(c) $2\sqrt{10}$
(d) 24
(e) not enough information.

19. Given the base 4 numbers 332 and 32, find their product in base 4.

(a) 100210
(b) 31210
(c) 30211
(d) 21301
(e) none of the above

20. A parallelogram is circumscribed about a circle. It is necessarily a

(a) rectangle
(b) square
(c) parallelogram with a 60° angle
(d) rhombus
(e) none of the above

1975

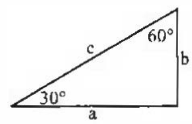
1. The sum of three consecutive positive integers is <u>always</u>

(a) odd
(b) even
(c) a perfect square
(d) divisible by 3
(e) none of the above

2. Which of the following holds true?

(a) $\log_3 2 < \log_2 3$
(b) $\log_3 2 = \log_2 3$
(c) $\log_3 2 > \log_2 3$
(d) $\log_3 2 = 1$
(e) $\log_2 3 = 1$

3. For the triangle shown, which of the following is true?



(a) a = b
(b) b = 2a
(c) c = 2a
(d) c = 2b
(e) none of the above is true

4. "The operation ° is commutative" means

(a) $x \circ 1 = 1$
(b) $x \circ x = x$
(c) $x \circ y = y \circ x$
(d) $x \circ (y \circ z) = (x \circ y) \circ z$
(e) none of the above

5. If $x = .1102$ (base 3), then x^2 is (base 3)

(a) .11021102
(b) .10102
(c) .01222111
(d) .010211
(e) none of the above

6. Given a square inscribed in a circle inscribed in an equilateral triangle, if each side of the triangle has length 6, what is the length of each side of the square?

(a) $\frac{1}{2}\sqrt{6}$
(b) $\sqrt{3}$
(c) $\sqrt{6}$
(d) $2\sqrt{3}$
(e) none of the above

7. Which is larger, the volume of a sphere of radius 1 or the volume of a right circular cone of height 1 and base radius 2?

(a) these volumes do not exist
(b) they are equal
(c) the sphere
(d) the cone
(e) none of the above are true

8. $\dfrac{a^4 + a^2b^2 + b^4}{a^2 + ab + b^2} =$

(a) $a^2 + ab + b^2$
(b) $a^2 + ab - b^2$
(c) $a^2 - ab - b^2$
(d) $a^2 - ab + b^2$
(e) none of the above

9. Five years from now Bill will be twice as old as he was two years after he was half as old as he will be in one year from now. His age is

(a) 16
(b) 13
(c) 8
(d) 41
(e) cannot be determined

10. The number $1.1313\overline{13}$ is the same as

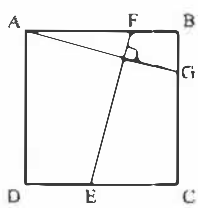
(a) $\frac{112}{99}$
(b) $\frac{113}{99}$
(c) $\frac{100}{99}$
(d) $\frac{1131313}{1000000}$
(e) none of the above

11. A jar contains 15 balls, of which 10 are red and 5 are black. If 3 balls are chosen at random the probability that all three will be red is

(a) 0
(b) $\frac{2}{3}$
(c) $\frac{4}{9}$
(d) $\frac{8}{27}$
(e) none of the above

12. The square ABCD has side length 1. Given that AG$\perp$EF and that $\overline{EF} = \sqrt{\frac{7}{16}}$, $\overline{BF} = \sqrt{\frac{1}{7}}$, then $\overline{AG}$ is

(a) $\sqrt{\frac{7}{6}}$
(b) $\sqrt{7}$
(c) $\sqrt{\frac{6}{7}}$
(d) $\sqrt{\frac{8}{7}}$



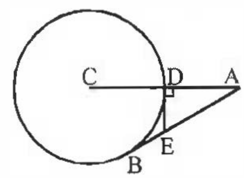
(e) none of the above

13. Which is the largest of

(a) 2^{4^3}
(b) 2^{3^4}
(c) 4^{2^3}
(d) 3^{2^4}
(e) 3^{4^2}

14. A circle of radius 3 has centre C. Let A be at a distance 5 from C and AB be a tangent to the circle. Let AC meet the circle at D and let E lie on AB with ED$\perp$AC. Then length $\overline{ED}$ is

(a) 1
(b) 2
(c) $\sqrt{2}$
(d) $\sqrt{3}$
(e) none of the above



15. The system of equations $2x - 3y = 4$, $2y - 4x = 8$ has

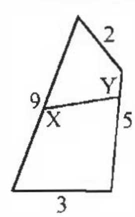
(a) ten solutions
(b) two solutions
(c) one solution
(d) no solutions
(e) none of the above

16. Suppose that a_1 is an integer not divisible by 3 and that $a_1^2 + a_2^2 \ldots + a_n^2$ is divisible by 3, where $a_2, \ldots, a_n$ are integers. Then n is

(a) arbitrary
(b) at least 3
(c) at most 2
(d) always odd
(e) none of the above

17. According to the diagram, $\overline{XY}$ cannot equal

(a) 2
(b) 4
(c) 7
(d) 10
(e) any of these



18. Assume the earth is a perfect sphere and a wire is stretched tightly around the equator. The

wire is lengthened one metre and then expanded uniformly so as to form a somewhat larger circle. The new radius will be approximately how many metres larger than the old one?

(a) .016
(b) .032
(c) .16
(d) .32
(e) 1

19. School X has 100 students and school Y has 50 students. These schools are to be replaced by a single school Z. If the students live in the immediate vicinities of their respective schools (X or Y), where should Z be placed so as to minimize the total distance travelled by all the students?

(a) at X
(b) at Y
(c) halfway in between
(d) one third of the way from X to Y
(e) at none of these

20. An urn contains 100 balls of different colors, 40 red, 27 green, 26 blue, and 7 white. What is the smallest number of balls that must be drawn without looking to guarantee that at least 15 balls have the same color?

(a) 86
(b) 50
(c) 43
(d) 39
(e) none of the above

1976

1. What is the value of $5^{\log_5 6}$?

(a) 1
(b) 5
(c) 6
(d) $\log_6 5$
(e) none of the above

2. Let S be the set of points, (x, y), in the plane satisfying both $x^2 + y^2 \leq 1$ and $x^2 + y^2 \geq r^2$. A value of r such that S is the empty set is:

(a) 1
(b) -1
(c) ½
(d) -½
(e) none of the above

3. If S, T, and V are sets, then $[(S \cap T) \cup (S \cap V)]$ is the same set as

(a) S
(b) $T \cup V$
(c) $T \cap V$
(d) $S \cap (T \cup V)$
(e) none of the above

4. A metal disc has one face marked "1" and the other face marked "2". A second metal disc has one face marked "2" and the other marked "3". Assume that, when tossed, the two faces of a disc are equally likely to turn up. If both discs are tossed, what is the probability that "4" is the sum of the numbers turning up?

(a) $\frac{1}{4}$
(b) $\frac{1}{2}$
(c) $\frac{3}{4}$
(d) $\frac{1}{3}$
(e) none of the above

5. Which of the following statements about is $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ true?

(a) it is irrational
(b) it is rational
(c) it is imaginary
(d) it is positive
(e) none of the above are true

6. If $i = \sqrt{-1}$, then i^6 is

(a) 1
(b) -1
(c) i
(d) $-i$
(e) none of the above

7. The solution set of the inequality $x^2 - x - 2 < 0$ is the interval

(a) $-2 \leq x \leq 1$
(b) $-2 < x < 1$
(c) $-2 \leq x < 1$

(d) $-2 < x \leq 1$
(e) none of the above

8. At the end of a party, everyone shakes hands with everyone else. Altogether there are 28 handshakes. How many people are there at the party?

(a) 8
(b) 14
(c) 20
(d) 56
(e) none of the above

9. Let $[x]$ denote the largest integer not exceeding x. Thus, e.g. $[2] = 2$, $[3.99] = 3$, $[-.5] = -1$. Which of the following statements are always true?

i) $[x + y] = [x] + [y]$
ii) $[2x] = 2[x]$
iii) $[-x] = -[x]$
(a) i) only
(b) ii) only
(c) iii) only
(d) all
(e) none

10. If the sum of the first n positive integers is $\frac{n(n+1)}{2}$, the sum of the first n positive odd integers is

(a) $\frac{n(n+1)}{4}$
(b) $\frac{n(2n+1)}{2}$
(c) n^2
(d) n^2-4
(e) none of the above

11. Suppose that $d=x^2-y^2$ where x and y are two odd integers. Which of the following statements are always true?

i) d is odd
ii) d is divisible by 4
iii) d is a perfect square
(a) i) only
(b) ii) only
(c) iii) only
(d) ii) and iii) only
(e) none of the above

12. If each term of the sequence $a_1, a_2, \cdots, a_n$ is either +1 or -1 then $a_1 + a_2 + \cdots + a_n$ is always

(a) 0
(b) 1
(c) $\begin{cases} \text{odd if n is odd} \\ \text{even if n is even} \end{cases}$
(d) $\begin{cases} \text{odd if n is even} \\ \text{even if n is odd} \end{cases}$
(e) none of the above

13. If x is a real number satisfying the equation $x^{x^x} = 2$ then x is equal to

(a) ∞
(b) 2
(c) $\sqrt[4]{2}$
(d) $\sqrt{2}$
(e) none of the above

14. The number of pipes of inside diameter 1 unit that will carry the same amount of water as one pipe of inside diameter 6 units of the same length is

(a) 6π
(b) 6
(c) 12
(d) 36
(e) none of the above

15. $2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k}$ is equal to

(a) 2^{-2k}
(b) $2^{-(2k-1)}$
(c) $-2^{-(2k-1)}$
(d) 0
(e) none of the above

16. Let P be the product of any 3 consecutive odd integers. The largest integer dividing all such P is

(a) 15
(b) 6
(c) 5
(d) 3
(e) none of the above

17. If $|x-\log y|=x+\log y$ where x and $\log y$ are real, then

(a) $x=0$
(b) $y=1$
(c) $x=0$ and $y=1$
(d) $x(y-1)=0$

(e) none of the above

18. Each of a group of 50 girls is blonde or brunette and is blue- or brown-eyed. If 14 are blue-eyed blondes, 31 are brunettes and 18 are brown-eyed, the number of brown-eyed brunettes is

(a) 7
(b) 9
(c) 11
(d) 13
(e) none of the above

19. After finding the average of 35 scores, a student carelessly included the average with the 35 scores and found the average of these 36 numbers. The ratio of the second average to the true average was

(a) 1:1
(b) 35:36
(c) 36:35
(d) 2:1
(e) none of the above

20. If the line $y = mx + 1$ intercepts the ellipse $x^2 + 2y^2 = 1$ exactly once, then m^2 is equal to

(a) $\frac{1}{2}$
(b) $\frac{2}{3}$
(c) $\frac{3}{4}$
(d) $\frac{4}{5}$
(e) none of the above

1977

1. If $a > b > 0$, then

(a) $\frac{a+1}{a} > \frac{b+1}{b}$
(b) $\frac{a+1}{a} \geq \frac{b+1}{b}$
(c) $\frac{a+1}{a} \leq \frac{b+1}{b}$
(d) $\frac{a+1}{a} < \frac{b+1}{b}$
(e) none of the above are true

2. Let AB be a diameter of a circle of radius 1 and let C be a point on the circumference such that $\overline{AC}=\overline{BC}$. Then the length $\overline{AC}$ is equal to

(a) 2
(b) $\frac{1}{2}$
(c) $\sqrt{2}$
(d) $\frac{1}{\sqrt{2}}$
(e) none of the above

3. Out of 100 people, 60 report that they receive the daily news by watching television, whereas 70 read the newspaper. Of those who read the newspaper, 70% also watch television. The number not receiving any news by television or newspaper is

(a) 15
(b) 19
(c) 23
(d) 27
(e) none of the above

4. $(64^{.9})(32)^{-.08}$ equals

(a) 64
(b) 32
(c) 24
(d) 8
(e) none of the above

5. Let $f(x)$ be a non-constant polynomial with real coefficients. If $f(x)=f(x-1)$ for all x then $f(x)$

(a) has exactly one root
(b) cannot exist
(c) has exactly two roots
(d) has either no roots or an infinite number of roots
(e) satisfies none of the preceding

6. Suppose k is a real number such that $0 < k < 1$. Of the two roots of the quadratic equation $kx^2 - 3x + k = 0$,

(a) both are positive
(b) both are negative
(c) both are zero
(d) one is positive and one is negative
(e) none of the above

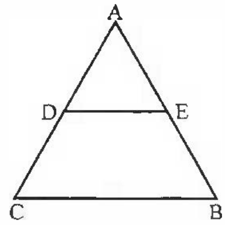
7. Let ℓ be a line in the real plane passing through the points (1, 1) and (3, 5). Then ℓ passes through the point $(2, y)$ where

(a) $y = 4$
(b) $y = 2$
(c) $y = 3$

(d) y = 5

(e) y is none of the above

8. ΔABC is an equilateral triangle with sides of length 1, and $DE \| CB$. If the area of ΔADE is equal to the area of the trapezoid DEBC, then the length DE equals



(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) $\frac{1}{\sqrt{2}}$

(d) $\frac{\sqrt{2}-1}{\sqrt{2}}$

(e) $\frac{\sqrt{3}-1}{\sqrt{3}}$

9. The inequality $(x + 1)(x - 1) \geq x^2$ is valid

(a) for all real x

(b) for no real x

(c) for all $x > 1$

(d) for all $x > 0$

(e) for none of the above

10. Suppose a bowl contains 3 red balls and 3 yellow balls. The probability that two balls drawn out without replacement will both be red is

(a) $\frac{1}{2}$

(b) $\frac{1}{4}$

(c) $\frac{1}{3}$

(d) $\frac{1}{6}$

(e) none of these

11. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}}$ equals

(a) 1

(b) $\frac{1}{2}$

(c) $\frac{2}{3}$

(d) $\frac{3}{5}$

(e) $\frac{5}{8}$

12. $\frac{xy-x^2}{xy-y^2} - \frac{xy}{x^2-y^2}$ can be simplified to

(a) $\frac{x^3}{y^3-yx^2}$

(b) $\frac{x^2}{y^2-x^2}$

(c) $\frac{x^2+y^2}{x^2-y^2}$

(d) $\frac{x^4+xy^3}{(x^2-y^2)(xy-y^2)}$

(e) none of the above

13. If the radius of a sphere is increased 100%, the volume is increased by

(a) 100%

(b) 200%

(c) 300%

(d) 400%

(e) none of the above

14. The expression x^4+16 equals

(a) $(x^2 + 4)(x^2 + 4)$

(b) $(x^2 + 4)(x^2 - 4)$

(c) $(x^2 - 4x + 4)(x^2 + 4x + 4)$

(d) $\left(x^2 - 2x\sqrt{2} + 4\right)\left(x^2 + 2x\sqrt{2} + 4\right)$

(e) none of the above

15. The price of a book has been reduced by 20%. To restore it to its former value, the last price must be increased by

(a) 25%

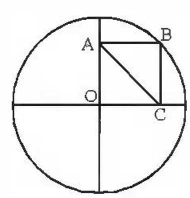
(b) 10%

(c) 15%

(d) 20%

(e) none of the above

16. OABC is a rectangle inscribed in a quadrant of a circle of radius 10. If OA=5, then AC equals



(a) $2\sqrt{5}$

(b) $\frac{1}{\sqrt{2}}$

(c) 8

(d) 12

(e) none of the above

17. The lengths of the medians of a right triangle which are drawn from the vertices of the acute angles are $\sqrt{73}$ and $2\sqrt{13}$. The length of the third median is

(a) $\sqrt{73+52}$
(b) $\sqrt{73}+2\sqrt{13}$
(c) 5
(d) 10
(e) none of the above

18. A car travels 240 miles from one town to another at an average speed of 30 miles per hour. On the return trip the average speed is 60 miles per hour. The average speed for the round trip is

(a) 35 mph
(b) 40 mph
(c) 45 mph
(d) 50 mph
(e) 55 mph

19. The expression $\log_3 6 + \log_3 3 - \log_3 2$ equals

(a) $\frac{5}{2}$
(b) 3
(c) 2
(d) 1
(e) 0

20. The slope of the line passing through the points (3, 4) and (1, 9) is

(a) $-\frac{5}{2}$
(b) $\frac{5}{2}$
(c) 5
(d) -2
(e) 6

1978

1. Which of the following inequalities is true for all positive numbers x?

(a) $x + \frac{1}{x} > 2$
(b) $x + \frac{1}{x} < 2$
(c) $x + \frac{1}{x} \geq 2$
(d) $x + \frac{1}{x} \leq 2$
(e) none of the above

2. A steamer was able to go twenty miles per hour upstream and twenty-five miles per hour downstream. On a return trip the steamer took two hours longer coming upstream than it took coming downstream. The total distance travelled by the steamer was

(a) 100 miles
(b) 200 miles
(c) 400 miles
(d) 800 miles
(e) 150 miles

3. If n is a positive integer, then $n^2 + 3n + 1$ is

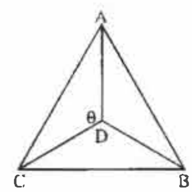
(a) always a perfect square
(b) never a perfect square
(c) sometimes a perfect square
(d) sometimes an even integer
(e) none of the above

4. The solution set of the inequality $x^2(x^2 - 1) \leq 0$ is

(a) an interval
(b) two intervals
(c) a point
(d) an interval and a point
(e) all real numbers

5. In the diagram, ΔABC is an equilateral triangle, ΔBCD is an isosceles triangle, and $\angle$CDB is a right-angle. Then the angle θ is

(a) 45°
(b) 90°
(c) 120°
(d) 135°
(e) none of the above



6. Given the binary operation * between two positive integers, m, n such that $m*n = mn + 1$ (mn is the usual multiplication of m and n), which of the following does <u>not</u> hold

(a) commutative law
(b) associative law
(c) $n*n$ is a positive integer
(d) $m*n \geq 2$
(e) $m*n$ is odd whenever m is even

7. Label the four quadrants of the (x, y) plane as

follows: $\begin{array}{c|c} \text{II} & \text{I} \\ \hline \text{III} & \text{IV} \end{array}$

Then the solution set of the simultaneous in equalities $x^2 - y < 0$, $x^2 + y^2 < 1$ lies entirely in quadrants

(a) I and II
(b) II and III
(c) III and IV
(d) IV and I
(e) none of the above are correct

8. If $f(n) = n^2$, where n is an integer, then $\frac{f(f(n+1)) - f(f(n-1))}{f(n+1) - f(n-1)}$ equals

(a) n^2
(b) $2n^2 + 2$
(c) $n^2 + 1$
(d) $n^4 + 1$
(e) none of the above

9. Which of the following inequalities hold for all pairs of real number x, y?

(a) $\sqrt{x^2 + y^2} \le x + y$
(b) $\sqrt{x^2 + y^2} \le x^2 + y^2$
(c) $\sqrt{x^2 + y^2} \le xy$
(d) $\sqrt{x^2 + y^2} \le |x| + |y|$
(e) none of the above

10. Two similarly proportioned boxes have their surface areas in the ratio 4:1. Their volumes are in the ratio

(a) 9:1
(b) 8:1
(c) 3:1
(d) 2:1
(e) none of the above

11. The roots of the quadratic polynomial $2x^2 + kx + 1$ are r and s. Which of the f ollowing are impossible

(a) $r = s$
(b) $r\,s = 1$
(c) $r + s = 1$
(d) $r + s = 0$
(e) all of the above

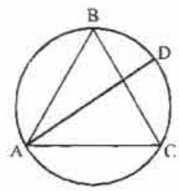
12. A hat contains three slips of paper, of which one bears the name John, one bears the name Diana and the other bears both names. If John and Diana each draw a slip, the probability that they each draw a slip with their own name is

(a) $\frac{1}{9}$
(b) $\frac{1}{6}$
(c) $\frac{1}{4}$
(d) $\frac{1}{3}$
(e) none of the above

13. The value of k such that $x^6 - kx^4 + kx^2 - kx + 4k + 6$ is divisible by x-2 is

(a) 1
(b) 5
(c) 7
(d) 11
(e) there is no such value of k

14. ΔABC is an equilateral triangle inscribed in a circle of diameter 1. If $\overline{AD}$ is a diameter of the circle, then the length $\overline{BD}$ is



(a) $\frac{1}{2}$
(b) 1
(c) 2
(d) $\frac{1}{3}$
(e) $\frac{1}{\sqrt{2}}$

15. The equation of the line through the point (1, 1) that is perpendicular to the line $y = -2x - 3$ is

(a) $y = \frac{2}{3}x + \frac{1}{3}$
(b) $y = \frac{1}{3}x + \frac{2}{3}$
(c) $y = 2x - 1$
(d) $y = \frac{1}{2}x + \frac{1}{2}$
(e) none of the above

16. If $\log_a b = c$, then $\log_a(b^c)$=

(a) bc
(b) 6^c
(c) c^c
(d) c^2
(e) $2c$

17. A circle and a square can never have in common exactly

(a) one point
(b) two points
(c) three points
(d) four points
(e) all of the preceding are possible

18. Let $\{a_1, a_2, a_3, \dots\}$ be a sequence of real numbers such that the sum of the first n of them is $n^2 + n$. Then a_n is equal to

(a) n
(b) $2n - 1$
(c) $2n + 1$
(d) 1
(e) none of the above

19. The domain of the function $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$ is

(a) a single point
(b) an infinite interval
(c) a finite interval
(d) an infinite interval with a point deleted
(e) none of the above

20. A polynomial whose graph passes through the points (-1, 7), (1, 0), (2, 0) is

(a) $x + 8$
(b) $x^2 - 3x + 2$
(c) $x^2 + 9$
(d) $x^3 - x^2 + x - 1$
(e) none of the above

1979

1. A triangle has sides of lengths 1, 2 and $\sqrt{3}$. Its area is

(a) $\frac{1}{2}$
(b) $\frac{\sqrt{3}}{2}$
(c) 2
(d) $2\sqrt{3}$
(e) none of the above

2. $P(x) = 4x^4 - kx^2 + 1$ has double roots if

(a) $k = 1$
(b) $k = 2$
(c) $k = 3$
(d) $k = 4$
(e) none of the above is true

3. For what values of a, b is the equation $(a^{\log_{10} b})^{ab} = (b^{\log_{10} a})^{ba}$ true?

(a) all values of a and b
(b) no values of a and b
(c) all positive values of a and b
(d) all negative values of a and b

4. The equation of the line that is perpendicular to the line $x + 2y = 3$ and passes through the point (4, 5) is

(a) $x - 2y = 3$
(b) $2x - y = 3$
(c) $2x + y = 3$
(d) $-2x + y = 3$
(e) none of the above

5. Let S be the set of points defined by the inequalities $x + y \geq 1$ and $x - y \leq 1$, $y \leq$ The area of the region determined by S is

(a) $\frac{3}{8}$
(b) $\frac{1}{2}$
(c) $\frac{5}{8}$
(d) 1
(e) none of the above

6. X is a square of diagonal 1, Y is an equilateral triangle of side 1 and Z is a right angled isosceles triangle whose equal sides have length 1. Comparing the areas of these figures

(a) X is larger than both Y and Z
(b) Y is larger than both Z and X
(c) Z is larger than both X and Y
(d) X, Y, Z all have the same area
(e) none of the above

7. A positive integer is squarefree if it cannot be divided exactly by the square of an integer larger than 1. The number of positive squarefree integers less than 20 is

(a) 0
(b) 3
(c) 9
(d) 13
(e) none of the above

8. The solutions of the equation $(\sin\theta + \cos\theta)^2 = 1$ are

(a) all multiples of $\frac{\pi}{2}$

(b) all multiples of π
(c) all odd multiples of $\frac{\pi}{2}$
(d) all even multiples of π
(e) none of the above

9. The sum of the first 27 odd positive integers is

(a) 153
(b) 196
(c) 144
(d) 216
(e) none of the above

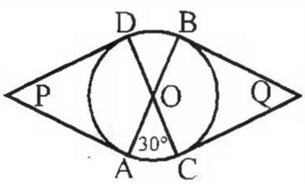
10. The expression $\left(\left(\left(\sqrt{2}\right)^{\sqrt{2}}\right)^{2}\right)^{\sqrt{2}}$ is equal to

(a) 2
(b) $\sqrt{2}$
(c) 4
(d) 8
(e) none of the above

11. If α, β are the roots of $x^2 + 7x - 5 = 0$, then $\alpha^2 + \beta^2$ is equal to

(a) 59
(b) 47
(c) -15
(d) 35
(e) none of the above

12. In the figure, AOB and COD are straight lines and O is the centre of the unit circle, while PD, PA, QB, QC are tangents to the circle. The distance from P to Q is



(a) 2
(b) 3
(c) 4
(d) $\frac{1}{2}$
(e) none of the above

13. For which values of x is a triangle with sides x, $x+1$, $x+2$ an acute-angled triangle?

(a) $x = 1$
(b) $x > 2$
(c) $x < 4$
(d) $x > 3$
(e) none of the above

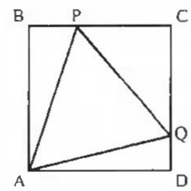
14. Which of the following inequalities is always true for any pair of real numbers x, y?

(a) $x + y \leq xy$
(b) $(x + y)^2 \geq xy$
(c) $x + y \geq xy$
(d) $(x + y)^2 \geq x + y$
(e) none of the above is always true

15. A twelve-hour digital watch displays the hours, minutes and seconds. During one complete day it registers at least one figure 3 for a total time of

(a) 1 hour and 5 seconds
(b) 1 hour, 15 minutes and 15 seconds
(c) 2 hours and 24 minutes
(d) 3 hours
(e) none of the above

16. In the diagram, ABCD is a square of side 1 and APQ is an equilateral triangle. The length DQ is equal to



(a) $\frac{1}{2}$
(b) $\sqrt{2} - 1$
(c) $\frac{5}{8}$
(d) $2 - \sqrt{3}$
(e) none of the above

17. The product of John's and Mary's ages is five more than four times the sum of their ages. If Mary is 4 years younger than John, John's age is

(a) 13
(b) 11
(c) 9
(d) 7
(e) none of the above

18. In a poll of 1000 coffee drinkers, 40% preferred their coffee with neither cream nor sugar and 60% of the remainder preferred their coffee with cream only. After deducting both of these groups, 40% of those left preferred their coffee with sugar only. The rest preferred coffee with cream and sugar and their number was

(a) 144
(b) 216
(c) 96
(d) 172
(e) none of the above

19. A sphere and a triangle <u>cannot</u> have in common exactly

    (a) 1 point
    (b) 2 points
    (c) 3 points
    (d) 4 points
    (e) all of the above are possible

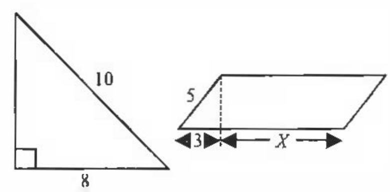
20. The picture cards are removed from a pack of 52 playing cards. The number of ways of drawing 2 cards from the remaining 40 so that the sum of the numerical values is 10 is

    (a) 100
    (b) 10
    (c) 50
    (d) 70
    (e) none of the above

1980

1. There are 5 roads between the towns A and B and 4 roads between B and C. The number of different ways of driving the round trip A→B→C→B→A without using the same road more than once is

   (a) 32
   (b) 240
   (c) 400
   (d) 16
   (e) none of the above



2. In the above diagrams, the area of the triangle is two fifths the area of the parallelogram. The value of x is therefore

   (a) 30
   (b) 12
   (c) 15
   (d) 24
   (e) none of the above

3. A die is thrown repeatedly until a 6 is obtained. Assuming the die to be fair, the probability that this will happen on the third throw is

   (a) $\frac{1}{6}$
   (b) $\frac{1}{216}$
   (c) $\frac{5}{36}$
   (d) $\frac{25}{216}$
   (e) none of the above

4. Find all real values of k for which kx^2+kx+1 has no real root

   (a) $-2 < k <$
   (b) $-2 < k < 4$
   (c) $0 < k < 2$
   (d) $0 < k < 4$
   (e) none of the above

5. For which value(s) of k are the lines $9x + ky = 7$ and $kx + y = 2$ parallel?

   (a) $k = 3$
   (b) $k = \pm 3$
   (c) $k = {}^1/3$
   (d) $k = \pm{}^1/3$
   (e) none of the above

6. Let $x = \log_{a^2} b$. The $\log_b a^x$ equals

   (a) $\sqrt{b}$
   (b) b^2
   (c) ½
   (d) 2
   (e) none of the above

7. If $\sin 3x=0$, then $\sin x$ equals

   (a) 0
   (b) 0 or $\pm\frac{1}{2}$
   (c) 0 or $\pm\frac{\sqrt{3}}{2}$
   (d) $0, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$
   (e) none of the above

8. In the right-angled triangle ABC, the lengths of the sides AB and BC are (x^2+y^2) and $2xy$ respectively. The length of the side AC is therefore

   (a) (x^2-y^2)
   (b) $(x-y)^2$
   (c) $(x^2-y^2)^2$

(d) $(x+y)^2$

(e) none of the above

9. In a certain examination, the average mark (out of 100) of the 30 boys in a class was 60. The girls did rather better, their average being 65. If the overall average for the class was 62, the number of girls who took the examination was

(a) 20

(b) 24

(c) 18

(d) 36

(e) none of the above

10. The following is true for all real numbers x, y:

(a) $(x+y)^2 \geq (x-y)^2$

(b) $|x+y| \geq x+y$

(c) $2\sqrt{5}$

(d) $xy \geq x+y$

(e) none of the above is true for all real x, y

11. The expression $\log_{10}(144)^{144}$ equals

(a) $576\log_{10}2 + 288\log_{10}3$

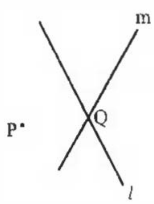
(b) $144\log_{10}2 + 144\log_{10}3$

(c) $(\log_{10}144)^{144}$

(d) $2\log_{10}144$

(e) none of the above

12. The line l has the equation $y=-2x-4$ and the line m has the equation $y=2x + 4$. If P is the point (0, 0) then the equation of the line through P and Q is



(a) $y = x$

(b) $y = -2x$

(c) $2x + 3y = 1$

(d) $-y - 2x = 1$

(e) none of the above

13. The maximum possible value of $3x - 3x^2$ where x is real is

(a) $\frac{5}{4}$

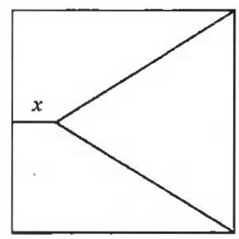
(b) 0

(c) -1

(d) $\frac{3}{4}$

(d) none of these

14. A unit square is divided into three parts of equal area as shown. Then x is



(a) $\frac{1}{3}$

(b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{4}$

(d) $\frac{1}{\sqrt{3}}$

(e) $\frac{1}{2}$

15. A unit square is divided into three parts by three lines of equal length as shown. The length of each line is

(a) $\frac{1}{4}$

(b) $\frac{5}{8}$

(c) $2-\sqrt{2}$

(d) $\frac{1}{\sqrt{2}}$

(e) $\frac{3}{5}$

16. The expression equals $\sqrt{\dfrac{\sqrt{5}-2}{\sqrt{5}+2}}$

(a) $\frac{1}{\sqrt{5}-\sqrt{2}}$

(b) $\frac{1}{\sqrt{2}+5}$

(c) $\sqrt{5}-2$

(d) $\sqrt{5}+2$

(e) $\frac{1}{5-\sqrt{2}}$

17. If $f(x) = x^2 - 5$ and $f(4+a) - f(4a) = 0$ then a is

(a) $\frac{4}{3}$ or $-\frac{4}{5}$

(b) 4 or $-\frac{4}{5}$

(c) $\frac{4}{3}$ or $\frac{4}{5}$

(d) -4 of $\frac{4}{3}$

(e) -4 or $-\frac{4}{5}$

18. Two circles and a straight line are drawn in the plane to form exactly N bounded regions. N <u>cannot</u> be

(a) 3

(b) 4
(c) 5
(d) 6
(e) all these are possible

19. A bowl contains 2 marbles of each of 4 colors. If you randomly remove 3 marbles from the bowl without replacing them, what is the probability that you have removed two of the same color?

(a) $\frac{1}{4}$
(b) $\frac{1}{7}$
(c) $\frac{80}{7}$
(d) $\frac{1}{2}$
(e) none of the above

20. If n is a positive integer so that $1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + 118 + 119$ then n is

(a) 60
(b) 69
(c) 76
(d) 84
(e) 89

1981

1. The circle $x^2+y^2=1$ and the ellipse $\frac{x^2}{4} + (y-2)^2 = 1$ intersect in how many real points?

(a) one
(b) two
(c) three
(d) four
(e) no real points

2. If $x°$ Fahrenheit equals $-x°$ Centigrade when x is

(a) 10
(b) -40
(c) $-\frac{16}{9}$
(d) $-\frac{80}{7}$
(e) none of the above

3. The values of k for which $x^2 + (x+1)(k+1) = 0$ has real solutions in x are

(a) $k = -1$ and $k = 3$
(b) $-1 \le k \le 3$
(c) $k \ge 3$ and $k \le -1$
(d) all values of k
(e) none of the above

4. sec x cosec x is equal to

(a) sec x + cosec x
(b) tan x + cot x
(c) sin x + cos x
(d) (sec x + cosec)$^{-1}$
(e) none of the above

5. The sum of the roots $3x^5 - 30x^4 - 105x^3 - 105x^2 + 72x$ is

(a) 3
(b) -30
(c) 10
(d) -35
(e) none of the above

6. For what values of k can the correct solution of $\log(3x+2) + \log(4x-1) = 2\log k$ be obtained by "cancelling the log" and solving $(3x+2)+(4x-1)=2k$?

(a) for no values of k
(b) for $k=3$
(c) for $k=11$
(d) for all values of k
(e) none of the above

7. Seven Canadian coins, none of which has a value greater than 25 cents, add up to 81 cents. The number of nickels is

(a) none
(b) one
(c) two
(d) five
(e) none of the above

8. Let ABCD be a square of side 1 and P be any point. The minimum total length of line-segments PA, PB, PC and PD is

(a) 2
(b) $2\sqrt{2}$
(c) $2+\sqrt{2}$
(d) 4
(e) none of the above

9. What is the value of

$$(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}+\sqrt{3})(1-\sqrt{2}+\sqrt{3})(1-\sqrt{2}-\sqrt{3})$$

(a) -8
(b) 4
(c) $2\sqrt{3}+2\sqrt{2}$
(d) 0
(e) none of the above

10. In how many ways can one choose 5 of the first 10 positive integers so that no two are consecutive?

(a) 2
(b) 4
(c) 6
(e) 8
(e) none of the above

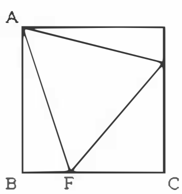
11. The only integer solution of $(3x)^{3x}.(2x)^{2x}=3^{18}.2^{9}$ is x =

(a) 1
(b) 2
(c) 3
(d) 0
(e) there is no solution

12. In each meeting, every pair of people shook hands once. At two successive meetings, the second one having a higher attendance, 100 handshakes in all took place. The second attendance was higher than the first by

(a) 1
(b) 10
(c) 11
(d) the situation is impossible
(e) none of the above

13. In the diagram, ABCD is a square of side 1 and AEF is an equilateral triangle. The length of AE is



(a) $\sqrt{2}$
(b) $\sqrt{3}$
(c) $\sqrt{6}-\sqrt{3}$
(d) $\sqrt{6}-\sqrt{2}$
(e) none of the above

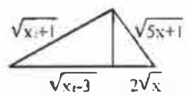
14. What is the sum of all the digits appearing in the first 99 positive integers?

(a) 900
(b) 1800
(c) 4950
(d) 9900
(e) none of the above

15. Joe tosses a fair coin three times in succession. Moe tosses three fair coins all at once. The probability that Joe gets more heads than Moe is

(a) $\frac{1}{2}$
(b) $\frac{3}{4}$
(c) $\frac{5}{8}$
(d) $\frac{7}{16}$
(e) none of the above

16. In the figure, the value of x must be



(a) 1
(b) 2
(c) 3
(d) 4
(e) none of the above

17. Let $f(x, y)$ be a function which is not identically zero and such that $f(x, y)=kf(y, x)$ for all x and y. The possible values of k are

(a) 0
(b) 1
(c) -1
(d) ± 1
(e) none of the above

18. The number of positive four digit integers that give the same integer when their digits are reversed is

(a) 81
(b) 90
(c) 100
(d) 121
(e) none of the above

19. The area of the triangle whose vertices have coordinates (0, 0), (2, 4) and (4, 2) is

(a) 12
(b) $6\sqrt{2}$
(c) $2\sqrt{10}$

(d) $2\sqrt{2}+4\sqrt{5}$

(e) none of these

20. The sum:
$$\frac{1}{2x+1}+\frac{1}{(2x+1)(4x+1)}+\frac{1}{(4x+1)(6x+1)}+\ldots+\frac{1}{(2(k-1)x+1)(2kx+1)}$$
is equal to

(a) $\frac{1}{2kx+1}$

(b) $\frac{k}{2kx+1}$

(c) $\frac{1}{(2x+1)(4x+1)\ldots(2kx+1)}$

(d) $\frac{k}{(2x+1)(4x+1)\ldots(2kx+1)}$

(e) none of the above

1982

1. A plane takes off from Edmonton, flies north for 500 kilometres, then east for 500 kilometres, next south for 500 kilometres and finally west for 500 kilometres. It will land

(a) north of Edmonton

(b) east of Edmonton

(b) South of Edmonton

(d) west of Edmonton

(e) right at Edmonton

2. Merle has an assortment of pennies, nickels, dimes and quarters but is unable to change a dollar bill. The largest possible amount Merle has in coins is

(a) \$0.94

(b) \$0.99

(c) \$1.19

(d) \$1.24

(e) none of the above

3. The angle between the hour hand and the minute hand at twenty minutes past one is

(a) 60°

(b) 70°

(c) 80°

(d) 90°

(e) none of the above

4. The number of solutions (x, y, z) of the equation $x+y+z=15$ where x, y, and z are integers is $0<x<y<z$ is

(a) 12

(b) 13

(c) 14

(d) 15

(e) none of the above

5. If a and b are the roots of $7x^2-4x+12$, then $\frac{1}{a}+\frac{1}{b}$ is equal to

(a) -3

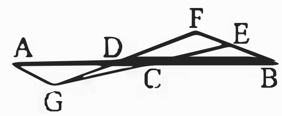
(b) $-\frac{1}{3}$

(c) $\frac{1}{3}$

(d) 3

(e) none of the above

6. In the diagram, $\overline{AC}=\overline{BC}$, $\overline{BE}=\overline{EF}$ and AG||FB. Then $\frac{\overline{BD}}{\overline{AD}}$ is



(a) less than 2

(b) equal to 2

(c) between 2 and 3

(d) equal to 3

(e) greater than 3

7. Stephen collects bugs. Some of them are spiders (8 legs each) and the rest are beetles (6 legs each). His collection consists of 8 bugs with a total of 54 legs. The number of spiders in his collection is

(a) 1

(b) 3

(c) 5

(d) 7

(e) none of the above

8. The minimum value of $x^2 + 2x - 3$ is

(a) -4

(b) -3

(c) -1

(d) 1

(e) none of the above

9. Kelly walks to school at a rate of 2 kilometres per hour. In order that the complete journey (home to school to home) is travelled at an average of 4 kilometres per hour, the rate at which the trip home must be made is

(a) 5 kilometres per hour

(b) 6 kilometres per hour

(c) 8 kilometres per hour

(d) depends on home-school distance

(e) none of the above

10. Let A, B, C, D and E be five points in space. If AB = 30, BC = 80, CD = 236, DE = 86 and EA = 40, then CE is equal to

(a) $\sqrt{236}-\sqrt{86}$

(b) $\sqrt{236^2-86^2}$

(c) 150

(d) 236

(e) none of the above

11. Let a, b, c and d be distinct real numbers. Let $x = \max[\min(a, b), \min(c, d)]$ and $y = \min[\max(a, c), \max(b, d)]$. Then

(a) x is always strictly greater than y

(b) x is always greater than or equal to y

(c) x is always less than or equal to y

(d) x is always strictly less than y

(e) none of the above

12. The sum of the positive divisors of 36 is

(a) 9

(b) 43

(c) 91

(d) 666

(e) none of the above

13. Peter and Trevor bought 300 grams and 500 grams of jelly beans respectively. They ate them together with Mr. Smith, each eating the same amount. Afterwards, Mr. Smith paid the boys 80¢. Peter's fair share amounted to

(a) 10¢

(b) 20¢

(c) 30¢

(d) 40¢

(e) none of the above

14. If $a = 6\sqrt{3}-3\sqrt{13}$ and $b = 6\sqrt{10}-15\sqrt{2}$, then a-b is

(a) at least 1

(b) positive and less than 1

(c) equal to 0

(d) negative and greater than -1

(e) at most -1

15. Five baskets contain 5, 12, 14, 22 and 29 eggs respectively. In each basket, some of the eggs are chicken eggs while the remaining ones are duck eggs. After one of the baskets is sold, the total number of chicken eggs remaining is equal to twice the total number of duck eggs remaining. The number of eggs in the basket sold is

(a) 5

(b) 12

(c) 14

(d) 22

(e) 29

16. An astronaut 2 metres tall walks once around the equator of a gigantic spherical planet. The top of his head describes a circle. The circumference of this circle is longer than the equator of the planet by

(a) less than 50 metres

(b) between 50 and 500 metres

(c) between 500 and 5000 metres

(d) between 5000 and 50000 metres

(e) greater than 50000 metres

17. The number of terms in the arithmetic series $8 + 16 + 24 + \cdots$ such that their sum first exceeds 1982 is

(a) 20

(b) 22

(c) 24

(d) 248

(e) none of the above

18. Rose and Mary play a series of three games. In each game, Rose's probability of winning is $^2/_3$. Rose's probability of winning at least two of the three games is

(a) less than $\frac{1}{2}$

(b) between $\frac{1}{2}$ and $\frac{2}{3}$

(c) equal to $\frac{2}{3}$

(d) between $\frac{2}{3}$ and $\frac{3}{4}$

(e) greater then $\frac{2}{3}$

19. If $\log_2 x = 2$ and $\log_{xy} z = \frac{1}{6}$, $\log_y z$ then is equal to

(a) $\frac{1}{4}$

(b) $\frac{1}{3}$
(c) 3
(d) 4
(e) none of the above

20. The expression $\frac{\cos x}{1-\sin x}$ is equal to

(a) $\sin x + \tan x$
(b) $\sin x \cdot \tan x$
(c) $\sec x + \tan x$
(d) $\sec x \cdot \tan x$
(e) none of the above

1983

1. Assume that the vertical distance between floors in a building is constant. The ratio of the vertical distance between the first and the third floors to the vertical distance between the first and the sixth floors is

(a) 2:5
(b) 1:2
(c) 2:1
(d) 5:2
(e) none of the above

2. The compound fraction $\frac{2}{3+\frac{2}{3+\frac{2}{3}}}$ is equal to

(a) $\frac{4}{9}$
(b) $\frac{22}{39}$
(c) $\frac{2}{3}$
(d) 1
(e) none of the above

3. A boy stands at the centre of a circle of radius 8 metres. A girl stands at a point 4 metres from the boy. The boy runs to some point on the circle and then to the girl. The shortest distance the boy must run in metres is

(a) 20
(b) $8+4\sqrt{5}$
(c) $8+4\sqrt{3}$
(d) $2\sqrt{68}$
(e) none of the above

4. The smallest positive integer which leaves a remainder of 2 whether it is divided by 12 or by 15 is

(a) 14
(b) 58
(c) 62
(d) 182
(e) none of the above

5. A bird in the hand is worth two in the bush. Five birds in hand and three birds in the bush together are worth \$30 more than three birds in hand and five birds in the bush together. A bird in hand is worth

(a) \$5
(b) \$7.50
(c) \$10
(d) \$15
(e) none of the above

6. In triangle ABC, $\angle A$ is at least $10°$ more than $\angle B$ and $\angle B$ is at least $25°$ more than $\angle C$. The maximum value of $\angle C$ is

(a) $35°$
(b) $40°$
(c) $45°$
(d) $50°$
(e) none of the above

7. If $f(x-1) = 3x^2 + 2x - 5$, then $f(x) =$

(a) $3x^2 + 2x - 4$
(b) $3x^2 + 2x - 6$
(c) $3x^2 + 8x$
(d) $3x^2 - 4x - 4$
(e) none of the above

8. Let $[x]$ denote the greatest integer not exceeding x. Then the set of all possible values of $[x] + [-x]$ is

(a) $\{0\}$
(b) $\{0, 1\}$
(c) $\{0, -1\}$
(d) $\{0, 1, -1\}$
(e) none of the above

9. In this problem, a, b, c are real constants. It is possible for all the points (1, 3), (2, 1) and (3, 5) to lie on a graph of the form

(a) $y = a^2x + b$

(b) $y = -a^2x + b$

(c) $y = a^2x^2 + bx + c$

(d) $y = -a^2x^2 + bx + c$

(e) none of the above

10. The sum of five numbers is 100. The sum of the first and the second number is 44, the sum of the second and third is 47, the sum of the third and the fourth is 37 and the sum of the fourth and the fifth is 35. The third number is

(a) 16

(b) 18

(c) 19

(d) 21

(e) none of the above

11. There were 16 participants in a mathematics contest. Of every two participants, at least one was right-handed. If the eventual winner was left-handed, then the number of right-handed participants was

(a) 7

(b) 8

(c) 15

(d) impossible to determine

(e) none of the above

12. If p and q are the roots of $x^2 + x + 1 = 0$, then $\frac{p}{q} + \frac{q}{p} =$

(a) -3

(b) -1

(c) 1

(d) 3

(e) none of the above

13. The smallest positive integer which can be expressed in the form $11x + 8y$ where xD and y are integers is

(a) 1

(b) 3

(c) 8

(d) 19

(e) none of the above

14. If $m = (2 + \sqrt{3})^{-1}$ and $n = (2 -)^{-1}$, then $(m + 1)^{-1} + (n+1)^{-1} =$

(a) 0

(b) 1

(c) $\sqrt{3}$

(d) 2

(e) none of the above

15. The expression $\log_{10}10 \div \log_{10}\sqrt{10}$ is equal to

(a) $\log_{10}\sqrt{10}$

(b) $\log_{10}(10 - \sqrt{10})$

(c) 2

(d) $\sqrt{10}$

(e) none of the above

16. The minimum value of $\sin^5 x + \cos^5 x$ for all $x > 0$ is

(a) 0

(b) $\frac{1}{2\sqrt{2}}$

(c) 1

(d) 2

(e) none of the above

17. A square root of the complex number $8+6i$ is

(a) $\sqrt{8} + \sqrt{6}i$

(b) $\sqrt{8} - \sqrt{6}i$

(c) $3+i$

(d) $3-i$

(e) none of the above

18. The four vertices of a regular tetrahedron consist of two opposite vertices of the top face of a unit cube together with two opposite vertices of the bottom face. The volume of this tetrahedron is

(a) $\frac{2\sqrt{2}}{3}$

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{1}{4}$

(e) none of the above

19. Kelly rolls a fair die and scores the number of spots showing on its upper face. Kerry rolls two fair dice and scores the total number of spots showing on their upper faces. The probability that Kelly's score is higher than Kerry's score is

(a) $\frac{10}{216}$

(b) $\frac{20}{216}$

(c) $\frac{25}{144}$

(d) $\frac{1}{3}$

(e) none of the above

20. Three unit circles are drawn with centres at (1, 1), (3, 1) and (5, 3). A line passing through (1, 1) is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. The slope of this line is

(a) $\frac{1}{3}$

(b) $\frac{1}{\pi}$

(c) $\frac{3}{\pi^2}$

(d) $\frac{1}{4}$

(e) none of the above

C. Sample Problems from the International Mathematics Tournament of the Towns

Students in Grades 7 to 10 take part in the Junior Tournament. Students in Grades 11 to 12 take part in the Senior Tournament. Each annual Tournament consists of a Fall Round and a Spring Round. Each Round has an O-Level Paper and an A-Level Paper. A student's score for the Tournament is the best of the four papers.

In an O-Level Paper, students are given four hours to attempt four problems. In an A-Level Paper, students are given five hours to attempt six or seven problems. Only the best three problems count for each paper. Problems may carry different weights, those in the A-Level Paper being worth more.

The questions are refreshing and unconventional, as can be seen in the following samples from the Junior Tournaments.

1. Each of 64 friends simultaneously learns one different item of news. They begin to phone one another to relate their news. Each conversation lasts exactly one hour, during which time it is possible for two friends to tell each other all of their news. What is the minimum number of hours needed in order for all the friends to know all of the news?

2. A game is played on an infinite plane. There are fifty-one pieces, one "wolf" and fifty "sheep". There are two players. The first commences by moving the wolf. Then the second player moves one of the sheep, the first player moves the wolf, the second player moves a sheep, and so on. The wolf and the sheep can move in any direction through a distance of up to 1 metre per move. Is is true that for any starting position, the wolf will be able to capture at least one sheep?

3. In a certain country, there are more than 101 towns. The capital of this country is connected by direct air routes with 100 towns, and each town, except for the capital, is connected by direct air routes with 10 towns. It is known that from any town, it is possible to travel by air to any other town, changing planes as many times as is necessary. Prove that it is possible to close down half of the air routes connected with the capital, and preserve the capability of traveling from any town to any other town within the country.

4. A pedestrian walked for 3.5 hours. In every period of one hour's duration, he walked 5 kilometres. Is it necessarily true that his average speed was 5 kilometres per hour?

5. In a ballroom dance class, 15 boys and 15 girls are lined up in parallel rows so that 15 couples are formed. It so happens that the difference in height between the boy and the girl in each couple is not more than 10 centimetres. The boys and girls are rearranged in their respective rows in descending order of height, and 15 new couples are formed, matching the tallest boy with the tallest girl. Prove that in each of the new couples, the difference in height is still not more than 10 centimetres.

6. A village consists of 9 blocks in a 3 by 3 formation, each block a square of side length 1. Each block has a paved road along each side. Starting from a comer of the village, what is the minimum distance

we must travel along paved roads, if each section of paved road must be passed along at least once, and we are to finish at the same corner?

7. Six musicians gathered at a chamber music festival. At each scheduled concert, some of these musicians played while others listened as members of the audience. What is the minimum number of such concerts that must take place in order to enable each musician to listen, as a member of the audience, to all the other musicians?

8. On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet, they both simultaneously change to the third color. Is it possible that they will eventually all be the same color?

9. There are 68 coins, each having a different weight. Show how to find the heaviest coin and the lightest coin in 100 weighings on a balance.

10. Three grasshoppers are on a straight line. Every second, one of the grasshoppers jumps across one, but not both, of the other two grasshoppers. Prove that after 1985 seconds, the grasshoppers cannot all be in their initial positions.

11. In a tournament, each of eight football teams plays every other team once. There are no ties. Prove that at the end of the tournament, it is always possible to find four teams A, B, C and D such that A beats B, C and D, B beats C and D and C beats D.

12. There are 20 football teams in a tournament. On the first day, all the teams play one game. On the second day, all the teams play one game again. Prove that after the second day, it is possible to select 10 teams, no two of which have played each other.

13. In a game with two players, there is a rectangular chocolate bar with 60 pieces arranged in a 6 by 10 formation. It can be broken only along the lines dividing the pieces. The first player breaks the bar along one line, discarding one section. The second player then breaks the remaining section, discarding one section. The first player repeats this process with the remaining section, and so on. The game is won by the player who leaves a single piece. In a perfectly played game, which player wins?

14. A machine gives out five pennies for each nickel and five nickels for each penny. Can Peter, who starts out with one penny, use the machine several times to end up with an equal number of nickels and pennies?

15. Nine pawns form a 3 by 3 square at the lower left corner of an 8 by 8 chessboard. Any pawn may jump over another one standing next to it onto an empty square directly beyond. Jumps may be horizontal, vertical or diagonal. We want to reform the 3 by 3 square at another corner of the chessboard by means of such jumps. Can the pawns be so rearranged at the

    (a) upper left hand corner?

    (b) upper right hand corner?

16. In a game, two players alternately choose larger positive integers. At each turn, the difference between the new and the old number must be greater than zero but smaller than the old number. The original number is 2. The player who chooses the number 1987 wins the game. In a perfectly played game, which player wins?

17. There are 2000 apples, contained in several baskets. One can remove baskets as well as apples from the baskets. Prove that it is possible to leave behind an equal number of apples in each of the remaining baskets, with the total number of apples not being less than 100.

18. It is known that the proportion of people with fair hair among people with blue eyes is greater than the proportion of people with fair hair among all people. Which is greater: the proportion of people with blue eyes among people with fair hair or the proportion of people with blue eyes among all people?

19. Two players alternately move a pawn on a chessboard from one square to another, subject to the rule that the distance of each move is strictly greater than that of the previous move. Distance is measured from the centre of the starting square to the centre of the destination square. A player loses when unable to make a legal move on his turn. Who wins if both use the best strategy?

20. (a) Prove that if 300 stars are placed on the squares of a 200 by 200 board, then it is possible to remove 100 rows and 100 columns in such a way that all stars will be removed.

    (b) Prove that it is possible to place 301 stars on the squares of a 200 by 200 board in such a way that after removing any 100 rows and 100 columns, at least one star remains.

Appendix III: A Selected Bibliography on Popular Mathematics

Part 1: Updates on the Bibliography in MfGS1

A. Martin Gardner's Scientific American Series

Two more volumes have appeared, and there will be a fifteenth and final volume. Several earlier volumes have also changed publishers. The Mathematical Association of America has acquired *Martin Gardner's New Mathematical Diversions from Scientific American, Martin Gardner's 6th Book of Mathematical Diversions from Scientific American, Mathematical Carnival, Mathematical Magic Show* and *Mathematical Circus*. The University of Chicago Press has acquired *The Scientific American Book of Mathematical Puzzles and Diversions, The 2nd Scientific American Book of Mathematical Puzzles and Diversions* and *The Unexpected Hanging and Other Mathematical Diversions*.

We are delighted that Martin Gardner has given us permission to print a comprehensive index of his Scientific American columns. This will constitute Part 2 of this Appendix.

Penrose Tiles to Trapdoor Ciphers, 1989,
W. H. Freeman.

Topics covered are Penrose tilings, Mandebrot's fractals, Conway's surreal numbers, mathematical wordplay, Wythoff's version of the game "Nim", mathematical induction, negative numbers, dissection puzzles, trapdoor ciphers, hyperbolas, the new version of the game "Eleusis", Ramsey theory, the mathematics of Berrocal's sculptures, curiosities in probability, Raymond Smullyan's logic puzzles, as well as two collections of short problems. The book also contains a surprise ending — the resurrection of Dr. Matrix!

Problem 1

Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to one another.

Fractal Music, Hypercards and More, 1992,
W. H. Freeman.

Topics covered are fractal music, the Bell numbers, mathematical zoo, Charles Sanders Peirce, twisted prismatic rings, colored cubes, Egyptian fractions, minimal sculptures, tangent circles, time, generalized ticktacktoe, psychic wonders and probability, mathematical chess problems, Hofstadter's *Gödel, Escher, Bach*, imaginary numbers, some accidental patterns, packing squares, Chaitin's irrational number Ω, as well as one collection of short problems.

Problem 2

Two players are engaged in a game of generalized ticktacktoe on an infinite board. The first player marks an X in a vacant cell, then the second player marks an O in a vacant cell, and the turns alternate thereafter. The first player wins if there is an X in each of four cells in a compact 2×2 configuration. Prove that the second player can prevent that from ever happening.

B. Raymond Smullyan's Logic Series

Satan, Cantor, and Infinity, 1992, Alfred A. Knopf.

In this book, a remarkable character known as the Sorcerer makes his debut. He escorts the readers on a wonderful guided tour, visiting familiar ground, such as the domains of the knights and the knaves, and those bordering on the land of Gödel. There are also ventures into new territories, including an island where intelligent robots create other intelligent robots that can continue this process *ad infinitum*. This eventually leads to the pioneering discoveries on infinity of the great mathematician, Georg Cantor. The readers may be amused to discover how Satan got into the picture.

Problem 3

Hilbert's Hotel has infinitely many rooms—one for each positive integer. The rooms are numbered consecutively—Room 1, Room 2, Room 3, and so on. Currently, each room is occupied by one person. A new guest arrives and wants a room. Neither she nor any of the other guests is willing to share a room, but the others are all cooperative in that they are willing to change their rooms, if requested to do so. How can the new guest be accommodated?

C. Oxford University Press Series on Recreations in Mathematics

The Mathematics of Games, by John D. Beasley, 1989.

This book mathematically analyzes some card and dice games, nim-type games, a version of John Conway's "Hackenbush", as well as providing a mathematical model for the study of some sports games. The principal techniques are counting, probability and game theory. Some mathematical puzzles are also considered.

Problem 4

You draw a card from a standard deck of 52 and claim to have a picture card, that is, a King, a Queen or a Jack. You can bet either \$5 or \$1. If your opponent concedes, she pays you the amount. If she challenges, then whoever is wrong pays the other double the amount. Your strategy is to bet \$5 whenever you have a picture card or an Ace, and \$1 otherwise. Do you win or lose on the average, and by how much?

The Puzzling World of Polyhedral Dissections, by Stewart T. Coffin, 1990.

This is a labor of love from an expert craftsman. Starting with two chapters of two-dimensional geometric puzzles, the author eases the readers gently into the third dimension and soon launches into his specialty, the burrs, which are assemblies of interlocking notched sticks. The book is profusely illustrated with black-and-white line drawings and photographs. It concludes with a chapter on woodworking techniques.

Problem 5

The surface of a $1 \times 2 \times 2$ block may be divided into 16 unit squares. Two such blocks are glued together so that the region common to both consists of 1, 2 or 4 unit squares. Find all the different solids that may be formed this way, and use a copy of each to build a solid rectangular block.

More Mathematical Byways, by Hugh ApSimon, 1990.

This book contains fourteen chapters. The first three form a sequence but the others are independent of each other. Unlike the earlier volume by the same author, the problems are of uneven difficulty levels, ranging from the relatively simple Alphametics to others which require considerable amounts of what the author calls "slog". One of the chapters, titled *Potential Pay*, is not really a problem but a commentary on a classic paradox.

Problem 6

There are four large trees in the plain: an oak, a pine, a quince and a rowan. Ceiling church lies directly between the pine and the rowan, and directly between the oak and the quince. The curate of Ceiling recommends two walks in the neighborhood. Each starts and finishes at the church. One visits in turn the oak, the pine and the quince. The other visits in turn the rowan, the quince and the pine. They are along straight paths, apart of course from turning the corners at the trees. The two walks are of the same total length. The oak and the quince are further apart than are the pine and the rowan. Which of the oak and the rowan is nearer to Ceiling church?

D. Dolciani Mathematical Expositions Series of the Mathematical Association of America

More Mathematical Morsels, by Ross Honsberger, 1991.

This is a collection of 57 problems, almost all of which are taken from the Canadian Mathematical Society's journal *Crux Mathematicorum*, plus further "gleanings" from its famed *Olympiad Corner*.

Problem 7

In a certain multiple-choice test, one of the questions was illegible, but the choice of answers, given below, was clearly printed. What is the right answer?

(a) all of the below

(b) none of the below

(c) all of the above

(d) one of the above

(e) none of the above

(f) none of the above

Old and New Unsolved Problems in Plane Geometry and Number Theory, by Victor Klee and Stan Wagon, 1991.

This book is divided into two halves, as suggested by the title, though the second half also covers problems about some interesting real numbers. Each half consists of two parts: in the first, twelve problems are presented, giving the statement, known results and background information; in the second, the same twelve problems are re-examined for further results and extensions. Each half concludes with a comprehensive bibliography. Although the problems are unsolved, and therefore difficult, it is not impossible for them to yield to an inspired attack. Even if this does not happen, gifted students who are willing to attempt them will find their mathematical talent enhanced.

Problem 8

What is the minimum number of points in the plane, no three on a line, such that some four of them will form the vertices of a convex quadrilateral?

Problems for Mathematicians Young and Old, by Paul Halmos, 1992.

The fourteen chapters of this book are titled Combinatorics, Calculus, Puzzles, Numbers, Geometry, Tilings, Probability, Analysis, Matrices, Algebra, Sets, Spaces, Mappings and Measures. The author, a ranking mathematician and master expositor, wrote this book for fun, and hopes that it will be read the same way.

Problem 9

Cucumbers are assumed to consist of 99% water. If 500 kilograms of cucumbers are allowed to stand overnight, and if the partially evaporated substance that remains in the morning is 98% water, how much is the morning weight?

Excursions in Calculus, by Robert Young, 1992.

The subtitle of this book is *An Interplay of the Continuous and the Discrete*. Using calculus as a unifying theme, the author branches into number theory, algebra, combinatorics and probability. The book contains a large collection of exercises and problems.

Problem 10

Suppose there are finitely many prime numbers and the largest is p. Let $M = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{p}{p-1}$, the product ranging over all prime numbers. Prove that $M = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$, the sum ranging over all positive integers. Deduce from this that there are infinitely many prime numbers.

The Wohascum County Problem Book, by George Gilbert, Mark Krusemeyer and Loren Larson, 1993.

This is a collection of problems, many of which are original compositions. It is mainly targeted at undergraduate students, but quite a few problems are accessible to high school students. The title refers to a recurrent group of problems based on country life in the fictitious locale.

Problem 11

Two ice fishermen set up their ice houses on a frozen circular lake, in exactly opposite directions from the centre, two-thirds of the way from the centre to the lakeshore. Assuming that any fish would go for the nearest lure, each has command of exactly half the lake. A third ice fisherman shows up, but the first two adamantly refuse to move. Is it possible for the newcomer to set up his ice house so that each has command of a region exactly one third the area of the lake?

Lion Hunting & Other Mathematical Pursuits, edited by Gerald Alexanderson and Dale Muggler, 1995.

The subtitle of this book is *A Collection of Mathematics, Verse, and Stories by Ralph P. Boas. Jr.*. A posthumous tribute to this great mathematician and teacher, it contains selected work of Boas and reminiscences by his friends and relatives. The title refers to a piece of mathematical folklore about various mathematical "methods" to trap a lion in a cage. The material chosen is most suitable for undergraduate students, but high school teachers and students should find it a most amusing and informative reading. Find out how the following "problem" arose.

Problem 12

Where in the world are the following places and what do the names mean?

(a) Llanfairpwilgwyngyllgogerychwyrndrobwllllandysiliogogogoch

(b) ChargogogogmanchargogogchaubunnagungamAug

(c) Taumatawhakatangihangakoauauotamateapokaiwhenuakitanatahu

The Linear Algebra Problem Book, by Paul Halmos, 1995.

The whole book is a sequence of structured problems. Like the preceding volume, most of this book is beyond high school level. However, the introductory problems are certainly not intimidating, and inquisitive students may be lured into a most rewarding exploration, laying a good foundation for their undergraduate studies.

Problem 13

If a new addition for real numbers, denoted by the temporary symbol $\oplus$, is defined by $a \oplus b = 2a + 2b$, is $\oplus$ associative?

E. New Mathematical Library of the Mathematical Association of America

In additional to the new volumes listed below, there is also a revised edition of an earlier volume, *Graphs and Their Uses.*

USA Mathematical Olympiads: 1972-1986, by Murray Klamkin, 1988.

This book collects the problems of the first fifteen USA Mathematical Olympiads. While they are presented chronologically, the solutions are grouped according to subject matters, which facilitates using this book for training sessions. There is a very useful 10-page glossary of mathematical terms and results, and an extensive bibliography.

Problem 14

Let $a_1, a_2, a_3, \ldots$ be a non-decreasing sequence of positive integers. For an integer $m > 1$, define $b_m = \min\{n | a_n > m\}$, that is, b_m is the minimum value of n such that $a_n > m$. If $a_{19} = 85$, determine the maximum value of $a_1 + a_2 + \cdots + a_{19} + b_1 + b_2 + \cdots + b_{85}$.

Exploring Mathematics with your Computer, by Arthur Engel, 1993.

The author has served as the leader and coach of the formerly West German I.M.O. team for many years, and is an acknowledged expert on problem-solving. This volume is a mathematics book, and not a programming book, even though the computer language *Pascal* is explained, and an I.B.M. diskette comes with the package. Problems in number theory, probability, statistics, combinatorics and numerical analysis are explored.

Problem 15

We have 15 boxes, each one with its own key which fits no other box. After mixing the keys at random, we drop one into each box. Now we break open 2 boxes. What is the probability that we are now able to open the remaining boxes with keys?

Game Theory and Strategy, by Philip Straffin, 1993.

This book consists of three chapters, dealing with two-person zero-sum games, two-person non-zero-sum games and N-person games. These terms are explained in details. Many applications are included, into such diverse fields as anthropology, warfare, philosophy, social psychology, biology, business, economics, politics and athletics. The book also presents concrete models such as Jamaican fishing.

Problem 16

Two people are jointly charged for a crime. If both confess, each will get a light sentence. If one confesses and the other does not, the first will receive a reward while the second will get a heavy sentence. If neither confesses, both will go free. Explain why when considering the situation individually, it is better to confess, and yet considering the situation collectively, it is better not to confess.

F. Mir Publishers' Little Mathematics Library Series

This excellent series has become an unfortunate casualty of the demise of the former Soviet Union. Lost also are *Mathematics Can Be Fun* and *Fun with Math and Physics* featured in Section I.

G. Books from W. H. Freeman & Company, Publishers

Note that the two most recent books in Martin Gardner's Scientific American Series are Freeman publications. The first of the books listed below has actually gone out of print, but fortunately, Dover Publications Inc. has decided to pick it up.

The Puzzling Adventures of Dr. Ecco, by Dennis Shasha, 1988.

The title character calls himself an omniheurist, solver of all problems (mathematical). The narrative is by a Watsonesque companion, Prof. Scarlet. Ecco's clients range from government officials, industrialists, and eccentric millionaires to no less than the president of a Latin American country. They bring him important, instructive and interesting problems in discrete mathematics, all of which Ecco solves to their satisfaction. The book concludes with the mysterious disappearance of Dr. Ecco.

Problem 17

A counselor took some campers on a wilderness trip. At 100 minutes before sunset, they were lost at a four-way cross-road. It was known that their campsite was down one of the four paths, 20 minutes away on foot. Thus there was enough time for two exploratory trips by the counselor and the campers, before they had to reassemble at the cross-road and head down the correct path. However, three unidentified campers would not necessarily tell the truth. Nevertheless, the counselor was able to deduce the correct location of the campsite. What was the minimum number of campers, counting the three clowns, and how should the counselor organize the explorations?

Codes, Puzzles, and Conspiracy, by Dennis Shasha, 1992.

In this delightful sequel, Prof. Scarlet and friends were hot on the trail of the kidnappers of Dr. Ecco. They went on a globe-trotting tour into exotic locales, where they found time to tackle mathematical problems which cropped up everywhere. The story turns into a mathematical thriller, complete with a Moriarty-like arch-villain, and a nefarious character in high places, shades of Ollie North.

Problem 18

The Amazing Sand Counter claims that if sand is put into a bucket, he can tell at a glance how many grains there are. However, he will not tell you. How can you test whether he indeed has this power, without him telling you anything which you do not already know?

New Book of Puzzles, by Jerry Slocum and Jack Botermans, 1992.

The subtitle of this book is *101 Classic and Modern Puzzles to Make and Solve*. It is the long-awaited sequel to *Puzzles Old and New* by the same dynamic duo, reviewed in **Mathematics for Gifted Students I (MfGS1)**. As with the earlier volume, it is full of superb colorful illustrations and photographs by Botermans.

Problem 19

Construct a $3 \times 3 \times 3$ cube with the following six pieces, each of unit thickness.

Another Fine Math You've Got Me Into..., by Ian Stewart, 1992.

This whimsical volume is by the current occupant of the exalted position long held by Martin Gardner, the editor of the Mathematical Games column in Scientific American. It has sixteen chapters, from *The Lion, the Llama, and the Lettuce* to *Sofa, So Good* ..., each treating a mathematical problem in some depth.

Problem 20

Seven varieties of grapes are to be arranged in plots. Each plot contains exactly three different varieties. Is it possible that any two plots have exactly one variety in common, and that any two varieties lie in exactly one common plot?

H. Books from Dover Publications, Inc.

Excursions in Number Theory,
by Stanley Ogilvy and John Anderson, 1988.

This book covers the basics of classical number theory. Topics include prime numbers, congruencies, Diophantine equations and Fibonacci numbers. The narrative style is very soothing. It concludes with 20 pages of elaboration and commentary on some finer points raised in the text.

Problem 21

Do there exist two irrational numbers a and b such that a^b is a rational number?

Excursions in Geometry, by Stanley Ogilvy, 1990.

The first half of this book is on inversive geometry, and the second half on projective geometry. These two topics are linked by the concept of cross-ratio and the study of the conic sections. It is in the same style as the preceding volume.

Problem 22

Given are two parallel lines and a segment on one of them. Construct the midpoint of this segment using a straight-edge but without using a compass.

Excursions in Mathematics, by Stanley Ogilvy, 1994.

The original title of this volume was *Through the Mathescope*. The opening chapter is titled *What Do Mathematicians Do?* It is followed by lively tours of number theory, algebra, geometry and analysis. The last chapter is titled *Topology and Apology*.

Problem 23

You toss a fair coin 2n times, hoping to get n heads and n tails. What is the value of n for which your probability of success is the highest?

I. Individual Titles

Selected Problems and Theorems in Elementary Mathematics has been acquired by Dover and renamed *The USSR Olympiad Problem Book*. Dover has also picked up *The Moscow Puzzles. All the Best from the Australian Mathematics Competition* has now become the incipient volume in a new series. The Mathematical Association of America has published *Five Hundred Mathematical Challenges*, comprising the first five booklets of the project, *1001 Problems in High School Mathematics*.

The Canadian Mathematical Olympiad,
1969—1993, edited by Michael Dobb and Claude Laflamme, Canadian Mathematical Society, 1993.

This book combines two earlier volumes, *The First Ten Canadian Mathematics Olympiads, 1969–1978* and *The Canadian Mathematics Olympiads, 1979–1985. The contests from 1986 to 1993, each consisting of five questions also appear.*

Problem 24

A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most, how many schools were represented by an odd number of players?

J. Addresses of Publishers

The Canberra College of Advanced Education has now become the University of Canberra. The publication of books is now done by the Australian Mathematics Trust. The address is still P.O. Box 1, Belconnen, ACT 2616, Australia

Part 2: Index of Martin Gardner's Scientific American Columns

Anthologies

1. The Scientific American Book of Mathematical Puzzles and Diversions
2. The Second Scientific American Book of Mathematical Puzzles and Diversions
3. New Mathematical Diversions from Scientific American
4. The Magic Numbers of Dr. Matrix
5. The Unexpected Hanging and other Mathematical Diversions
6. The Sixth Book of Mathematical Diversions from Scientific American
7. The Mathematical Carnival
8. The Mathematical Magic Show
9. The Mathematical Circus
10. Wheels, Life, and other Mathematical Amusements
11. Knotted Doughnuts and other Mathematical Entertainment
12. Time Travel and other Mathematical Bewilderments
13. Penrose Tiles to Trapdoor Ciphers
14. Fractal Music, Hypercards and more Mathematical Recreations

Note: The fifteenth and last volume should appear some time in 1996.

Monthly Columns

01	Dec 56	Hexaflexagon
01	Jan 57	Magic with a Matrix
01	Apr 57	Probability Paradoxes
01	May 57	The Icosian Game and the Tower of Hanoi
01	Jun 57	Curious Topological Models
01	Jul 57	The Game of Hex
01	Aug 57	Sam Loyd: America's Greatest Puzzlist
01	Sep 57	Mathematical Card Tricks
01	Oct 57	Memorizing Numbers
01	Nov 57	***Nine Problems
01	Dec 57	Polyominoes
01	Jan 58	Fallacies
01	Feb 58	Nim and Tac Tix
01	Mar 58	Left or Right
02	Apr 58	The Monkey and the Coconuts
02	May 58	Tetraflexagons
02	Jun 58	Henry Ernest Dudeney: England's Greatest Puzzlist
02	Jul 58	Digital Roots
02	Aug 58	***Nine Problems
02	Sep 58	The Soma Cube
02	Oct 58	Recreational Topology
02	Nov 58	Squaring the Square
02	Dec 58	The Five Platonic Solids
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02	Feb 59	Recreational Logic
02	Mar 59	Magic Squares
02	Apr 59	James Hugh Riley Shows, Inc.
02	May 59	***Nine Problems
02	Jun 59	Eleusis: The Induction Game
02	Jul 59	Origami
02	Aug 59	Phi: The Golden Ratio
02	Sep 59	Mechanical Puzzles
02	Oct 59	Probability and Ambiguity
03	Nov 59	Euler's Spoilers: An Order-10 Graeco-latin Square
03	Dec 59	Group Theory and Braids

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04	Jan 60	—Dr. Matrix
03	Feb 60	***Eight Problems
03	Mar 60	The Games and Puzzles of Lewis Carroll
03	Apr 60	Board Games
03	May 60	Packing Spheres
03	Jun 60	Paper Cutting
03	Jul 60	The Transcendental Number Pi
03	Aug 60	Victor Eigen: Mathemagician
03	Sep 60	The Four-color Map Theorem
03	Oct 60	***Nine Problems
03	Nov 60	Polyominoes and Fault-free Rectangles
03	Dec 60	The Binary System

		***
04	Jan 61	—Dr. Matrix
03	Feb 61	The Ellipse
03	Mar 61	The 24 Color Squares and the 30 Color Cubes
03	Apr 61	H. S. M. Coxeter
03	May 61	Mr. Apollinax Visits New York
03	Jun 61	***Nine Problems
03	Jul 61	Bridg-it and Other Games
03	Aug 61	The Calculus of Finite Differences
05	Sep 61	Knots and Borromean Rings
05	Oct 61	The Transcendental Number *e*
05	Nov 61	Geometric Dissections
05	Dec 61	Scarne on Gambling
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05	Jan 62	The Church of the Fourth Dimension
05	Feb 62	***Eight Problems
05	Mar 62	A Matchbox Game-learning Machine
05	Apr 62	Spirals
05	May 62	Rotations and Reflections
05	Jun 62	Peg Solitaire
05	Jul 62	Flatlands
05	Aug 62	Tests of Divisibility
05	Oct 62	***Nine Problems
05	Nov 62	The Eight Queens and Other Chessboard Diversions
05	Dec 62	A Loop of String
		***
04	Jan 63	—Dr. Matrix
05	Feb 63	Curves of Constant Width
05	Mar 63	The Paradox of the Unexpected Hanging
05	Apr 63	***Thirty-seven Quickies
05	May 63	Rep-tiles: Replicating Figures on the Plane
06	Jun 63	The Helix
06	Jul 63	Klein Bottles and Other Surfaces
06	Aug 63	Combinatorial Theory
06	Sep 63	Bouncing balls in Polygons and Polyhedrons
06	Oct 63	Four Unusual Board Games
06	Nov 63	***Nine Problems
06	Dec 63	Parity Checks
		***
04	Jan 64	—Dr. Matrix
06	Feb 64	Sliding-block Puzzles
06	Mar 64	Patterns and Primes
06	Apr 64	Graph Theory
06	May 64	The Ternary System
06	Jun 64	*** Eight Problems
06	Jul 64	The Cycloid: Helen of Geometry
06	Aug 64	Mathematical Magic Tricks
06	Sep 64	Word Play
06	Oct 64	The Pythagorean Theorem
06	Nov 64	Limits of Infinite Series
06	Dec 64	Polyiamonds
		***
04	Jan 65	—Dr. Matrix
06	Feb 65	Tetrahedrons
06	Mar 65	***Nine Problems
06	Apr 65	Infinite Regress
06	May 65	The Lattice of Integers
06	Jun 65	O'gara, The Mathematical Mailman
06	Jul 65	Op Art
06	Aug 65	Extraterrestrial Communication
07	Sep 65	Piet Hein's Superellipse
08	Oct 65	Polyominoes and Rectification
08	Nov 65	***Nine Problems
07	Dec 65	Magic Stars and Polyhedrons
		***
04	Jan 66	—Dr. Matrix
07	Feb 66	Penny Puzzles
07	Mar 66	Aleph-null and Aleph-one
07	Apr 66	The Art of M. C. Escher
07	May 66	Cooks and Quibble-cooks
07	Jun 66	How to Trisect an Angle
07	Jul 66	The Numerology of Dr. Fliess
07	Aug 66	The Rising Hourglass and Other Physics Puzzles
07	Sep 66	Mrs. Perkins' Quilt and Other Square-packing Problems
07	Oct 66	Card Shuffles
07	Nov 66	Hypercubes
07	Dec 66	Pascal's Triangle
		***
04	Jan 67	—Dr. Matrix
07	Feb 67	Jam, Hot and Other Games
07	Mar 67	***Nine Problems
07	Apr 67	Calculating Prodigies
07	May 67	Tricks of Lightning Calculations
08	Jun 67	Polyhexes and Polyaboloes
07	Jul 67	Sprouts and Brussels Sprouts
08	Aug 67	Factos

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04	Jan 69	—Dr. Matrix
09	Feb 69	Boolean Algebra
09	Mar 69	Fibonacci and Lucas Numbers
09	Apr 69	***Seven Problems and Ten Quickies
09	May 69	Random Walks and Gambling
09	Jun 69	Random Walks on the Plane and in Space
09	Jul 69	Matches
09	Aug 69	Simplicity
09	Sep 69	Mascheroni Constructions
04	Oct 69	—Dr. Matrix
09	Nov 69	Patterns of Induction
09	Dec 69	Dominoes

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09	Jan 70	The Abacus
09	Feb 70	***Nine Problems
09	Mar 70	Cyclic Numbers
09	Apr 70	Solar System Oddities
09	May 70	Optical Illusions
09	Jun 70	Elegant Triangles
10	Jul 70	Diophantine Analysis and Fermat's Last Theorem
09	Aug 70	Palindromes: Words and Numbers
10	Sep 70	Wheels
10	Oct 70	The Game of Life, Part 1
10	Nov 70	***Ten Problems
10	Dec 70	Nontransitive Dice and Other Probability Paradoxes

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04	Jan 71	—Dr. Matrix
10	Feb 71	The Game of Life, Part 2
10	Mar 71	Alephs and Supertasks
10	Apr 71	Geometrical Fallacies
10	May 71	The Combinatorics of Paper Folding
09	Jun 71	Can Machines Think?
10	Jul 71	***Thirty-six Quickies
10	Aug 71	Ticktacktoe Games
10	Sep 71	Plaiting Polyhedrons
10	Oct 71	The Game of Halma
10	Nov 71	Advertising Premiums
10	Dec 71	Salmon on Austin's Dog

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10	Jan 72	Nim and Hackenbush
04	Feb 72	—Dr. Matrix
10	Mar 72	Golomb's Graceful Graphs
10	Apr 72	***Nine Problems
10	May 72	Chess Tasks
10	Jun 72	Slithers, $3x + 1$, and Other Curious Questions
10	Jul 72	Mathematical Tricks with Cards
11	Aug 72	The Binary Gray Code
11	Sep 72	Polycubes
11	Oct 72	Coincidence
11	Nov 72	Bacon's Cipher
11	Dec 72	Doughnuts: Linked and Knotted

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11	Jan 73	Sim, Chomp and Race Track
11	Feb 73	Elevators
11	Mar 73	Napier's Bones
11	Apr 73	Napier's Abacus
11	May 73	***Nine Problems
11	Jun 73	Crossin Numbers
11	Jul 73	Newcomb's Paradox
04	Aug 73	—Dr. Matrix
11	Sep 73	Point Sets on the Sphere
11	Oct 73	Look-see Proofs
11	Nov 73	Worm Paths
11	Dec 73	Waring's Problems

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11	Jan 74	The I Ching
11	Feb 74	Cram, Bynum and Quadraphage
11	Mar 74	Reflections on Newcomb's Paradox
11	Apr 74	***Nine Problems
12	May 74	Time Travel
04	Jun 74	—Dr. Matrix
12	Jul 74	Hexes and Stars
12	Aug 74	Tangrams, Part 1
12	Sep 74	Tangrams, Part 2
12	Oct 74	Nontransitive Paradoxes
12	Nov 74	Combinatorial Card Problems
12	Dec 74	Melody-making Machines

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12	Jan 75	Anamorphic Art
08	Feb 75	Nothing
12	Mar 75	***Eight Problems
12	Apr 75	Six Sensational Discoveries
12	May 75	The Csaszar Polyhedron
12	Jun 75	Dodgem and Other Simple Games
12	Jul 75	Tiling with Convex Polygons
12	Aug 75	Tiling with Polyominoes, Polyiamonds and Polyhexes
04	Sep 75	—Dr. Matrix
xx	Oct 75	Extrasensory Perception by Machine

12	Nov 75	Curious Maps
12	Dec 75	***Five Problems

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12	Jan 76	Magic Squares and Cubes
12	Feb 76	Block Packing
12	Mar 76	Induction and Probability
xx	Apr 76	Snarks and Boojums
08	May 76	Everything
12	Jun 76	Catalan Numbers
12	Jul 76	Fun with a Pocket Calculator
12	Aug 76	Tree-plant Problems
13	Sep 76	Conway's Surreal Numbers
13	Oct 76	Back From the Klondike and Other Problems
04	Nov 76	—Dr. Matrix
13	Dec 76	Mandelbrot's Fractals

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13	Jan 77	Penrose Tiling
13	Feb 77	The Oulipo
13	Mar 77	Wythoff's Nim
13	Apr 77	***Eight Problems
13	May 77	Mathematical Induction and Colored Hats
13	Jun 77	Negative Numbers
13	Jul 77	Cutting Shapes into *n* Congruent Parts
13	Aug 77	Trapdoor Ciphers
13	Sep 77	Hyperbolas
13	Oct 77	The New Elussis
13	Nov 77	Ramsey Theory
04	Dec 77	—Dr. Matrix

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13	Jan 78	From Burrs to Berrocal
13	Feb 78	Sicherman Dice, Kruskal Count and Other Curiosities
13	Mar 78	Raymond Smullyan's Logic Puzzles
14	Apr 78	White, Brown, and Fractal Music
14	May 78	The Tinkly Temple Bells
14	Jun 78	Mathematical Zoo
14	Jul 68	Charles Sanders Peirce
14	Aug 78	Twisted Prismatic Rings
14	Sep 78	The Thirty Color Cubes
14	Oct 78	Egyptian Fractions
14	Nov 78	Minimal Sculpture
04	Dec 78	—Dr. Matrix

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14	Jan 79	Tangent Circles
14	Feb 79	***Six Problems
14	Mar 79	Does Time Ever Stop? Can the Past be Altered?
14	Apr 79	Generalized Ticktacktoe
14	May 79	Psychic Wonders and Probability
14	Jun 79	Mathematical Chess Problems
14	Jul 79	Douglas Hofstadter's Godel, Escher, Bach
14	Aug 79	Imaginary Numbers
14	Sep 79	Pi and Poetry: Some Accidental Patterns
14	Oct 79	Packing Squares
14	Nov 79	Chaitin's Omega
xx	Dec 79	***Seven Problems

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xx	Jan 80	Checkers
xx	Feb 80	The Coloring of Unusual Maps
xx	Mar 80	Graphs that can Help
xx	Apr 80	Fun with Eggs
xx	May 80	Combinatorial Designs
xx	Jun 80	The Monster Group
xx	Jul 80	Science and Technology in the Planiverse
xx	Aug 80	The Pigeonhole Principle
04	Sep 80	—Dr. Matrix
xx	Oct 80	The Mathematics of Elections
xx	Nov 80	Taxicab Geometry
xx	Dec 80	The Strong Law of Small Numbers

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xx	Feb 81	Gauss's Congruence Theory
xx	Apr 81	***Seven Problems
xx	Jun 81	Geometrical Symmetries of Scott Kim
xx	Aug 81	The Abstract Parabola Fits the Concrete World
xx	Oct 81	Euclid's Parallel Postulate
11	Dec 81	The Laffer Curve

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xx	Aug 83	Tasks You Cannot Help Finishing
xx	Sep 83	The Topology of Knots
xx	Jun 86	Casting a Net on a Checkerboard

Note:
Columns denoted by "xx" have not yet been anthologized in this series.

Part 3: An Extension of the Bibliography in MfGS1

K. Mathematics Competition Enrichment Series

As mentioned earlier, *All the Best from the Australian Mathematics Competition* has become the incipient volume of this new series, published by the Australian Mathematics Trust. The AMT was established by the late **Peter O'Halloran**, one of his most significant accomplishments among many outstanding achievements in a distinguished career. The Australian Mathematics Competition has become the model for many around the world. This series, under the leadership of **Graham Pollard** and **Peter Taylor**, will consolidate the Trust's leadership role in the world of mathematics competitions.

Mathematical Tool chest, edited by A. W. Plank and N. H. Williams, 1992.

This is a compilation, started by the late **Jim Williams**, of the basic results in mathematics most often called upon in competitions. It is organized into the following nine chapters: combinatorics, arithmetic & number theory, algebra, inequalities, analysis, plane geometry, solid geometry, analytic & vector geometry, and geometric transformations.

Problem 25

Nine real numbers are arranged in a circle. Their sum is 90. Prove that the sum of a block of four adjacent numbers is at least 40.

International Mathematics Tournament of the Towns: 1984-1989, edited by Peter Taylor, 1992.

For information about the Tournament, see Appendix II, which also features a sampling of Junior problems. This book contains one of the best collections of problems of all time. With a few exceptions, the solutions are worked out independently of the Russian proposers, with Edmonton student **Calvin Li** contributing quite a few.

Problem 26

Inside a rectangle is inscribed a quadrilateral, which has a vertex on each side of the rectangle. Prove that the perimeter of the inscribed quadrilateral is not smaller than double the length of a diagonal of the rectangle.

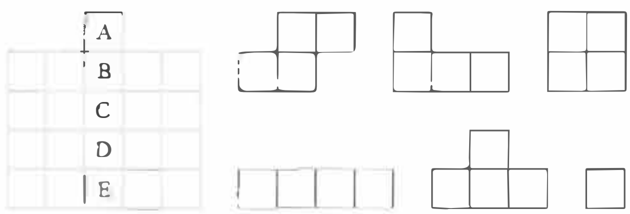
More of All the Best from the Australian Mathematics Competition, edited by Peter O'Halloran, Graham Pollard and Peter Taylor, 1992.

The earlier volume covers the AMC from 1976 to 1984, and the present one from 1985 to 1991. Here, the 483 problems are also grouped by subject.

Problem 27

A toy in six pieces, which can be turned over, came in a box as shown in the diagram. The unit square was in one of the marked spaces. This space was marked by the letter

(A) A (B) B (C) C (D) D (E) E



Problem Solving via the Australian Mathematics Competition, by Warren Atkins, 1992.

This book takes 149 problems from the Australian Mathematics Competition for 1978—1991 and develops strategies for their solutions. The problems are about Diophantine equations, counting techniques, speed, time and distance, and geometry. In each area, problems are presented, along with discussions and some solutions, with further solutions deferred to an appendix.

Problem 28

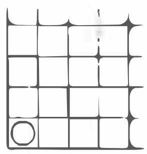
Warren and Naida have a straight path, 1 metre wide and 16 metres long, from the front door to their front gate. They decide to pave it. Warren brings home 16 paving stones, each 1 metre square. Naida brings home 8 paving stones, each 1 metre by 2 metres. Fortunately, they can get refunds on the unused paving stones. Assuming that they would consider using all rectangular or all square paving stones, or some of each, how many patterns are possible in paving the path?

International Mathematics Tournament of the Towns: 1980-1984, edited by Peter Taylor, 1993.

This volume appeared after the volume on the Tournaments from 1984 to 1989 because much time and energy were required to compile the older contests from the archives. The effort was amply rewarded with this book which is of the highest quality.

Problem 29

The entire first quadrant of the coordinate plane is divided into unit squares and serves as a checkerboard. The diagram shows a single counter inside the region consisting of six squares designated as home squares. In each move, we may replace a counter by two others, occupying the square to the north and to the east of its current position, provided that those squares are vacant. Is it possible to find a finite sequence of moves after which no counters remain on any home squares?



International Mathematics Tournament of the Towns: 1989-1993, edited by Peter Taylor, 1994.

This third volume on the Tournament covers only four years because the number of problems per year has increased. It is thicker than either of the other two books. Edmonton students **Jason Colwell, Peter Laffin, Steven Laffin, Calvin Li** and **Matthew Wong** all contributed solutions.

Problem 30

Find ten different positive integers such that each of them is a divisor of their sum.

The Asian Pacific Mathematics Olympiad, 1989-1993, edited by Hans Lausch, 1994.

This regional contest was another brainchild of the late **Professor Peter O'Halloran**. From a modest beginning with Australia, Canada, Hong Kong and Singapore in 1989, the number of participating countries had grown to thirteen in 1993. In addition to the problems, the book gives a detailed account of the organization of the contest.

Problem 31

Suppose there are 997 points given in the plane. If every two points are joined by a line segment with its midpoint colored red, prove that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

L. Enrichment Series from the Center for Excellence in Mathematics Education

The Center is the creation of **Alexander Soifer**, who emigrated from the former Soviet Union to the United States in 1978. He brought with him the tradition of the famed Mathematics Circles, and his own expertise in problem solving, particularly in combinatorial geometry.

Mathematics as Problem Solving, by A. Soifer, 1987.

This book has five chapters. The first one is on general techniques and useful tools in problem solving. The remaining four give illustrations from Number Theory, Algebra, Geometry and Combinatorics.

Problem 32

The numbers a, b and $\sqrt{a}+\sqrt{b}$ are all rational. Prove that so are $\sqrt{a}$ and $\sqrt{b}$.

How Does One Cut a Triangle, by A. Soifer, 1990.

Two principal problems are discussed in this book. The first is to find all positive integers n such that every triangle can be cut into n triangles congruent to each other. In the second, "congruent" is replaced by "similar". Necessary tools are developed along the way, and other related problems are raised.

Problem 33

Given nine points in a triangle of area 1, prove that three of them form a triangle of area not exceeding $\frac{1}{4}$.

Geometric Etudes in Combinatorial Mathematics, by V. Boltyanski and A. Soifer, 1991.

This book contains four chapters. The first one is on tiling rectangles with polyominoes. The second is on the application of the Pigeonhole Principle in Geometry. The third is on the Theory of Graphs. The fourth, which is quite substantial and more advanced, deals with Convex Sets and Combinatorial Geometry.

Problem 34

The V-tromino is a shape obtained from a 2×2 square by removing one of the four unit squares. Show how to pack 15 copies of the V-tromino into a 5×9 box.

Colorado Mathematical Olympiad, by A. Soifer, 1994.

The first part of the book contains the problems of the first ten olympiads, plus personal reminiscences of its early history. The second part contains the solutions and further explorations. Some of the problems are Russian folklore, while others are the creations of the author and his wide network of friends.

Problem 35

On a 3×3 chessboard, two white knights are on adjacent corners and two black knights are on the other two corners. Is it possible to move them so that the two knights of each color occupy opposite corners?

M. Birkhäuser Mathematics Series

This series contains three volumes to be published later on geometry, calculus and combinatorics. The first two volumes were written in Russian more than twenty-five years ago, and were the notes of a mathematics correspondence school in the former Soviet Union. This School was organized by **I. M. Gelfand** and continues to be directed by him to this date.

Functions and Graphs, by I. M. Gel'fond, E. G. Glagoleva and E. E. Shnol, 1990.

The important concept of a function is carefully introduced via examples. Graph sketching is explored using little more than high school algebra. The main part of the book is the study of the polynomial and rational functions, built up one step at a time, starting with the linear and quadratic functions. There are many exercises. Hints and answers are given to the more difficult ones.

Problem 36

Two roads intersect at a right angle. Two cars drive towards the intersection, one on the first road at a speed of 60 kilometres per hour and the other on the second road at a speed of 30 kilometres per hour. At noon, both cars are 10 kilometres from the intersection. At what moment will the distance between the cars be least? Where will the cars be at this moment?

The Method of Coordinates, by I. M. Gel'fond E. G. Glagoleva and A. A Kirilov, 1990.

Algebra and geometry, which most students today consider completely different subjects, are in fact quite closely related. The method of coordinates transforms geometric images into algebraic formulae. In the first part of the book, coordinate systems on the line, in the plane and in space are studied. In the second part, four-dimensional spaces are discussed.

Problem 37

A four-dimensional cube has 16 vertices. How many edges, two-dimensional faces and three-dimensional faces does it have?

Algebra, by I. M. Gelfand and A. Shen, 1993.

Starting from basic arithmetic, many important properties of the number system are molded into general results in algebraic terms. In seventy-two sections, the book takes the readers through the essential parts of high school algebra, including exponents, factorization, progressions, polynomial equations, and the often neglected topic of inequalities.

Problem 38

Let $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n$ be real numbers. The famous Cauchy-Schwarz Inequality states that $(p_1^2 + p_2^2 + \cdots + p_n^2)(q_1^2 + q_2^2 + \cdots + q_n^2) \geq (p_1 q_1 + p_2 q_2 + \cdots + p_n q_n)^2$. Give a proof by considering the quadratic expression $(p_1 + q_1 x)^2 + (p_2 + q_2 x)^2 + \cdots + (p_n + q_n x)^2$.

N. Cambridge University Press Puzzle Books

Mathematical Amusement Arcade, by Brian Bolt, 1984.

This book contains 130 mathematics puzzles, many of which are taken from the popular literature. The statements are presented with attractive story lines, clear diagrams and amusing cartoons. The solutions are given in red, and comprise half the book.

Problem 39

Four points in the plane determine six distances. In six configurations, there are only two different values among these distances. One of them consists of the centre and the three vertices of an equilateral triangle. Find the other five configurations.

Mathematical Funfair, by Brian Bolt, 1989.

This book contains 128 mathematics puzzles.

Problem 40

Arrange the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 in some order to form a nine-digit number such that it is divisible by 9, the number formed of the first 8 digits is divisible by 8, the number formed of the first 7 digits is divisible by 7, and so on.

Mathematical Cavalcade, by Brian Bolt, 1992.

This book contains 131 mathematical puzzles.

Problem 41

Sixteen points are given in a 4×4 array. Draw a broken line consisting of six line segments joined end to end, passing through each of the points at least once.

A Mathematical Pandora's Box, by Brian Bolt, 1993.

This book contains 142 mathematical puzzles.

Problem 42

Make a single straight cut through some of five squares each of area 1 and arrange the resulting pieces into a square of area 5.

Challenging Puzzles, by Colin Vout and Gordon Gray, 1993.

After 18 starter puzzles, the reader goes on an adventure with Snow White and the Seven Dwarfs (20 problems), visits the Isle of Maranga (28 problems), consults the Martian Dating Agency (18 problems) and finishes up by playing some sports and games (16 problems). Hints, solutions and further explorations are provided in separate sections.

Problem 43

Jock, Doc, Grumpy and Bossy were playing cards, one of which was lying on the table. "Who led that?" asked Grumpy. Bossy muttered something. Doc said, "Bossy led it." Jock said that Bossy had told him that Grumpy had led it. "No, it wasn't me or Jock," said Grumpy. "As I said before, it was Doc," said Bossy. In fact, only one of the dwarfs had been speaking truthfully. So who led that card?

O. The Spectrum Series of the Mathematical Association of America

This consists of a number of independent titles written for or acquired by the Mathematical Association of America. It contains over twenty volumes, including five titles from Martin Gardner's Scientific American Series.

The Last Problem, by Eric Temple Bell, 1991.

This book was originally published by Simon and Schuster in 1961, a year after the death of the author. While the central theme is Pierre Fermat and his Last Problem or Theorem, there are many historical references to western civilization in general and related mathematical problems in particular.

Problem 44

Find all positive integer solutions of $x^2 + 12 = y^4$.

Journey into Geometries, by Marta Sved, 1991.

In Part 1 of the book, the reader embarks on a geometric tour with Alice, Lewis Carroll and Dr. Whatif. The company is amicable, the conversation lively and the mathematics engaging. In Part 2, the tour is reexamined in a more formal setting, and solutions to the problems raised in Part 1 are given.

Problem 45

The power of a point P with respect to a circle with center O and radius r is defined as $OP^2 - r^2$. Prove that if a line passing through P cuts the circles at A and B, then the product of PA and PB is equal to the absolute value of the power of the point P.

Polyominoes, by George Martin, 1991.

While there is some overlap with Solomon Golomb's classic of the same title, reviewed later (see Section P), there is also interesting new material in this book, which can be read independently or as a supplement to the Golomb text.

Problem 46

Can a 6×6 square be tiled with 18 dominoes so that it is impossible to divide it into two subrectangles without splitting any dominoes?

The Lighter Side of Mathematics, edited by Richard Guy and Bob Woodrow, 1994.

This book reprints the proceedings of the Eugene Strens Memorial Conference on Recreational Mathematics and its history, held in 1986 at Calgary on the occasion of Richard Guy's seventieth birthday. This occasion also marked Guy's acquisition of the Strens collection of material on recreational mathematics on behalf of the University of Calgary. The contributed papers are grouped under Tiling & Coloring, Games & Puzzles, and People & Pursuits.

Problem 47

A county has five towns, A, B, C, D and E. There is a road connecting every pair of them. None of the roads meet except at the towns. Five are one-way paved roads, going from C to D, D to E, E to A, A to B and B to D. The other five are one-way country roads, going from D to A, A to C, C to E, E to B and B to C. A tourist is lost in a snow storm in one of the towns, and there is no one in sight. He cannot find any sign with the name of the town. The directions of the four roads are clearly marked, but they do not say where the road leads. The tourist phoned the Travel Bureau which is in D. The operator gives the sequence of instructions: "From wherever you are, sir, take the paved road leading out of town. When you get to the other end, take the country road leading out of that town, ..." Complete the sequence so that the tourist will end up in D, even though he may already be there or pass through it while following the instructions.

Circles: A Mathematical View, by Dan Pedoe, 1995.

This book was originally published by Pergamon Press in 1957 and later picked up by Dovers. In this third version, the author starts off with a new Chapter 0, introducing basic terminology and results about the circle which unfortunately have become increasing foreign to the current generation of students. The four chapters of the original work then follow, investigating in-depth the fundamental properties of this most pleasing curve, with explorations into inversive geometry, complex numbers and non-Euclidean geometry. A new appendix reprints an engaging story about the tragedies and triumphs of the short life of Karl Wilhelm Feuerbach. It was written by Laura Guggenbuhl in 1953 but is not generally known in the mathematics circle.

Problem 48

Using a compass without the straight-edge, construct the midpoint of a segment given only its two endpoints.

P. More Individual Titles

The first part of this section describes books related to mathematics competitions and Olympiads, and the second decribes books on popular mathematics.

Index to Mathematical Problems: 1980–1984, by Stan Rabinowitz, Math Pro Press, 1992.

This monumental volume contains all problems posed in a large number of mathematics journals with problem sections, as well as many mathematics competitions, in that five-year period. In this book only the statements are given, and they are classified systematically by subject. The book contains many useful indices with well-constructed cross-references. Among other things, they can be used to track down the solutions to the problems. The price for this large and valuable book is extremely reasonable.

Problem 49

Grandpa is 100 years old and his memory is fading. He remembers that last year — or was it the year before? — there was a big birthday party in his honor, each guest giving him a number of beads equal to his age. The total number of beads was a five-digit number x67y2, but to his chagrin he cannot recall what x and y stand for. How many guests were at the party?

An Olympiad Down Under,
by the late Peter O'Halloran, Australian Mathematics Trust, 1988.

This is the report on the 29th International Mathematical Olympiad, held in Australia in conjunction with their bicentennial celebrations in 1988. It is one of the crowning achievements of the author, and shows an enormous organizational effort. It contains the six contest problems, all other problems that were submitted, as well as solutions.

Problem 50

In a multiple choice test, there were 4 questions and 3 possible answers for each question. A group of students was tested and for any 3 of them, there was a question which the 3 students answered differently. What is the largest possible number of students tested?

Chinese Mathematical Olympiads, 1986-1993, by C. Li and Z. Zhang, Chiu Chang Mathematics Publishers, 1994.

This English translation was donated by the publishers to the Organizing Committee of the 1994 International Mathematical Olympiad in Hong Kong, as gifts to the leaders and deputy leaders of all participating teams. This book can be obtained from the publishers directly as well as from the Australian Mathematics Trust.

Problem 51

PQRS is a convex quadrilateral inside triangle ABC. Prove that the area of one of the triangles PQR, PQS, PRS and QRS is not less than ¼ of the area of triangle ABC.

Mathematical Challenges, edited by the Scottish Mathematical Council, Blackie and Sons, 1989.

Since 1975–76, **David Monk** has run a Mathematical Challenge for Scottish students. It is a problem solving contest conducted by correspondence. The problems are at varying levels of difficulty, but all are very attractive. This book contains the first twelve contests.

Problem 52

Prove that the sum of the squares of five consecutive integers is never a perfect square.

Mathematical Challenge!, by Leroy Mbili, Mathematics Digest, 1978.

This little booklet contains, as suggested by its subtitle, *100 problems for the Olympiad enthusiast*, with a healthy dose of geometry.

Problem 53

AB and EF are equal segments on a line parallel to CD. AC intersects BD at P, and CE intersects DF at Q. Prove that PQ is parallel to CD.

Cariboo College High School Mathematics Contest Problems, 1973–1992, edited by Jim Totten, University College of the Cariboo, 1992.

This book contains over 1000 problems for students in Grades 8 through 12. There is a companion Solution Manual by Leonard Janke and Jim Totten.

Problem 54

ABFE, BCHG and CADJ are squares constructed outside triangle ABC. The combined area of the first two squares is one half that of the hexagon DEFGHJ. Prove that the area of ABC is equal to that of BFG and find $\angle ABC$.

The Art of Problem Solving - Volume 1, by Sandor Lehoczky and Richard Rusczyk, Greater Testing Concepts, 1993.

The book was written when the authors were senior undergraduate and beginning graduate students. The style is that of an informal discussion, with plenty of opportunities for on-hand experience in problem solving. The twenty-nine chapters cover most topics in the standard high school curriculum, along with many others that are often neglected. There is a companion Solution Manual by the same authors.

Problem 55

Let l, m and n be positive integers. Prove that their product is equal to their greatest common divisor times the least common multiple of lm, mn and nl.

The Art of Problem Solving - Volume 2, by Sandor Lehoczky and Richard Rusczyk, Greater Testing Concepts, 1994.

This book continues where the previous one left off, with another twenty-six chapters covering more advanced topics. There is also a companion Solution Manual by the same authors.

Problem 56

Let x, y and z be real numbers with $xyz = 1$.
Evaluate $\frac{1}{1+x+xy}+\frac{1}{1+y+yz}+\frac{1}{1+z+zx}$.

Problem Solving Through Problems, by Loren Larson, Springer-Verlag, 1983.

The first two chapters cover general methods in problem solving, and the remaining chapters cover their application in arithmetic, algebra, infinite series, introductory analysis, inequalities and geometry.

Problem 57

Let S be a set and $\otimes$ be a binary operation on S such that $x \otimes x = x$ for all x in S and $(x \otimes y) \otimes z = (y \otimes z) \otimes x$ for all x, y and z in S. Prove that $x \otimes y = y \otimes x$ for all x and y in S.

The Green Book, by Kenneth Williams and Kenneth Hardy, Integer Press, 1985.

The subtitle of the book is *100 practice problems for undergraduate mathematics competitions*, but many problems are at the high school level. Some are original compositions while others are compiled from various sources. Hints and solutions are provided.

Problem 58

Prove that the equation $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24$ has no solution in integers x, y and z.

The Red Book, by Kenneth Williams and Kenneth Hardy, Integer Press, 1988.

This book has the same subtitle as the Green Book. There are no geometry problems in either.

Problem 59

Prove that there exist infinitely many positive integers which are not expressible in the form $n^2 + p$ where n is a positive integer and p is a prime.

Principles and Techniques in Combinatorics, by C. C. Chen and K. M. Koh, World Scientific, 1992.

Although intended principally as an undergraduate textbook on combinatorics, the book can also be used for a problem solving course in the subject. It has a wide selection of problems from various journals and contests.

Problem 60

Six scientists are working on a secret project. They wish to put the documents in a cabinet with many locks, all of which must be open before the document can be retrieved. Each scientist is given a number of keys, such that the cabinet can be opened if and only if at least three scientists are present. What is the smallest number of locks required, and what is the smallest number of keys each scientist must carry?

The Japanese Temple Problem Book, by Hidetosi Fukagawa and Dan Pedoe, Charles Babbage Research Center, 1989.

This book contains a selection of problems in Euclidean geometry displayed on *Sangaku*, which are mathematical tablets which were hung under the roofs of shrines or temples in Japan during the Edo period (1603—1867). Solutions to selected problems are given in the second part of the book, which also contains photographs of some Sangaku, plus other historical references.

Problem 61

Two circles touch each other externally and an external common tangent touches them at the points A and B respectively. Prove that AB^2 is equal to four times the product of the radii of the circles.

Polyominoes — Puzzles, Patterns, Problems and Packings, by Solomon W. Golomb, Princeton University Press, 1994.

This is the long awaited reprinting of the definitive treatise, originally published in 1965 by Charles Scribners Sons. It contains two new chapters, two new appendices and a vastly expanded bibliography section. Written by the founder of the subject, this book is a must for all who are interested in polyominoes, either for recreation or as a subject for serious research work.

Problem 62

It is well known that if two opposite corners were removed from a chessboard, the remaining part cannot be covered by 31 dominoes. This is because the two cells removed are of the same color. If two cells of opposite colors are removed, can the remaining part of the chessboard always be covered by 31 dominoes?

Dictionary of Curious and Interesting Numbers, by David Wells, 1986.

The first entry is -1 and i, and the remaining entries are real numbers in increasing order. The second entry 0 is followed immediately by Liouville's number $10^{-(1!)} + 10^{-(2!)} + 10^{-(3!)} + \cdots$. Each entry contains useful information about each number's mathematical properties as well as its place in the history of mathematics.

Problem 63

Let $\sigma(n)$ denote the sum of all positive divisors of the positive integer n. It is easy to show that neither 2 nor 5 is equal to $\sigma(n) - n$ for any positive integer n. Find the next integer with this property.

Dictionary of Curious and Interesting Geometry, by David Wells, 1991.

The entries are mostly geometric objects and theorems named after famous mathematicians, such as the Euler line and Menelaus' Theorem. Terms are clearly defined and relevant information is provided, with references where appropriate. The whole book is profusely illustrated with well-drawn line diagrams. It is a fertile ground for exploration in geometry.

Problem 64

Prove Viviani's Theorem, which states that in an equilateral triangle, the sum of the perpendiculars from any point to the sides is equal to the altitude of the triangle.

Book of Curious and Interesting Puzzles, by David Wells, 1992.

This is a compilation of 568 puzzles from various sources, including quite a few ancient civilizations from around the world. The second half of the book contains the answers and solutions.

Problem 65

Some rabbits and chicken have among them 35 heads and 94 feet. How many of each kind of animal are there?

Mathematics in Education, edited by Themistocles Rassias, University of La Verne Press, 1992.

This volume is published as part of the centennial celebration of the University of La Verne. It contains 10 articles, 8 research notes plus a problem section.

Problem 66

Prove that for any integer $n > 1$, there exists a permutation $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ such that for $1 \leq k < a_1 + a_2 + \cdots + a_k$ is divisible by a_{k+1}.

Hoffmann's Puzzles Old and New, edited by Edward Hornden, L. E. Hornden, 1993.

This is a modern edition of the original work published in 1893 by Frederick Warne and Company. It was the first book which featured mechanical puzzles prominently, and it attempted to catalog all that were known at the time. The new edition adds color pictures of many of the now rare puzzles.

Problem 67

Arrange nine counters so that there are ten rows each with three counters.

The Book of Ingenious and Diabolical Puzzles, by Jerry Slocum and Jack Botermans, Times Books (a division of Random House), 1994.

In this third book by the same two authors, the puzzles are grouped into chapters according to an earlier version of Jerry Slocum's Taxonomy reproduced later in this issue (see Appendix IV).

Problem 68

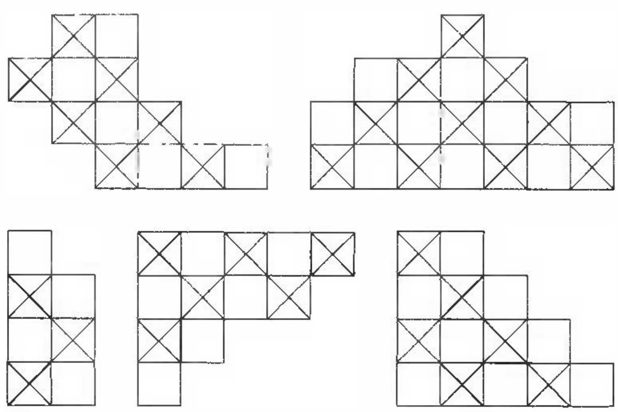
There are 10 pentacubes which consist of more than one layer, have no planes of symmetry and do not fit in a $2 \times 2 \times 2$ box. Use them to construct a $2 \times 5 \times 5$ box.

Compendium of Checkerboard Puzzles, by Jerry Slocum and Jacques Haubrich, Slocum Puzzle Foundation, 1993.

This is a comprehensive listing of dissection problems organized according to the number of pieces into which the checkerboard is divided. The number of distinct solutions to each problem is given, and if it is not too large, all solutions are featured.

Problem 69

Use the following five pieces to form a checkerboard.

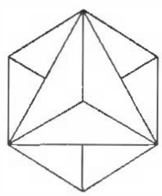


Unit Origami, by Tomoko Fusé, Japan Publications (Canadian distributors Fitzhenry & Whiteside), 1990.

This book introduces a new twist to the popular pastime of origami, or paper folding. The underlying idea is to build many copies of relatively simple building blocks, which are then put together like Lego pieces into fantastically complex designs. Full instructions are given, illustrated with line drawings, and there are also color photographs of many finished products. The book is also carried by Key Curriculum Press.

Problem 70

The diagram shows what appears to be three interlocking squares, defining the skeleton of a regular tetrahedron. Construct it using two pieces of origami paper of each of three colors, all six of the same size and folded in an identical manner. No cutting or gluing is allowed.



Exploring Math Through Puzzles, by Wei Zhang, Key Curriculum Press, 1996.

This book describes a number of mechanical puzzles and explains how they can be constructed and used in the classroom setting.

Problem 71

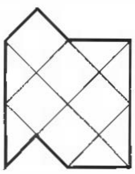
Find all hexominoes which can be folded into empty cubes.

Super-Games, by Ivan Moscovitch, Hutchinson, 1984.

This book consists mainly of dissection puzzles, and is full of superb color illustrations.

Problem 72

Dissect the octagon in the diagram into four pieces congruent to one another, in two different ways.



Q. More Addresses of Publishers

Birkhäuser,
675 Massachusetts Avenue,
Cambridge, MA, USA, 02139.

Charles Babbage Research Center,
P.O. Box 272,
St. Norbert Postal Station,
Winnipeg, MB, R3V 1L6.

Center for Excellence in Mathematics Education,
885 Red Mesa Drive,
Colorado Springs, CO, USA, 80933.

Chiu Chang Mathematics Publishers,
4/F, #3, Alley 15, Lane 147,
Section 3, Shin-Yi Road,
Taipei, Taiwan.

Fitzhenry & Whiteside,
195 Allstate Parkway,
Markham, ON, L3R 4T8.

Greater Testing Concepts,
P. O. Box A—D, Stanford,
CA, USA, 94309.

Hutchinson Publishing Group,
17—21 Conway Street,
London, W1P6JD,
United Kingdom.

Integer Press,
P.O. Box 6613,
Station J, Ottawa, ON, K2A 3Y7.

Key Curriculum Press,
P.O. Box 2304,
Berkeley, CA, USA, 94702.

L. E. Hornden,
Cane End House,
Cane End, Reading,
RG49HH, United Kingdom.

Math Pro Press,
P.O. Box 713,
Westford, MA, USA, 01886-0021.

Mathematics Digest,
Department of Mathematics,
University of Cape Town,
7700 Rondebosch, South Africa.

Penguin Books,
10 Alcorn Avenue,
Toronto, ON, M4V 3B2.

Slocum Puzzle Foundation,
257 South Palm Drive,
Beverly Hills, CA, USA, 90212.

Springer-Verlag,
175 Fifth Avenue,
New York, NY, USA, 10010.

University College of the Cariboo,
Box 3010,
Kamloops, BC, V2C 5N3.

University of La Verne Press,
1950 Third Street, La Verne,
CA, USA, 91750.

World Scientific,
Suite 1B, 1060 Main Street,
River Edge, NJ, USA, 07661.

Appendix IV: A Selection of Resource Material

A. The World of Puzzle Inventors and Collectors

Partly because of the review of mathematical games and puzzles in Appendix IV of **MfGS1**, the guest editor was invited to attend the eleventh International Puzzle Party in 1991. This is an annual event. In recent years, it is held cyclically in North America, Asia and Europe. The founder of the I.P.P. is **Jerry Slocum**, a retired aircraft executive who lives in Beverly Hills.

There are actually three parties in one. Typically, the Exchange Party takes place on a Saturday afternoon. This is open only to puzzle inventors who have pre-registered, and they come with enough copies of their latest brainchild for exchange with the others.

The Magic Party is a buffet dinner on Saturday evening. This is called the Magic Party because many puzzle people are also professional or amateur magicians. Even before the dinner, various members are showing off their new tricks, right at individual tables so that the audience has a close-up look. After the dinner some formal magic presentations are performed on stage by professional magicians.

The General Party commences on Sunday morning. Many companies which manufacture and distribute puzzles have display tables, as do individual inventors. The whole floor is a glorious mosaic of wood, acrylic, glass and other material in the most ingenious configurations. Those of us who do not have the privilege of attending the Exchange Party finally get to see what is brand new.

Apart from the main events, the I.P.P. provides a wonderful opportunity to renew acquaintances or meet those previously known only by reputation. There are also guided and unguided tours to local puzzle stores. The guest editor was thrilled to have received an invitation to visit the Slocum home.

It consists of two houses, only one of which is for normal domestic purpose. The main floor of the other houses Jerry's library of over 3000 puzzle books, and serves as his workshop. One of Jerry's specialities is the collection of ancient and old puzzles, which require exquisite care in their handling and preservation.

Most of Jerry's collection of over 20000 puzzles are in his puzzle museum upstairs. Many are prominently displayed on shelves or in glass cases. Others are systematically filed away in drawers and folios. No word can describe the place. One simply has to be there.

In 1993, the guest editor attended the thirteenth I.P.P. at Breukelen, the Netherlands, along with Daniel van Vliet and Matthew Wong, on our way home from the Problem-Solving Workshop in Russia. At the party, Jerry was inducted into the new Puzzle Hall of Fame, along with **Nob Yoshigahara** of Japan and **L. E. Hordern** of the United Kingdom.

Prior to the party, Jerry had established the **Slocum Puzzle Foundation** as a charitable and educational corporation. Its purpose is to educate the public on puzzles, their history, development and use in various cultures of the world. The Foundation will actively support the use of puzzles for education. It has published a *Directory of Puzzle Collectors and Puzzle Sellers*, containing useful information as well as addresses of collectors, retail puzzle shops and mail-order sources for puzzles.

Nob Yoshigahara, principal organizer of the twelfth and the fifteenth I.P.P., is another person whose puzzle collection numbers over 20000. He is the most prolific puzzle designer in the world, and runs a large puzzle company in Japan. Nob edits an informative and humorous newsletter *puzzletopia* which appears at irregular intervals, surprising and delighting friends when they receive it unexpectedly.

Mike and **Conni Green**, two of the organizers of the fourteenth I.P.P. in Seattle, run a specialty store called **Puzzletts** in that lovely city. They have recently introduced **The Great Puzzle Saga**, a quarterly mail-order catalog through which they present puzzles they have found from around the world. Many of them are in stock in their store. Puzzletts recently opened a page on the internet's world wide web. The address is http://puzzletts.com. Mike and Conni also offer to locate puzzles people have heard about but cannot find themselves. Their long-term plan includes no less than a puzzle theme park!

B. Jerry Slocum's Puzzle Taxonomy

Jerry has developed a systematic classification of puzzles. It is reproduced here with his kind permission.

Mechanical Puzzles

1. **Put-together Puzzles**
   - Two-dimensional Assembly Puzzles
   - Three-dimensional Assembly Puzzles
   - Miscellaneous Put-together Puzzles
   - Matchstick Puzzles
2. **Take-apart Puzzles**
   - Trick or Secret Opening Puzzles
   - Secret Compartment Puzzles
   - Trick Locks and Keys
   - Trick Matchboxes
   - Trick Knives
3. **Interlocking Solid Puzzles**
   - Figural
   - Geometric Objects
   - Three-dimensional Jigsaw Puzzles
   - Burr Puzzles
   - Keychain Puzzles
   - Miscellaneous Interlocking Solid Puzzles
4. **Disentanglement Puzzle**
   - Cast Iron and Sheet Metal Puzzles
   - Wire Puzzles
   - String Puzzles
   - Miscellaneous Disentanglement Puzzles
5. **Sequential Movement Puzzles**
   - Solitaire Puzzles
   - Counter Puzzles
   - Sliding Piece Puzzles
   - Rotating Cube Puzzles
   - Maze and Route Puzzles
   - Miscellaneous Sequential Movement Puzzles
   - Mazes and Labyrinths for People
6. **Dexterity Puzzles**
   - Throw and Catch
   - Rolling Ball Puzzles
   - Maze Dexterity Puzzles
7. **Puzzles Vessels**
8. **Vanish Puzzles**
9. **Folding Puzzles**
10. **Impossible Objects**
11. **Other Mechanical Puzzles**

Mathematical and Logic Puzzles

- Arithmetical Puzzles
- Magic Squares Puzzles
- Geometric Puzzles
- Logic Puzzles
- Other Mathematical Puzzles

Word Puzzles and Riddles

- Rebus Puzzles
- Anagram Puzzles
- Word Square Puzzles
- Acrostic Puzzles
- Charade Puzzles
- Crossword Puzzles
- Riddles
- Conundrums
- Other Word Puzzles

Visual Puzzles

- Hidden Image Puzzles
- Optical Illusion Puzzles

Computer Puzzles

- Text-only Puzzles
- Graphical Puzzles

C. More Resource Material

In **MfGS2**, less effort will be made to identify sources of puzzles. This is partly because a much wider selection is now available, and partly because there are reliable places where one can get them. Apart from the Slocum Puzzle Foundation and Puzzletts, **Bits and Pieces** also has a very good mail-order catalog.

There is one source, however, whose products are not available elsewhere. Quite a number of items from **Kadon Enterprises** were featured in **MfGS1**. More will be presented here, because the guest editor considers their products of the highest physical, aesthetic and intellectual quality. **Kate Jones**, the same Kathy Jones who contributed an article to **MfGS1**, will not compromise by cutting cost. Thus direct mail-order is the only option to keep prices affordable.

From time to time, Kate distributes products for other people. Often, they are individuals who have essentially single products, and must rely on a more secure distribution network. Kate is always willing to encourage their efforts. Kadon has acquired the puzzle set **Kaliko** reviewed in **MfGS1**.

Kate also handles products of exceptional value which are not readily available. A most exciting example is **Perplexing Poultry** by Pentaplex of the United Kingdom. This puzzle is a fanciful rendition of the famous non-periodic tilings of **Roger Penrose**. They come in black-and-white or in colors. Each set consists of many tiles of two shapes, a "fat" chicken and a "skinny" one. They interlock in unbelievable **Escher** style.

Under licensing from Pentaplex, Kadon manufactured two other sets in **The Penrose Universe**, called **Penrose "Kites and Darts"** and **Penrose Diamonds**. Then they developed their own **Collidescape** and **Puzzling Pentagon**, all based on the Penrose tilings. Two other related sets are **Rombix** and **Rombix Jr.**

Kadon's polyomino series also includes the **Octominoes**, 369 pieces plus 6 monominoes which tile a 51 x 58 rectangle. Actually, this item no longer appears in their catalog, but inquiries may be made regarding possible special order. For the first time, polyiamonds and polyhexes are offered, in **Lamond Ring** and **Hex Nut**. A smaller set called **Hex Nut Jr** is also available.

Stockdale Super Square, rviewed in **MfGS1**, is based on the McMahon squares. Two other variations have appeared in **Multi-Match I** and **II**. The corresponding triangular versions are called **Trifolia, Multi-Match III** and **IV**.

A triangular version of the famous game of Hex is now available from Kadon. It is called **The Game of Y**. In this two-player game, each tries to connect to all three sides of a triangular board. There is a companion book called **Mudcrack Y & Poly Y**.

There is one more item from Kadon which must be mentioned. It is a classroom set called **Combinatorix**. It consists of many colorful wooden tiles in the shapes of squares, equilateral triangles, hexagons and isosceles right triangles. The three manuals contain numerous exploratory activities using these tiles. It is ideal for group investigations.

Many of Kadon's puzzle sets fall into the *Put-together* category, mostly in two-dimensions. A very imaginative three-dimension puzzle is Nob Yoshigahara's **Pineapple Delight**. It is really the pentominoes but in cylindrical form, to be assembled inside a glass. The pieces look good enough to eat.

The guest editor got another outstanding puzzle from Nob Yoshigahara. It consists of seven rectangular blocks in a wooden box. After they are dumped out, it is quite easy to put six of them back in. There is apparently no room at all for the seventh. It is a very bewildering paradox.

Another puzzle along this line is **Dragon's Egg**. It consists of a box containing four wooden pieces, each with a few hemispherical cavities on its faces. When they are dumped out, three marbles show up, but although there are enough cavities for all, one marble seems to persistently stick out of the box.

In the *Take-apart* category, two recent puzzles are **Escape from Alcatraz** and **YOT**, both available from the Slocum Puzzle Foundation. In the former, a metal ball is literally behind bars, and the object is to spring it from captivity. In the latter, there is a silver dollar partly visible inside a circular container which has a handle. The object is to get the money.

The *Interlocking* category consists primarily of puzzles called burrs, the smallest of which has six pieces. Two expert craftsmen are **Stewart Coffin** and **Bill Cutler**. The latter won a design contest with his **Blockhead**, now available commercially under the mysterious name of **Sneaky Squares**! It consists of four plastic blocks in a box. They can be dumped out quite readily, but it takes some doing to get all four back in. The underlying principle here is different from two apparently similar puzzles mentioned earlier.

Puzzletts offers an outstanding selection of *Disentanglement* puzzles. **Oskar's Disks**, available from Kadon, consists of two wooden circular mazes intertwined with each other. From the same inventor, **Oskar van Deventer** of the Netherlands comes the more widely available **Oskar's Cube**. It is an ingenious three-dimensional maze which projects into three planar mazes, the "mouse" being the point of intersection of a triple-cross which sticks out of the hollow box.

The *Sequential Moves* category is dominated by the Rubik-type puzzles, and **Cubes International** has added to its offerings which were reviewed in **MfGS1**. The most exciting new family is the puzzle balls of **Uwe Mèffert**. Two of them feature Disney characters while a third features Sonic and Tails of Sega video-game fame. **Christoph Bandelow**, who *is* Cubes International, has written a solution manual for this puzzle, including the mathematics behind it. The booklet has so far been published in seven languages. Other companies which still make Rubik-type puzzles are **I-Development Institute** of Hong Kong and **International Puzzles & Games** of Taiwan.

Called the best *Dexterity* puzzle in decades, the **Elverson Bottle** is sealed and contains a wooden ball. The latter must be manipulated into the neck of the bottle around a wooden stick which has indentations and protrusions. It is available from the Slocum Puzzle Foundation.

The **Impossible Bottle**, also available from the Slocum Puzzle Foundation, has a wooden arrow passing through two tiny holes on the side of a Coke bottle, much smaller than the head and tail of the arrow. Despite careful examination, the guest editor has still not discovered how it is made.

To conclude this review, mention must be made of the fantastic computer puzzles from **Soleau Software**. Winner of the 1995 Shareware Industry Awards for Best Entertainment Software, **William Soleau** designs non-violent strategy games and challenging puzzles at varying levels of difficulty. The prices for upgrading from the shareware versions to the full versions are very reasonable.

D. Addresses:

Stewart Coffin,
79 Old Sudbury Road,
Lincoln, MA 01773.

Bill Cutler Puzzles,
405 Balsam Lane,
Palatine, IL 60067.

Bits and Pieces,
1 Puzzle Place, B8016, Stevens Point,
WI 54481-7199.

Cubes International,
An der Wabeck 37, D-58456,
Witten, Germany.

I-Development Institute, B.V.I.,
P.O. Box 24455, Aberdeen,
Hong Kong.

International Puzzles and Games,
3/F, #192, Sec. 2, Chung Ching Road,
Taipei, Taiwan.

Kadon Enterprises,
1227 Lorene Drive, #16,
Pasadena, MD 21122.

Puzzletts,
24843 144th Pl. S.E.,
Kent, WA 98042.

Slocum Puzzle Foundation, 2
57 South Palm Drive,
Beverly Hills, CA 90212.

Soleau Software,
163 Amsterdam Avenue, #213,
New York, NY 10023.

Nob Yoshigahara,
4-10-1-408 Iidabashi,
Tokyo 102, Japan.

Appendix V: Answers and Solutions

A. Supplementary Problems in Appendix I

An Imaginary Postal Service:

The only perfect design with the five-stamp sets is the one shown in the diagram, apart from the switching of the one and two cent stamps.

1	3
8	2
7	

Dissecting Rectangular Strips into Dominoes:

A recurrence relation is $f_n = f_{n-2}$ with initial conditions $f_0 = 1, f_1 = 4, f_2 = 2$ and $f_3 = 3$.

A Space Interlude:

1. We already know that if each Space Cannon is divided into 3 parts, we need nine Space Pods. With only eight available, we divide the Space Cannons into A, B, C and D. The Space Pods carry (A,B), (A,B), (A,C), (B,C), (C), (D), (D) and (D) respectively. To get all four parts, the Space Octopus must grab one of the last three, but will not get all of A, B and C by grabbing only one other.
2. With seven Space Pods, we divide the Space Cannons into A, B, C, D and E. A working scheme has the Space Pods carrying (A,B,C), (A,B,D), (A,C,D), (B,C,D), (E), (E) and (E) respectively. Suppose we divide the Space Cannons into A, B, C and D only. Clearly, none of them should be carrying 3 parts or more. Let one of them carry A and B. Then none of them can carry C and D. Thus three other Space Pods carry C and the remaining three carry D. We still have 2 A parts and 2 B parts. We may assume that one of the Space Pods carrying C also carries A. Then none of those carrying D can carry B, so that the 2 B parts go to the other two Space Pods carrying C. Now the remaining A part must go to one of the Space Pods carrying D, but this allows the Space Octopus to get all parts by grabbing just two Space Pods.
3. With six Space Pods, we divide the Space Cannons into A, B, C, D, E and F. A working scheme has the Space Pods carrying (A,B,C), (A,B,D), (C,D,E), (C,D,F), (E,F,A) and (E,F,B) respectively. Suppose we divide the Space Cannons into A, B, C, D and E only. Then there are 15 parts in all. Hence some Space Pod must carry at least 3 parts. Certainly, it should not carry 4 or more. Hence we may assume that it carries A, B and C. We have 6 D and E parts to be carried by the other five Space Pods, so that one of them must carry both of them. Hence it is possible for the Space Octopus to grab just two Space Pods and get all 5 parts.

How to Flip without Flipping:

There are three such triangles which are not similar to one another. The measures of their angles are (20°, 40°, 120°), $(22\frac{1}{2}°, 45°, 112\frac{1}{2}°)$ and $(25\frac{5}{7}°, 51\frac{3}{7}°, 102\frac{6}{7}°)$.

A Tale of Two Cities:

Obviously, such a negative circle must be in the interior of the inner chain. It has a neighbor of each kind if and only if it has an integration point.

B. Contest Questions in Appendix II B

These are worked out by **Daniel Robbins**, a Grade 12 student at École Secondaire Beaumont.

1967	ecede	abeda	ccdcd	baebb	adbec
1968	dbcca	dedec	ebadb	beaca	aedeb
1969	ceaea	cedeb	cdabc	cbdce	dbced
1970	cbadb	deebc	acadd	bcbba	edacd
1971	baadd	cdcbb	debce	abecd	
1972	ddaca	ecabb	dbcde	cebae	

1973	cbaca	cdbcd	eacbc	cdbcb
1974	cccbd	bedda	becbe	bcdba
1975	dadcc	cbdea	dabec	bddab
1976	cedba	beaec	bcedc	dddaa
1977	ddbbb	accbe	daeda	ecbca
1978	ccbad	babdb	beead	deece
1979	bdcbe	edaec	acdbe	dbaed
1980	bbddb	ccaab	aedab	caecd
1981	aecbc	ccbaa	eadae	ebbeb
1982	bccac	bbaec	caad	abdac
1983	abece	bcccd	cbabc	eccba

C. Sample Contest Problems in Appendix II C

1. More generally, suppose there are 2^n friends. After n rounds, the most anyone can learn are 2^n pieces of gossip. Hence n rounds are necessary. We now prove by induction on n that n rounds are also sufficient. For $n = 1$, the result is trivial. Suppose the result holds up to $n - 1$ for some $n > 2$. Consider the next case with 2^n friends. Have them call each other in pairs in the first round. After this, divide them into two groups, each containing one member from each pair who had exchanged gossip. Each group has 2^{n-1} friends who know all the gossip among them. By the induction hypothesis, $n - 1$ rounds are sufficient for everyone within each group to learn everything. This completes the induction argument. In particular, with 64 friends, 6 rounds are both necessary and sufficient.

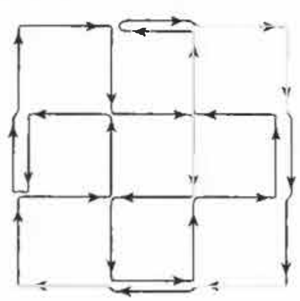
2. The "sheep" player wins. Place one of the 50 sheep on each of the lines $y = 3m$, $1 < m < 50$, so that initially, no sheep is within 1 metre of the wolf. The sheep will stay on their respective lines, which are 3 metres apart. Since the wolf's maximum speed is 1 metre per move, it can threaten at most 1 sheep at a time. In a one-to-one scenario, the wolf cannot run down the sheep even if the sheep is confined to move along a fixed line.

3. Construct a graph G in which each city is represented by a vertex and each direct air-route by an edge. Let G' be the graph obtained from G by removing M, the vertex representing the capital, and all edges incident with it. By hypothesis, G is a connected graph, but G' may consist of a number of components. However, each component must contain at least one vertex connected to M in G. In G', such vertices have degree 9 while all others have degree 10. However, each component must have an even number of vertices with odd degree. Hence at least two vertices in each component are connected to M in G. Since M has degree 100, the number of components in G' is at most 50. Hence we can reconnect G' by restoring M and one edge connecting it to each component of G'. This is just a different way of saying that we can remove from G at least 50 edges incident with M without disconnecting it.

4. The answer is no. Divide the 3.5 hours into 7 periods each of 0.5 hours. The pedestrian walks at 6 kilometres per hour in periods 1, 3, 5 and 7 and at 4 kilometres per hour in periods 2, 4 and 6. For any 1 hour interval, the pedestrian walked at each speed for exactly 0.5 hours. Hence the distance covered is exactly 5 kilometres. However, the total distance covered is 18 kilometres, yielding an average speed of more than 5 kilometres per hour.

5. Let $b_1, b_2 \ldots b_{15}$ be the heights of the boys and $g_1, g_2 \ldots g_{15}$ be the heights of the girls. Suppose for some k, $1 < k < 15$, $|b_k - g_k| > 10$. Without loss of generality, we may assume $g_k - b_k > 10$. Then $g_i - b_j > 10$ for all i and j where $1 < i < k$ and $k < j < 15$. Consider the boys of height b_j, $k > j > 15$ and the girls of height g_i, $1 < i < k$. By the Pigeonhole Principle, some two of these 16 must form a couple in the original lineup. However, $g_i - bj > 10$ contradicts the hypothesis.

6. The diagram shows a closed tour of length 28 with fourfold symmetry. We claim that it has minimum length. Each of the four corners is incident with two roads and requires at least one visit. Each of the remaining twelve intersections is incident with three or four roads and requires at least two visits. Hence the minimum is at least $4 + 12 \times 2 = 28$.



7. Let the musicians be A, B, C, D, E and F. Suppose there are only three concerts. Since each of the six must perform at least once, at least one concert must feature two or more musicians. Say both A and B perform in the first concert. They must still perform for each other. Say A performs in the second concert for B and B in the third for A. Now C, D, E and F must all perform in the second concert, since it is the only time B is in the audience. Similarly they must all perform in the third. The first concert alone is not enough to allow C, D, E and F to perform for one another. Hence we need at least four concerts. This is sufficient, as we may have A, B and C in the first, A, D and E in the second, B, D and F in the third and C, E and F in the fourth.

8. Note that the total number of chameleons is divisible by 3. When divided by 3, the initial numbers of the three kinds of chameleons leave remainders of 0, 1 and 2. Of course, the sum of these three remainders will always be divisible by 3, so that their collective values must be one of (0, 0, 0), (1, 1, 1), (2, 2, 2) and (0, 1, 2). In a multi-color meeting, all three remainders change value. Hence one of the remainders is not 0, one of them is not 1 and one is not 2. It follows that they must always be 0, 1 and 2 in some order, meaning that there are chameleons of at least two different colors at any time.

9. More generally, we show that $3n - 2$ weighings are sufficient for 2n coins. We first divide the coins into n pairs, and use n weighings to sort them out into a "heavy" pile and a "light" pile. The heaviest coin is among the n coins in the "heavy" pile. Since each weighing eliminates 1 coin, $n - 1$ weighings are necessary and sufficient for finding it. Similarly, $n - 1$ weighings will locate the lightest coin in the "light" pile. Thus the task can be accomplished in $3n - 2$ weighings. For $2n = 68$, $3n - 2 = 100$.

10. There are six permutations of the grasshoppers: 123 (132) 312 (321) 231 (213). They are arranged so that each alternate one is in brackets. A jump changes a permutation into either one of its neighbors, where the first and last permutations are also considered as neighbors of each other. Suppose that the initial permutation is not one in brackets. Note that 1985 is an odd number. Then after an odd number of jumps, the resulting permutation must be in brackets. Hence after 1985 jumps, the grasshoppers cannot regain even their initial relative positions.

11. Of the 8 teams, there must be a champion who has won the most games. We denote that team by A. This team must have won at least 4 games. We denote such teams which A has defeated by B, C, D and E. Among these teams, there must also be a champion that has won at least 2 games, for example team B, which beats teams C and D. By symmetry, we may assume that C beats D. This yields the desired ordering.

12. Construct a graph with 20 vertices representing the 20 teams. Two vertices are joined by a red edge if the two teams they represent play each other on the first day, and by a blue edge if they play on the second day. Since each vertex is incident with one red edge and one blue edge, each component of the graph is an even cycle. By taking every other vertex in each cycle, we have 10 independent vertices, representing 10 teams, no two of which have yet played each other.

13. The key observation is that from a non-square formation, one can always leave a square formation, but from a square formation, one must leave a non-square formation. Since the starting formation is non-square and the winning move consists of leaving a 1 by 1 square formation, the first player has a sure win by leaving a square formation on every move.

14. Each use of the machine increases the total number of coins by 4, an even number. Since Peter starts with 1 coin, he will always have an odd number of coins. Thus it is impossible for him to have an equal number of nickels and pennies.

15. (a) Note that 5 of the pawns start on black squares while the other 4 start on white squares. All permitted moves preserve square color. It is thus impossible for the pawns to be moved to the upper left hand corner as this contains 5 white squares and 4 black squares.

    (b) If we number the columns starting from the left, 6 pawns commence on odd-numbered columns while the other 3 commence on even-numbered columns. All permitted moves preserve the parity of the column number. It is impossible for the pawns to be moved to the upper right hand corner as this contains 6 squares in even-numbered columns and 3 squares in odd-numbered columns.

16. The first player wins. The sequence 1987, 993, 496, 248, 124, 62, 31, 15, 7 and 3 is obtained by dividing each term by 2, ignoring any remainder. We claim that these are winning positions. By the rules, 1987 is a winning position. Suppose $2k$ or $2k + 1$ is a winning position. We claim that so is k. On the opponent's next move, the largest number that can be chosen is $2k - 1$, which falls short of $2k$. The smallest number that can be chosen is $k+1$, after which $2k+1$ can be reached. This justifies our claim. Since the initial number is 2, the first player can win by choosing 3 and the next winning position thereafter.

17. Assume that the task is impossible. Then the total number of baskets is not more than 99, as otherwise we could leave 1 apple in each of 100 baskets and remove the rest. Now the largest basket has at most 99 apples. The second largest has at most 49, as otherwise we leave 2 baskets each with 50 apples. Similarly, the next largest four have at most 33, 24, 19 and 16 respectively. Even if each of the remaining 93 has 16 apples each, the total is at most $99 + 49 + 33 + 24 + 19 + 16 \times 94 = 1828$, which is a contradiction.

18. Let a, b, c and d denote the respective numbers of people with neither blue eyes nor fair hair, those with blue eyes but not fair hair, those with fair hair but not blue eyes, and those with blue eyes and fair hair. Then the proportion of people with fair hair among people with blue eyes is $\frac{c}{b+c}$ and the proportion of people with fair hair among all people is $\frac{c+d}{a+b+c+d}$. We are given that $\frac{c}{b+c} \times \frac{c+d}{a+b+c+d}$, so that $\frac{c}{c+d} \times \frac{c+d}{a+b+c+d}$. It follows that the proportion of people with blue eyes among people with fair hair is more than the proportion of people with blue eyes among all people.

19. The key to this problem is to consider each square of the chess board to be one of a set of four, symmetric about a horizontal and a vertical line through the centre of the board. For example, the four central squares form such a set, as do the four corner squares. The first player wins regardless of the starting position, by simply placing the pawn in the opposite square of the current "set of four". The second player is always forced to move to an outer "set of four" and the first player responds by placing the pawn in the opposite square of that set. The second player 2 must ultimately place the pawn in a corner square whereupon the first player wins by placing the pawn in the opposite corner.

20. (a) Given any distribution of the 300 stars, consider a set S of 100 rows containing the highest number of them. We claim that this number is at least 200. Otherwise, there is a row in S containing at most 1 star, and a row not S containing at least 2 stars. This contradicts the maximality of S. It follows that if we remove the 100 rows in S, at most 100 stars are left. We can make them disappear by removing at most 100 columns.

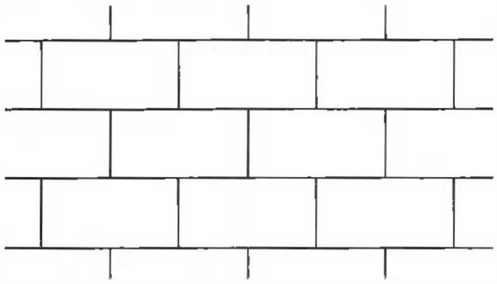
    (b) Place 200 stars on the main diagonal, 100 stars on the diagonal immediately above in rows 1 to 100, and the last star on the first square of the 101st row. Denote by S_1 the 99 rows each with 1 star, and by S_2 the 101 rows each with 2 stars. Suppose we remove k-1 rows from S_1 and 101-k rows from S_2. The 100-k stars in remaining rows in S_1 occupy 100-k columns with no other stars. The $2k$ stars in the remaining rows in S_2 occupy at least k+1 columns collectively. Thus we cannot get rid of all the stars remaining by removing any 100 columns.

D. Sample Problems in Appendix III

1. Consider a particular person. Suppose she has more acquaintances than strangers among the other five. Then she must be acquainted with at least three of them. If no two of these three are acquainted, then they form a triangle of mutual strangers. If some two of them are acquainted, then they form with the first person a triangle of mutual acquaintances. The case where the first person has fewer acquaintances than strangers among the other five can be handled in the same way.

2. The second player can force a draw with the following strategy. Divide the infinite board into dominoes as shown in the diagram. Whenever

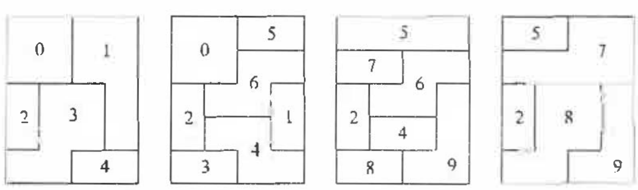
the first player takes a cell, the second player takes the other cell of the same domino. Since every compact 2×2 configuration must contain a domino, the first player cannot win.



3. If every guest moves to the next room, then the first room is available for the new guest. Since the hotel is infinite, there is no other "end" from which a guest would have been expelled.

4. Your opponent will always challenge a \$1 bet, because there is no reason why you should place such a bet with a picture card. Hence you expect to lose 2 dollars in 9 of 13 games. Suppose your opponent concedes a \$5 bet with probability p. With a picture card, you win $5p + 10(1 - p) = 10 - 5p$ dollars, in 3 of 13 games. With an Ace, you win $5p - 10(1 - p) = -10+15p$ dollars, in 1 of 13 games. Since $9(-2) + 3(10 - 5p) + (-10 + 15p) = 2$, you expect to win on the average 2 dollars every 13 games.

5. There are ten such pieces, and together they form a $4 \times 4 \times 5$ block as shown in the diagram, drawn in four layers. Parts with identical labels belong to the same piece.



6. Since $OQ > PR$, there exists a point E between O and Q such that $OE = PR$. Then $OP = QR - (OQ - OE) = QR - QE < RE$ by the Triangle Inequality. Consider the triangles EOR and PRO. We have $OE = PR$, $OR = RO$ and $REOP$. Hence $\angle EOR > \angle PRO$ by the Side-Angle-Side Inequality. It follows from the Angle-Side Inequality that $CR > CO$.

7. If (a) is true, then so is (e) which claims that (a) is not. Hence (a) is false. Since (c) claims that (a) is true, (c) is also false. If (d) is true, then one of (a), (b) and (c) is true. Since (a) and (c) are false, (b) must be true, but it claims that (d) is not. Hence (d) is also false, and so is (b). If (f) is true, then all of (a), (b), (c), (d) and (e) are false, but that would make (e) true. Hence (f) is false. Hence (e) is the only one that can be true, and this leads to no contradictions.

8. Clearly, four points are not enough. Suppose we have five. Consider the smallest convex polygon containing them. If it is a pentagon or a quadrilateral, we have a convex quadrilateral as desired. Suppose it is a triangle ABC, with D and E inside. The line DE must intersect two sides of this triangle, say AB and AC, with D closer to AC. Then $BCDE$ is a convex quadrilateral.

9. The amount of solid substance remains unchanged. Since its percentage has doubled, the total weight must have halved to 250 kilograms.

10. The geometric series $1 + x + \cdots + x^n$ sums to $\frac{1}{1-x}$ if $x < 1$. Hence
$M = \left(1+\frac{1}{2}+\cdots\right)\left(1+\frac{1}{3}+\cdots\right)\left(1+\frac{1}{5}+\cdots\right)\cdots\left(1+\frac{1}{p}+\cdots\right)$.
When we multiply out the right side, we will get the reciprocals of all positive integers. For instance, $\frac{1}{60}$ will come from $\left(\frac{1}{2}\right)^2\frac{1}{3}\frac{1}{5}1 1\cdots$. Hence
$M = 1+\frac{1}{2}+\frac{1}{3}+\cdots$. Now $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{2}$, and so on. If we go far enough, the right side will exceed M. This contradiction shows that there must be infinitely many prime numbers.

11. Let AB be the diameter dividing the domains of the first two ice fishermen. By symmetry, these remain equal in area as long as the newcomer sets up his ice house on AB. If it is set up at A, clearly he will be at a disadvantage. If it is at the centre O, he will have gained the upper hand. Hence there is a point somewhere between A and O at which the third ice fisherman can set up his ice house so that all three domains have equal area.

12. (a) This is a village in Wales. The name means *Church of St. Mary in a hollow of white hazel, near to a rapid whirlpool, to St. Tysilio Church and to a red cave.*

(b) This is a lake in Massachusetts, USA.
The name means *I fish on my side and you fish on your side and nobody fishes in the middle*.

(c) This is a place in New Zealand.
The name means *The brow of the hill where Tamatea who sailed all round the land played his nose flute to his lady love*.

13. The answer is "no". For instance, $(1 \oplus 1) \oplus 2 = 4 \oplus 2 = 12$ while $1 \oplus (1 \oplus 2) = 1 \oplus 6 = 14$.

14. Consider first the special case where $a_3 = 85$ for $1 \le i \le 19$. Then the desired sum is clearly $19 \cdot 85 = 1700$. If not, let j be the largest index such that $a_j < 85$. Suppose $a_j = k$. If we increase a_j by 1, then all the b's remain unchanged except for b_{k+1}, which decreases by 1. This balances out the gain by a_j, so that the sum remains unchanged. By repeating this increment process, we will eventually arrive at the special case considered earlier. Hence the desired sum is always equal to 1700.

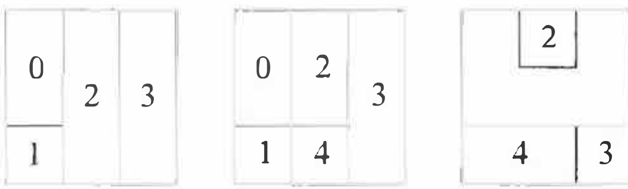
15. We claim that the probability is $\frac{2}{15}$. We prove this by induction on the number n of boxes. For $n = 2$, the result is trivial. Suppose the claim holds for some $n \ge 2$. Consider the next case with $n + 1$ boxes. Suppose the $(n + 1)$st key is in the $(n + 1)$st box. This happens $\frac{1}{n+1}$ of the time, and the probability of success now is 0. Suppose the kth key is there instead for some k, $1 \le k \le n$. This happens $\frac{n}{n+1}$ of the time. If we get to the $(n + 1)$st key, we can open the $(n + 1)$st box and retrieve the kth key. Hence we may pretend that the $(n + 1)$st key and box were not there, and that the kth key is where the $(n + 1)$st key was. By the induction hypothesis, the probability of success now is $\frac{2}{n}$. It follows that the overall probability is $\frac{1}{n+1}(0)+\frac{n}{n+1}\left(\frac{2}{n}\right)=\frac{2}{n+1}$. This completes the inductive argument.

16. If a prisoner thinks only of himself, he would reason as follows: suppose the other guy confesses. Then either I get a light sentence by also confessing, or get a heavy sentence otherwise. If he does not confess, I can get an award by ratting on him. No matter what he does, I am better off confessing. The irony is that both would end up getting a light sentence, whereas they could have gone free if neither confesses.

17. Five campers will be sufficient. The counsellor sends all of them down path A while she explores path B. If the campers return a 5:0 or 4:1 decision, it can be believed and the counsellor will know the true situation about paths A and B. She will have time to explore path C if necessary. Suppose the campers return a 3:2 decision and the campsite is not down path B. If two of them say "No", the counsellor sends one of them down path C while she checks out path A. If she does not find the campsite herself, the report from path C will be reliable. Suppose after the first exploration, three campers say "No". The counsellor sends all of them down path C while she checks out path A. If she does not find the campsite herself, both of those who say "Yes" are liars, and the majority decision from path C will be correct. Four campers are not sufficient because the one who always tells the truth cannot outnumber the liars unless he is alone. This means that the counsellor must be able to identify him after the first exploration. However, one of the liars can confuse the issue by telling the truth up to that point.

18. First, ask the Amazing Sand Counter to mentally note down the number of grains of sand in the bucket. Second, ask him to turn around while you remove a number of grains. Third, ask him to glance at the bucket again. If he has the power claimed, he will be able to tell you how many grains are missing.

19. The solution is shown in the diagram, where the cube is drawn in three layers. Parts with identical labels belong to the same piece.



20. With seven varieties, there are $\frac{7 \cdot 6}{2} = 21$ pairs of them. Each plot accounts for 3 pairs, so that there must be 7 plots. Labelling the varieties 1 to 7, the plots may contain (1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 4, 7) and (3, 5, 6), rspectively. It is routine to verify that all conditions are met.

21. First, take both a and b to be the irrational number $\sqrt{2}$. If a^b is rational, we have what we want. If $\sqrt{2}^{\sqrt{2}}$ is irrational, take it to be a and keep b unchanged. Then $a^b = 2$ is rational.

22. Let the segment be AB. Take a point C not on the two parallel lines. Join C to A and B, cutting the other line at E and D respectively. Join BE and AD, intersecting at P. The point F of intersection of the lines CP and AB is the midpoint of AB. To see this, let Q be the point of intersection of the lines CP and DE. Then triangles PBF and PEQ are similar, as are PAF and ADQ. Hence $\frac{FB}{QE}=\frac{FP}{QP}=\frac{FA}{QD}$. Now the triangles CBF and CDQ are also similar, as are CAF and CEQ. Hence $\frac{FB}{QE}=\frac{FC}{QC}=\frac{FA}{QE}$. Multiplication yields $FB^2 = FA^2$ or $FB = FA$.

23. The total number of possible outcomes of tossing $2n$ coins is 2^{2n}. The number of those with exactly n heads is equal to the binomial coefficient $\frac{(2n)!}{n!n!}$. Hence the probability is $p_n = \frac{(2n)!}{2^{2n}n!n!}$. Now

$$\frac{p_{n+1}}{p_n} = \frac{(2n+2)!}{2^{2n+2}(n+1)!(n+1)!}\cdot\frac{2^{2n}n!n!}{(2n)!} = \frac{2n+1}{2n+2} < 1.$$

Hence p_n decreases as n increases. It follows that the maximum occurs when $n = 1$.

24. We show first that it suffices to consider the case in which all students of each school are of the same sex. Indeed, if some school has both a boy and a girl, then the number of singles matches played by this boy is the same as the number of mixed single matches played by the girl, and vice versa. It follows that sending both the boy and the girl home alters neither of the conditions of the problem. Now, suppose that, as above, each school has either all girls or all boys, and that k schools have an odd number of students. Suppose there are, in all, B boys, G girls, S singles and M mixed singles, with $|B - G|$ and $|S - M|$. Then $M = BG$ and

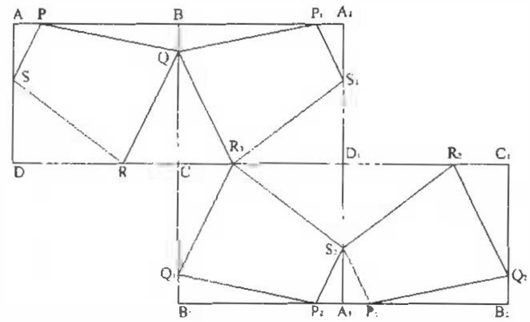
$$-1 \le S - M \le \frac{1}{2}(B(B-1)+G(G-1)) - BG = \frac{1}{2}$$

$$((B-G)^2-(B+G)) \le \frac{1}{2}(1-(B+G)).$$

It follows that $-3 \le -(B + G)$ or $B + G \le 3$. Since each of the k schools has at least one student, $k \le B + G \le 3$, and there are at most 3 schools with an odd number of students. The upper bound $k = 3$ is attained if two of them have 1 girl and 0 boys each, and a third school has 0 girls and 1 boy.

25. There are nine blocks of four adjacent numbers. Each number appears in four of these blocks. Now four times the sum of the numbers is 360. Hence the average sum of each block is 40, and at least one block is no less than 40.

26. Reflect the rectangle $ABCD$ and its inscribed quadrilateral $PQRS$ three times as shown in the diagram. By the Triangle Inequality, $PQ + QR + RS + SP = PQ + QR_1 + R_1S_2 + S_2P_3 \ge PP_3 = AA_2$, and the desired result follows immediately.



27. If we color the spaces black and white in checkerboard fashion, with A black, then there are 11 black spaces and 9 white ones. Each of the first, second, fourth and fifth pieces covers 2 black and 2 white spaces, leaving 3 black and 2 white spaces. Since the third piece must cover 3 squares of one color and 1 of the other, it covers 3 black and 1 white spaces. It follows that the unit square is on a white space, which can only be B or D. Since it obviously cannot be in B, the answer is D, and it is not hard to see how the other five pieces can fit in around it.

28. Let g_n be the number of ways of paving a straight path, 1 metre wide and n metres long. Clearly, $g_1 = 1$ and $g_2 = 2$. For $n \ge 3$, we can either start with a square paving stone or a rectangular one. In the first case, we are left with an unpaved path of length n-1, and there are g_{n-1} ways of completing the paving. In the second case, we are left with an unpaved path of length n - 2, and there are g_{n-2} ways of completing the paving. Hence $g_n = g_{n-1} + g_{n-2}$ for $n \ge 3$. It follows that $g_3 = 3$, $g_4 = 5$, $g_5 = 8$, $g_6 = 13$, $g_7 = 21$, $g_8 = 34$, $g_9 = 55$, $g_{10} = 89$, $g_{11} = 144$, $g_{12} = 233$, $g_{13} = 377$, $g_{14} = 610$, $g_{15} = 987$ and $g_{16} = 1597$.

29. Assign values to the squares as shown in the diagram. Initially, the single counter occupies a square of value 1. Note that the value of each square is equal to the sum of the values of the squares to the north and east of it. It follows that in any move, the total value of the squares occupied by counters is unchanged, and must be equal to 1 all the time. The total value of all the squares, counting one row at a time, is equal to $2+1+\frac{1}{2}+\frac{1}{4}+\cdots=4$. The total value of the six home squares is $2\frac{3}{4}$. The total value of the squares on the first row, not counting the home squares, is $\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{4}$. By symmetry, this is also the total value of the squares on the first column, not counting the home squares. It follows that the total value of the remaining squares is $\frac{3}{4}$. Suppose after a finite number of moves, the home squares are cleared of all counters. Now there must be exactly one counter on the first row and exactly one on the first column, each occupying a square of value at most $\frac{1}{8}$. It follows that the total value of the occupied squares is strictly less than $\frac{1}{8}+\frac{1}{8}+\frac{3}{4}=1$, and we have a contradiction.

$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

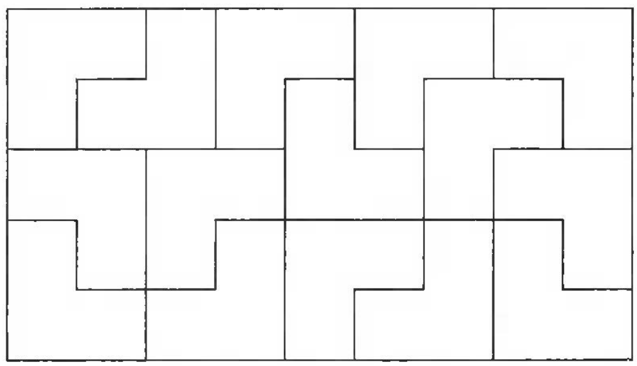
30. We start with 1, 2 and 3, each of which divides their sum 6. If we throw in 6, and each of these four numbers will divide their sum 12. This eventually leads to 1, 2, 3, 6, 12, 24, 48, 96, 192 and 384.

31. First, if we take 997 evenly spaced points on a line, then there are exactly 1991 red points, consisting of the 996 midpoints between consecutive points as well the 995 points other than the 2 at the ends. In general, let M and N be 2 points at the greatest distance apart. Let their midpoint be Q and let the perpendicular bisector of MN be l. Let the other points be P_i, $1 \leq i \leq 1995$, and let X_i and Y_i be the midpoints of MP_i and NP_i respectively. Note that the X_i are distinct, as are the Y_i. Moreover, none coincides with Q. Now $MX_i \leq MQ$ since $MP_i \leq MN$. It follows that each X_i is on the same side of l as M. Similarly, each Y_i is on the same side of l as N. Hence no X_i can coincide with any Y_i. Thus we have at least 1991 red points, consisting of the X_i, the Y_i and Q.

32. First note that $\sqrt{a}-\sqrt{b}=\frac{a-b}{\sqrt{a}+\sqrt{b}}$ is also rational. Hence $\sqrt{a}=\frac{\sqrt{a}+\sqrt{b}}{2}+\frac{\sqrt{a}-\sqrt{b}}{2}$ is rational, as is $\sqrt{b}=\frac{\sqrt{a}+\sqrt{b}}{2}-\frac{\sqrt{a}-\sqrt{b}}{2}$.

33. Divide the triangle into four congruent ones by drawing the three segments joining midpoints of the sides. With 9 points, at least 3 will be in the same small triangle. Hence they determine a triangle of area at most $\frac{1}{4}$.

34. A solution is shown in the diagram.



35. Note that the knights can never move to the central square of the chessboard. Number the remaining squares from 1 to 8 in clockwise order, with the white knights on squares 1 and 3, and the black knights on squares 5 and 7. Now they can only move along the cycle (1, 6, 3, 8, 5, 2, 7, 4), and their relative positions on this cycle can never change. Hence the desired task is not possible.

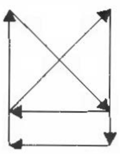
36. After t hours, the first car will be at a distance of $10 - 60t$ kilometres and the second car, $10 - 30t$ kilometres from the intersection. Instead of minimizing directly the distance between them, we may minimize the square of this distance, which by Pythagoras' Theorem is equal to $(10 - 60t)^2 + (10 - 30t)^2 = 200 - 1800t + 4500t^2 = 5(4 + (6 - 30t)^2)$. The minimum occurs at $t = \frac{1}{5}$, which is 12 minutes past noon. At that moment, the first car is at 10 - 12 = -2 or 2 kilometres past the

intersection, and the second at $10 - 6 = 4$ kilometres before the intersection.

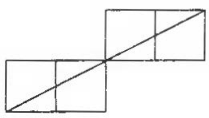
37. It has 32 edges, 24 two-dimensional faces and 8 three-dimensional faces.

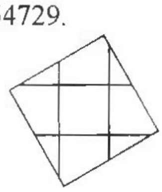
38. The quadratic expression expands into $Ax^2 + Bx + C$, where $A = q_1^2 + q_2^2 + \cdots + q_n^2$, $B = 2(p_1q_1 + p_2q_2 + \cdots + p_nq_n)$ and $C = p_1^2 + p_2^2 + \cdots + p_n^2$. Since it is a sum of squares, it is never negative for any real values of x. It follows that its discriminant $B^2 - 4AC$ must be non-positive. Hence $(\frac{B}{2})^2 \leq AC$, which is equivalent to the desired result.

39. Take an equilateral triangle ABC. Let D be on the perpendicular bisector of BC, with $AD = BC$. Then $BD = CD \neq AD$. This gives rise to two configurations as D can be on either side of BC. A third consists of two equilateral triangles ABC and CDA, and a fourth consists of a square $ABCD$. The fifth is an isosceles trapezoid $ABCD$ with $AB = BC = CD$ and $AD = AC = BD$.



40. The only solution is 381654729.





41. A solution is shown in the diagram.

42. A solution is shown in the diagram.

43. Jock led it, and was in fact the only one who spoke truthfully.

44. We have $12 = y^4 - x^2 = (y^2 - x)(y^2 + x)$. One of the factors on the right side must be even, which implies that both are even. The only possibility is $y^2 - x = 2$ and $y^2 + x = 6$, leading to $x = y = 2$.

45. We may assume that PA ≤ PB. Drop the perpendicular OD from O onto AB. Then AD = BD. Suppose P in inside the circle. By Pythagoras' Theorem, $PA \cdot PB = (AD - PD)(BD + PD) = AD^2 - PD^2 = (OA^2 - OD^2) - (OP^2 - OD^2) = r^2 - OP^2$. If P is outside the circle, we have $PA \cdot PB = (PD - AD)(PD + BD) = OP^2 - r^2$.

46. There are 10 potential lines of division. If the task is possible, each must be blocked by a domino across its path. However, if exactly one domino is in the way, then the line divides the remaining part of the square into two regions each containing an odd number of cells. This is impossible because each is supposed to be tiled with dominoes. Hence each line is blocked by at least two dominoes. Clearly, each domino can block only one line. Since there are only 18 dominoes, at most 9 lines can be blocked. Hence the task is impossible.

47. After taking a paved road and a country road, the lost tourist should take three more country roads and finally another paved road. It is routine to verify that no matter where he starts, the lost tourist will always end up in D.

48. Let the two points be O and A. Draw a circle γ with centre O passing through A, and a circle λ with centre A passing through O. Use the common radius to mark off three successive arcs on λ, starting from O, so that we end up at point B which is at the other end of the diameter from O. Draw a circle with centre B passing through O, cutting γ at the points P and Q. With these two points as centres, draw two circles passing through O, and intersecting each other again at a point M. Then M is the desired midpoint of OA. To see that it is so, note by symmetry that M indeed lies on the line OA. Now POM and BOP are isosceles triangles. Since they have a common base angle, they are similar to each other. Hence $\frac{MO}{OP} = \frac{OP}{OB}$ which leads to $MO = \frac{1}{2}OA$.

49. If the party were held when grandpa was 99, then $x67y2 \equiv x + 10y + 69 \equiv 0 \pmod{99}$. Since x and y are single-digit numbers, we must have $x + 10y = 30$. However, this means that x is a multiple of 10, and $x67y2$ cannot be a five-digit number. It follows that the party was held when grandpa was 98, and $x67y2 \equiv 4x + 10y + 38 \equiv 0 \pmod{98}$. This leads to $2x + 5y = 30$ and we must have $x = 5$. Then $y = 4$ and the number of guests was $56742 \div 98 = 579$.

50. For each question, denote the answers 0, 1 and 2 in non-ascending order of their numbers of respondents. Suppose there were 10 or more students. Then at least 7 of them answered 0 or 1 in Question 4, which would not be useful for distinguishing triples of them. Similarly, at least 5 of these 7 answered 0 or 1 in Question 3, at least 4 of these 5 answered 0 or 1 in Question 2 and at least 3 of these 4 answered 0 or 1 in Question 1.

Students	1	2	3	4	5	6	7	8	9
Question 1	0	0	0	1	1	1	2	2	2
Question 2	0	1	2	0	1	2	0	1	2
Question 3	0	1	2	2	0	1	1	2	0
Question4	0	1	2	1	2	0	2	0	1

These 3 did not answer any of the four questions differently. It follows that the largest possible number of students is 9, and this is indeed possible, as shown in the table above.

51. A triangle inside a parallelogram has at most one half its area, and the same goes for a parallelogram inside a triangle. Let the angles at P, Q, R and S be α, β, γ and δ respectively. Since $(\alpha + \beta) + (\gamma + \delta) = 360°$, we may assume that $\alpha + \beta \geq 180°$, and similarly that $\alpha + \delta \geq 180°$. If we complete the parallelogram $PQTS$, then T is inside $PQRS$. Now PQS has at most one half the area of $PQTS$ which in turn has at most one half the area of ABC.

52. When divided by 4, an even square leaves remainder 0 while an odd square leaves remainder 1. When the sum of the squares of five consecutive integers is divided by 4, the remainder is either $1 + 0 + 1 + 0 + 1 = 3$ or $0 + 1 + 0 + 1 + 0 = 2$. Thus this sum cannot be a square itself.

53. Triangles PAB and PCD are similar, as are QCD and QEF. It follows that $\frac{AP}{CP} = \frac{AB}{CD} = \frac{EF}{CD} = \frac{EQ}{CQ}$. Hence PQ is parallel to AE and BD.

54. Let $BC = a$, $CA = b$, $AB = c$ and $\angle ABC = \beta$. Then the area of ABC is $\frac{1}{2}ca\sin\beta$ and that of BFG is $\frac{1}{2}ca\sin(180°-\beta)$. Hence they have the same area. Similarly, ADE and CHJ also have the same area as ABC. Now the area of $ACJD$ is $b^2 = c^2 + a^2 - 2ca\cos\beta$. The total area of $ABFE$ and $BCHG$ is $c^2 + a^2$ while that of the remaining part of $DEFGHJ$ is $c^2 + a^2 - 2ca\cos(\beta) + 2ca\sin\beta$. Equating these two values, we have $\theta = 45°$.

55. Let p be any prime number and for any positive integer k, let $H(k)$ denote the highest powers of p which divides k. Let $H(l) = a$, $H(m) = b$ and $H(n) = c$. Then $H(l\,m) = ab$, $H(mn) = bc$ and $H(nl) = ca$. It follows that $H(\gcd(l,m,n)) = \min\{a,b,c\}$ and $H(\operatorname{lcm}(l\,m, mn, nl)) = \max\{ab,bc,ca\}$. The desired result follows immediately.

56. Note that

$$\frac{1}{1+x+xy} = \frac{z}{z+zx+1} = \frac{yz}{yz+1+y} \text{ and } \frac{1}{1+z+zx} = \frac{y}{y+yz+1}.$$

It follows that the desired sum is equal to

$$\frac{yz}{yz+1+y} + \frac{1}{1+y+yz} + \frac{y}{y+yz+1} = 1$$

57. We have

$$\begin{aligned} x \odot y &= (x \odot y) \odot (x \odot y) \\ &= ((x \odot y) \odot x) \odot y \\ &= ((y \odot x) \odot x) \odot y \\ &= ((x \odot x) \odot y) \odot y \\ &= (y \odot y) \odot (x \odot x) \\ &= y \odot x. \end{aligned}$$

58. Suppose we have an integer solution. Clearly, $x^4 + y^4 + z^4$ is even. We may assume that x is even and y and z are either both odd or both even. If they are even, then the left side of the equation is divisible by 16, but 24 is not. Hence y and z are both odd. However, the left side of the equation is congruent modulo 16 to $0 + 1 + 1 - 2 - 2x^2 - 2x^2$. This leads to $-4x^2 \equiv 8 \pmod{16}$ or $x^2 \equiv 2 \pmod 4$, which is impossible.

59. Suppose $(3m + 2)^2 = n^2 + p$ where m and n are positive integers and p is a prime. Then $p = (3m + 2 - n)(3m + 2 + n)$, which means that $3m + 2 - n = 1$ and $3m + 2 + n = p$. Solving this system of equations yields m = $\{p - \frac{1}{2}\}$ or $p = 3(2m + 1)$. Thus p cannot be a prime. Hence $(3m + 2)^2$ does not have the desired form for any positive integer m.

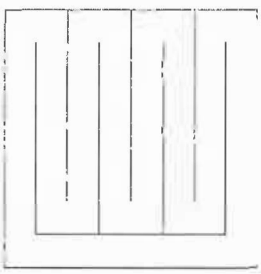
60. For every two scientists, there must be a lock which neither can open. However, each of the other four must have a key, as otherwise some three of them will not be able to open it. Hence we should have one lock for each pair

of scientists and give keys only to the other four. This requires 15 locks and 10 keys for each scientist.

61. Let OA and PB be the respective radii. We may assume that $OA \geq PB$. Drop the perpendicular PC from P onto OA. By Pythagoras' Theorem, $AB^2 = OC^2 = OP^2 - CP^2 = (OA + PB)^2 - (OA - PB)^2 = 4OA{\cdot}PB$.

62. The dominoes can tile the chessboard by following the path shown in the diagram. The removal of two cells of opposite colors will result in the removal of one domino and the shifting of one group of the remaining dominoes between the two removed cells along the path.

63. The next such integer is 52.

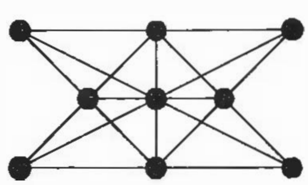


64. Let P be the point inside the equilateral triangle ABC of side 1. Let $2a$, $2b$ and $2c$ be the respective distances of P from BC, CA and AB. Then the area of PBC, PCA and PAB are a, b and c respectively, so that the area of ABC is $a + b + c$. Since its base is 1, its altitude is indeed $2a + 2b + 2c$.

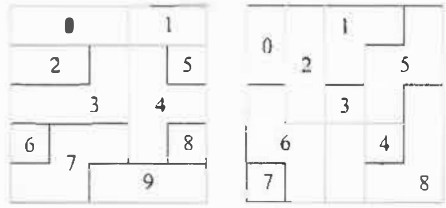
65. Ask the chicken to stand on one foot and the rabbits to put up their front paws. Then there will be $94 \div 2 = 47$ feet, exceeding the number of heads by $47 - 35 = 12$. Each rabbit contributes one foot to this total, so that the number of rabbits is 12, and the number of chickens is $35 - 12 = 23$.

66. If $n = 2k$, a permutation which works is $(a_{k+1}, a_1, a_{k+2}, a_2, \ldots, a_{2k}, a_k)$. If $n = 2k + 1$, just add a_{2k+1} at the end.

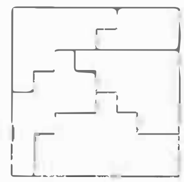
67. A solution is shown in the diagram.



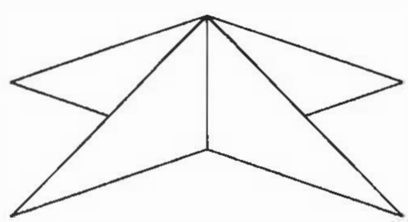
68. The solution is shown in the diagram, where the block is drawn in two layers. Parts with identical labels belong to the same piece.



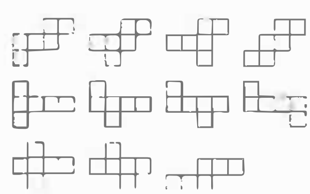
69. A solution is shown in the diagram.



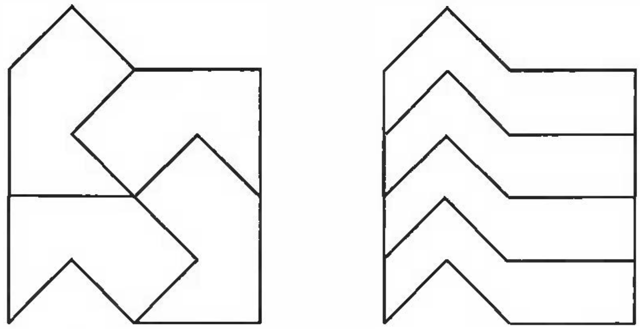
70. Make four creases as follows. The first two are along the diagonals, with the colored side showing. The other two are along the segments joining midpoints of opposite sides, with the plain side showing. Then the paper is folded into the four-winged shape shown in the diagram, with the midpoints of the four sides coming together. Two opposite wings serve as "pockets" while the other two serve as "tongues". Let the colors be red, yellow and blue. Put the red tongues in the yellow pockets, the yellow tongues in the blue pockets and the blue tongues in the red pockets. Paper clips can help during assembly, but they are not needed to hold the structure together once it is completed.



71. The eleven hexominoes are shown in the diagram.



72. The two solutions are shown in the diagram.



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