# Conceptual Understanding and Computational Skill in School Mathematics

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Evaluation Standard 9 of the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) addresses the issue of mathematics procedures including, but not limited to, computational methods and algorithms. The standards document relates that

> a knowledge of procedures involves much more than simple execution. Students must know when to apply them, why they work, and how to verify that they give correct answers; they also must understand concepts underlying a procedure and the logic that justifies it. (NCTM 1989, 228)

It is easy to agree with these conclusions, particularly the view that conceptual understanding is as important as computational skill or manipulative ability (see Kulm 1994).

In contrast, the inaugural issue of *Mathematics Teaching in the Middle School* contains a short letter to the editor that tells of a trick for subtracting mixed fractions. Unfortunately, "Brian's Method" (Curtis 1994) appears to be the exact opposite of the recommendations of the *Curriculum and Evaluations Standards*. Not only that, but as far as we can tell, the usual method of subtracting mixed fractions requires exactly the same amount of computation as Brian's method—one addition and one subtraction, as opposed to two subtractions, respectively—and so the latter hardly seems justified, either computationally or conceptually. (See Sasser 1994 for some discussion of Brian's method and Howe 1995 for reactions and a rejoinder.)

Equipping students with quick tricks might give them the edge in mathematical competitions, but we fear it will also give them a misleading impression of the nature of mathematics and may even hinder their progress in subsequent courses. Mathematics is not a bag of tricks or even a list of formulas. It is a way of thinking, a thought process that we seek to cultivate in our students, no matter what their age.

Emphasizing the "why" over the "how" in a mathematics classroom is an admirable but sometimes difficult goal. A situation taken from our own classroom illustrates this point.

A common word problem in a first-year algebra course is the following:

> Two numbers are in the ratio of 2 to 5. One of the numbers is 21 more than the other. What are the two numbers?

A typical solution follows.

***Solution 1***

Let $x$ and $y$ denote the two unknown numbers. Since they are in the ratio of 2 to 5, we may write

(1) $\qquad \dfrac{x}{y} = \dfrac{2}{5}$

and since one is 21 more than the other, we also have

(2) $\qquad y = x + 21.$

Equation (2) *cannot* be written as $x = y + 21$, since $x < y$ by our choice of variables in (1). Substituting (2) into (1), we find that

$$\frac{x}{x+21} = \frac{2}{5} \Rightarrow 5x = 2x + 42$$

(3) $$\Rightarrow 3x = 42$$
$$\Rightarrow x = 14.$$

Finally, substituting (3) into (2) yields the solution

$$\begin{cases} x = 14 \\ y = 35, \end{cases}$$

which satisfies both (1) and (2).

We usually do not expect our students to produce such detailed solutions, but we at least want them to write down equations (1) and (2) and substitute one into the other. We also want them to get into the habit of checking their work.

So what do we do when a bright young student comes up with this surprising alternative solution?

***Solution 2***

$$5 - 2 = 3$$

$$\frac{21}{3} = 7$$

$$2 \times 7 = 14\ (= x)$$

$$5 \times 7 = 35\ (= y)$$

Not only does this series of sample computations give the correct answer, but *it gives the correct answer every time*, for any constants! Because the student checked his answer to make sure it was correct, and because we do not usually require a detailed explanation like that given in solution 1, we feel compelled to give him credit for a purely mechanical process that just happens to work—every time. So we sit down and try to figure out *why* solution 2 works, in hopes that we can return to this student, and perhaps to the whole class, with some kind of explanation.

Our first attempt to justify the unorthodox solution resulted in a complicated pair of algebraic identities that we could never hope to explain to our first-year-algebra students. For a long time, that solution was the best we could do. Then later we learned of a relatively simple justification of solution 2 that allowed us to refine our arguments such that we could finally explain them to our eighth graders. By that time, however, our students had graduated and gone on to high school. If they were still with us today, we would offer the following justification.

## Justification of Solution 2

This method of solving the problem simultaneously justifies the unorthodox solution. First, let us generalize the problem. We want to solve the equations

$$\text{(4)} \qquad \frac{x}{y} = \frac{a}{b} \qquad y - x = d$$

given constants $a$, $b$, and $d$ with $b \neq 0$. From the first equation in (4) we have

$$\text{(5)} \qquad \frac{x}{y} = \frac{a}{b} \Rightarrow \frac{x}{a} = \frac{y}{b} \ (= c)$$

assuming that $a \neq 0$. The constant $c$ corresponds to the "magic" number 7 in solution 2. From the right-hand side of (5) we see that

$$\text{(6)} \qquad x = ac \qquad y = bc$$

for some unknown constant $c$. The equations in (6), together with the second equation of (4), yield

$$\text{(7)} \qquad y - x = d \Rightarrow bc - ac = d \Rightarrow (b - a)\,c = d \Rightarrow c = \frac{d}{b-a}$$

provided that $b - a \neq 0$. Substituting (7) into (6), we obtain

$$\text{(8)} \qquad x = a \times \frac{d}{b-a} \qquad y = b \times \frac{d}{b-a},$$

which solves (4). For our particular problem,

$$\text{(8)} \qquad x = 2 \times \frac{21}{5-2} \qquad y = 5 \times \frac{21}{5-2},$$

which matches exactly the sequence of steps in solution 2.

Did we miss a golden opportunity? Perhaps so, but we may have learned something in the process. A teacher may not know how to explain a student's unorthodox solution, which was our initial reaction to solution 2, or may choose not to reveal it, but in either situation the student must realize that the problem is not completely solved until a proof is found that justifies the method in all cases. The teacher will certainly want to find such a proof—by reading books, talking to colleagues or posting to the Internet—in a form that students can understand and appreciate.

We *want* our students to discover patterns in problems. This discovery is an important part of the mathematical process. Because few students feel the need to generalize and prove their findings, however, part of our job as teachers is to create an atmosphere that encourages abstraction and proof as well as experimentation and conjecture, a process even our first-year-algebra students can appreciate. For example, we can tell them the story of the student who discovered solution 2 and how a justification was found. We can also point them to the literature for other more spectacular examples, such as the discovery by a Grade 9 student reported in Morgan (1994). But most of all, we should encourage the mathematical process in our classrooms and be prepared to take advantage of any situation that arises.

## References

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