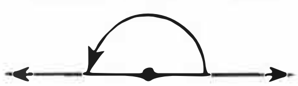
What's My Angle?

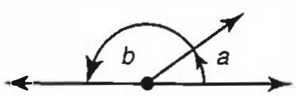
Jane Barnard

An angle is the union of two rays sharing only a common endpoint. Angles are frequently measured using degrees. The ancient Babylonians used a sexagesimal, or base 60, numeral system and assigned a measure of 360 degrees to one revolution of the circle. They divided a circular region into 6 sectors and subdivided each sector into 60 more sectors, giving 360 subdivisions. Each small subdivision is called a *degree*.

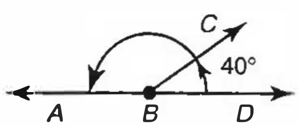
A *straight angle* is an angle formed by opposite rays and has a measure of 180 degrees. Two angles, and exactly two, are *supplementary* if they form a straight angle or the sum of their measures is 180 degrees.



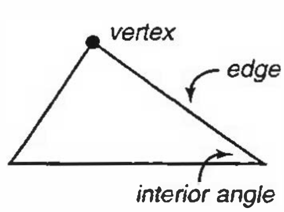
1. What is the sum of the measures of the two angles at the right?



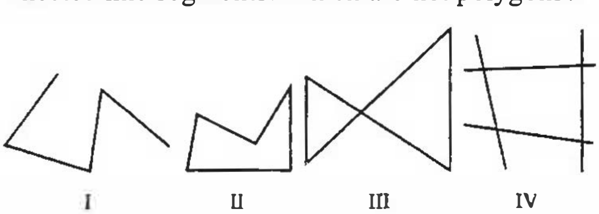
2. Angles *ABC* and *CBD* are supplementary. Find the measure of angle *ABC*, written $m\angle ABC$.



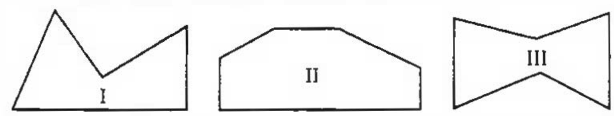
Polygons are composed of connected line segments, or *edges*, which meet only at endpoints and enclose a single portion of the plane, called the *interior* of the polygon. Some people call the edges sides. A *vertex* is a point where exactly two edges meet. The angle formed at each vertex and measured inside the polygon is an *interior angle*.



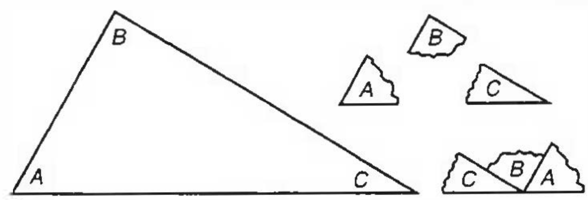
3. Each of the following shapes is composed of connected line segments. Which are not polygons?



4. A polygon is *convex* if every interior angle has a measure less than 180 degrees. Which of the following polygons are not *convex*? Such polygons are sometimes called concave.



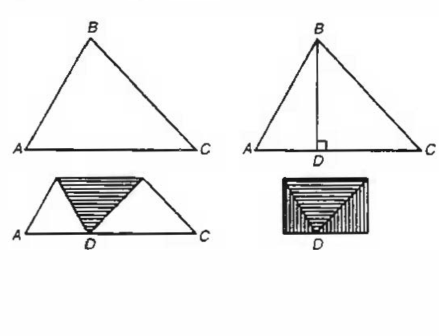
A triangle is a polygon having exactly three edges. Cut out a paper triangle like the one below and label the angles as shown. Tear off the corners of the triangle, that is, tear off $\angle A$, $\angle B$ and $\angle C$. Put the three angles together, as shown, so they share a common vertex.



5. What kind of angle is formed?
6. What is the degree measure of this angle?
7. What is the sum of the measures of $\angle A$, $\angle B$ and $\angle C$?
8. Recall the definition of supplementary angles. Are $\angle A$, $\angle B$ and $\angle C$ supplementary? Why or why not?
9. Repeat this activity using a different triangle. What do you observe?

Another way to look at the sum of the measures of the angles of a triangle is by paper folding. *Cut* a paper triangle like the one shown. Fold the triangle to create an altitude, the segment perpendicular to $\overline{AC}$ through the vertex *B*. Mark *D* as the point of intersection of the altitude and the base. Fold *B* down to *D*. Fold *A* and *C* over to *D*.

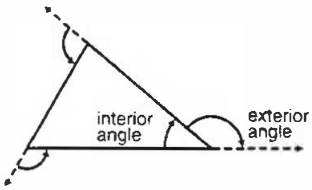
10. The three angles of the triangle fold to point *D* to form what kind of angle?



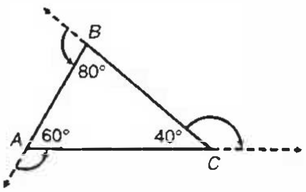
Exterior Angles

In addition to three interior angles, a triangle has exterior angles. To form one set of exterior angles, extend each edge beyond the vertex in one direction.

11. What kind of angle is formed by an interior angle of the triangle and its exterior angle? What is the sum of their measures?

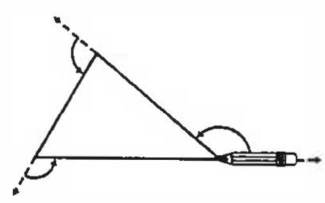


12. Find the measure of each exterior angle of triangle ABC. What is the sum of these measures?

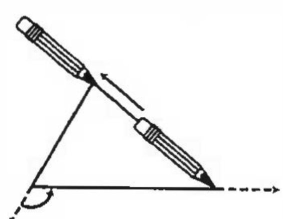


To check your answer, place a pencil flat on your paper along the extended edge as shown. Rotate the pencil counterclockwise about the vertex, turning it through the exterior angle until it lies along the edge of the triangle. Slide the pencil along the edge until its point lies on the next vertex. Rotate through the exterior angle again and slide along the edge of the triangle two more times. The pencil should come back to its original position.

13. Your pencil made a complete turn, or revolution, of how many degrees?

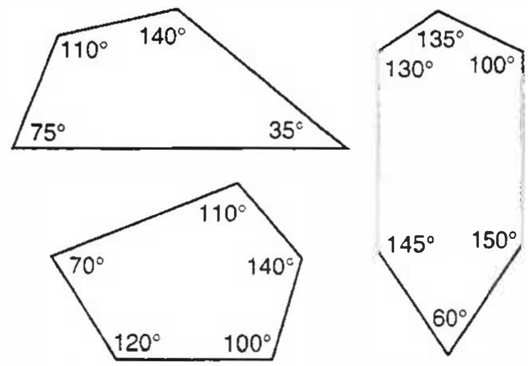


14. Find the measures of the exterior angles of the following convex polygons and use your findings to complete the chart. Use your information from the triangle in question 12.



15. For a convex dodecagon, which has 12 edges, what is the sum of the measures of the exterior angles?

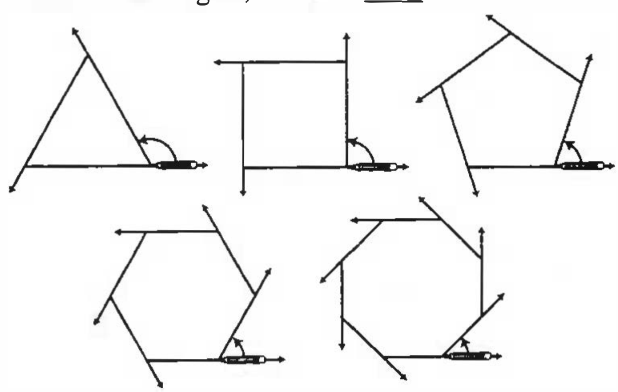
Polygon	Sum of Measures of Exterior Angles
Triangle	
Quadrilateral	
Pentagon	
Hexagon	



16. Complete the conjecture: In a convex polygon, the sum of the measures of the exterior angles is ____.

A polygon is *regular* if all edges are *congruent*, that is, have the same length, and all interior angles are *congruent*, that is, have the same measure.

17. For the following regular polygons, use the "pencil method" to verify the sum of the measures of the exterior angles, which is ____.



If interior angles of a regular polygon are congruent, then the exterior angles must also be congruent. An equilateral triangle has three congruent exterior angles, the sum of whose measures is 360 degrees. So each exterior angle has a measure of 360/3, or 120, degrees. Since interior and exterior angles are supplementary, the measure of each interior angle of the equilateral triangle is 180 – 120, or 60, degrees.

18. Using this method, find the measure of each exterior and each interior angle of these regular polygons in Table 1.

Table 1

Regular Polygon	No. of Edges	Measure of Each Exterior Angle	Measure of Each Interior Angle	Sum of the Measures of Interior Angles
Triangle	3	$360°/3 = 120°$	$180°- 360°/3 = 180° - 120° = 60°$	$3 \times 60° = 180°$
Quadrilateral	4	$360°/4 = 90°$	$180° - 360°/4 = 180° - 90° = 90°$	$4 \times 90° = 360°$
Pentagon	5			
Hexagon	6			
Heptagon	7	$360°/7 = 51\ 3/7°$	$180° – 360°/7 = 180° – 51\ 3/7° = 128\ 4/7°$	$7 \times 128\ 4/7° = 900°$
Octagon	8			
Nonagon	9			
Decagon	10			
Dodecagon	12			
Icosagon	20			
n-gon	n			

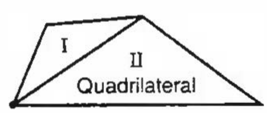
19. If a regular polygon has 30 edges, what is the measure of each exterior angle? Of each interior angle?

20. If the measure of an interior angle of a regular polygon is 170 degrees, how many edges does the polygon have?

21. Describe how to find the measure of each interior angle of a regular polygon with n edges, usually called a regular n-gon.

22. Write a generalization to accompany this explanation: The measure of each interior angle of a regular n-gon is ______________________ .

Triangulation



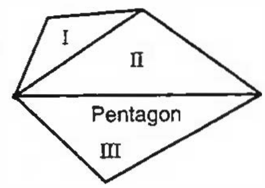
Another way to think about interior angles of any convex polygon is to look at triangles. Every such polygonal region can be *triangulated*, that is, it can be separated into triangular regions by drawing all the diagonals from any one vertex. For example, consider the quadrilateral shown, which is separated into two triangular regions.

23. The sum of the measures of the angles in triangle I is ______________________ .

24. The sum of the measures of the angles in triangle II is ______________________ .

25. The sum of the measures of the angles in the quadrilateral is the sum of the measures of the angles in triangles I and II, which is ____° + ____° = 2 × ____ °.

26. Triangulating the pentagon gives ____ triangular regions (see diagram that follows).



27. The sum of the measures of the angles in the pentagon is the sum of the measures of the angles in triangles I, II and II, which is ____° + ____° + ____° = 3 × ____°.

28. Draw diagonals from the given vertex to triangulate the following polygons. Complete the chart, then look for a pattern.

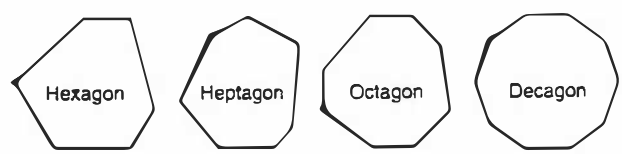


Table 2

Polygons	No. of Edges	No. of Triangles	Sum of Measures of Interior Angles
Triangle	3	1	$180° \times 1 = 180°$
Quadrilateral	4	2	$180° \times 2 = 360°$
Pentagon	5		
Hexagon	6		
Heptagon	7		
Octagon	8		
Decagon	10		
Icositetragon	24		
n-gon	n		

29. Using the pattern you discovered, find the measure of one interior angle of a regular n-gon:_____.

30. You have found two generalizations for determining the measure of one interior angle of a regular n-gon. Algebraically, show that $180 - 360/n$, which is a generalization of the exterior-angle method in question 18, is equal to $(180\,(n - 2))/n$, which is the generalization of the triangulation method.

Can you . . .

- count all the diagonals of a regular triangle, quadrilateral, pentagon, . . . , n-gon?
- relate interior and exterior angle relationships to triangular numbers?
- establish a relationship between an exterior angle of a quadrilateral and its three nonadjacent interior angles?
- determine the number of edges of a regular polygon if you know the measure of either an interior or an exterior angle?
- determine the maximum number of intersection points of the diagonals of a convex n-gon?
- draw the inscribed and circumscribed circles for the regular polygons?
- describe a means of finding out when two convex polygons are similar? Two triangles are similar if their corresponding angles are congruent.
- determine the ratio of the interior angles of a triangle if you know the ratio of the exterior angles?

Did you know . . .

- that pilots give the direction in which the airplane is traveling by using degrees measured clockwise from north?
- that celestial navigation is a means of using the angle of elevation of certain stars to determine one's position on water?
- that Eratosthenes (275–195 B.C.) approximated the circumference of Earth by measuring lengths of shadows and sizes of angles?
- that certain regular polygons can be constructed by compass and straightedge and others cannot?
- that R. Buckminster Fuller used regular polygons to create his geodesic dome?
- that Aristarchus of Samos (310–230 B.C.) approximated the distance from Earth to the sun by using angle measures?
- that angle measure is essential in accurately portraying land descriptions?
- that a polygon with 11 edges is a hendecagon; with 13, a tridecagon or triskaidecagon (note that "kai" is Greek for "and"); with 14, a tetradecagon or tetrakaidecagon; and with 15, a pentadecagon or pentakaidecagon?

Mathematical Content

- Geometry

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Answers

1. 180 degrees.
2. 140 degrees.
3. I, III, IV.
4. I, III.
5. Straight angle.
6. 180 degrees.
7. 180 degrees.
8. No; three angles are used and the definition requires "exactly two."
9. The three angles form a straight angle.
10. Straight angle.
11. Straight; 180 degrees.
12. 120 degrees, 100 degrees, 140 degrees; 360 degrees.
13. 360 degrees.
14.

Polygon	Sum of Measures of Exterior Angles
Triangle	120 + 100 + 140 = 360
Quadrilateral	145 + 40 + 70 + 105 = 360
Pentagon	80 + 40 + 70 + 110 + 60 = 360
Hexagon	30 + 80 + 45 + 50 + 35 + 120 = 360

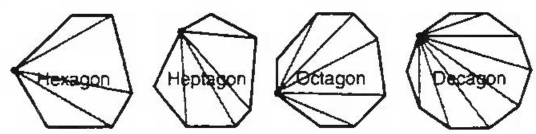
15. 360 degrees.
16. Always 360 degrees.
17. 360 degrees.

18.

No. of Edges	Measure of Each Exterior Angle	Measure of Each Interior Angle	Sum of the Measures of Interior Angles
5	$\frac{360^\circ}{5} = 72^\circ$	$180^\circ - \frac{360^\circ}{5} = 180^\circ - 72^\circ = 108^\circ$	$5 \times 108^\circ = 540^\circ$
6	$\frac{360^\circ}{6} = 60^\circ$	$180^\circ - \frac{360^\circ}{6} = 180^\circ - 60^\circ = 120^\circ$	$6 \times 120^\circ = 720^\circ$
8	$\frac{360^\circ}{8} = 45^\circ$	$180^\circ - \frac{360^\circ}{8} = 180^\circ - 45^\circ = 135^\circ$	$8 \times 135^\circ = 1080^\circ$
9	$\frac{360^\circ}{9} = 40^\circ$	$180^\circ - \frac{360^\circ}{9} = 180^\circ - 40^\circ = 140^\circ$	$9 \times 140^\circ = 1260^\circ$
10	$\frac{360^\circ}{10} = 36^\circ$	$180^\circ - \frac{360^\circ}{10} = 180^\circ - 36^\circ = 144^\circ$	$10 \times 144^\circ = 1440^\circ$
12	$\frac{360^\circ}{12} = 30^\circ$	$180^\circ - \frac{360^\circ}{12} = 180^\circ - 30^\circ = 150^\circ$	$12 \times 150^\circ = 1800^\circ$
20	$\frac{360^\circ}{20} = 18^\circ$	$180^\circ - \frac{360^\circ}{20} = 180^\circ - 18^\circ = 162^\circ$	$20 \times 162^\circ = 3240^\circ$
n	$\frac{360^\circ}{n}$	$180^\circ - \frac{360^\circ}{n}$	$n\left[180^\circ - \frac{360^\circ}{n}\right] = 180n - 360^\circ$

19. $360^\circ/30 = 12$ degrees; 178 degrees.
20. 36 edges.
21. Find the measure of an exterior angle by dividing 360 degrees by the number of edges, and then subtract this result from 180 degrees.
22. $180 - (360^\circ/n)$.
23. 180 degrees
24. 180 degrees.
25. $180^\circ + 180^\circ = 2 \times 180^\circ$.
26. Three
27. $180^\circ + 180^\circ + 180^\circ = 3 \times 180^\circ$.

28.



Polygons	No. of Edges	No. of Triangles	Sum of Measures of Interior Angles
Pentagon	5	3	$180^\circ \times 3 = 540^\circ$
Hexagon	6	4	$180^\circ \times 4 = 720^\circ$
Heptagon	7	5	$180^\circ \times 5 = 900^\circ$
Octagon	8	6	$180^\circ \times 6 = 1080^\circ$
Decagon	10	8	$180^\circ \times 8 = 1440^\circ$
Icositetragon	24	22	$180^\circ \times 22 = 3960^\circ$
n-gon	n	$n - 2$	$180^\circ \times (n - 2)$

29. $(180^\circ \times (n - 2))/n$.

30. $\frac{180^\circ \times (n-2)}{n} = \frac{180n^\circ - 360^\circ}{n} = \frac{180n^\circ}{n} - \frac{360^\circ}{n} = 180^\circ - \frac{360^\circ}{n}$

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