Some Interesting Facts About Euler's Number

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Euler's number e, the base of natural logarithms, has various unusual properties which seem to be possessed by actions occurring in nature. It has an intimate relationship with natural phenomena such as radioactive decay, exponential growth and so on.

If a certain amount of material would produce an equal amount of the same material over a period of time, and the newly created material would produce still more new material at the same rate, then at the end of the period there would be approximately 2.7 times the original material. It can be shown that rate of growth involves the number, e.

Assuming that we start with one unit of this material and take a time halfway through some designated period, the original material will have increased to 1.5 units. For the second half of the period, there would be 1.5 pounds of the material producing still more material. At the end of the period, there would be a total of 2.25 pounds. As we divide the period into smaller parts, we take more and more account of the increase due to the newly created material itself.

If we are to find the exact amount of material at the end of the period, we must divide the period into an infinite number of infinitely small periods and add them together. The sum of this infinite series is numerically equivalent to e which is approximately 2.71828 . . .

The binomial formula can be used to derive the series for e as follows. Let b be any variable and apply the binomial formula to $(1 + b)^n$.

$$(1+b)^n = 1+nb + \frac{n(n-1)}{2!} b^2 + \frac{n(n-1)(n-2)}{3!} b^3 + \ldots$$

Now let k be a variable such that $b = 1/k$ and let x be a variable such that $kx = n$.

$$\left(1+\frac{1}{k}\right)^{kx} = 1+kx\left(\frac{1}{k}\right) + \frac{kx(kx-1)}{2!}\left(\frac{1}{k}\right)^2 + \frac{kx(kx-1)(kx-2)}{3!}\left(\frac{1}{k}\right)^3 + \ldots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{k}\right)}{2!} + \frac{x\left(x-\frac{1}{k}\right)\left(x-\frac{2}{k}\right)}{3!} = \ldots$$

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{kx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

Now, let $x = 1$. Then

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

is defined as e.

Euler's number, e, can therefore be defined as the sum of the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

If x is a variable, then e^x can be approximated using the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

We can find e yet another way. We know that a given function $y = f(x)$ can be approximated by a sequence of polynomials $f_n(x)$ of the form

$$f_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

in which

$$f(0) = a_0,$$
$$f'(0) = a_1,$$
$$\frac{f''(0)}{2} = a_2,$$
$$\vdots$$
$$\frac{f^{(n)}(0)}{n!} = a_n$$

We would like to represent the function $f(x) = e^x$ near $x = 0$ by such a sequence to get the general polynomial of $f(x) = e^x$.

For the given function, the derivatives at $x = 0$ are

$$f(0) = e^0 = 1$$
$$f'(0) = 1$$
$$\vdots$$
$$f^{(n)}(0) = 1$$

Therefore

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

If $f(x) = e^x$, then at $x = 1$

$$f_n(x) = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}$$

and the original series,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is called the power series of e.

Often money is lent with the understanding that when earnings accumulate they are to be added to the original investment at specified times and thus become part of a new principal. Interest accrued on this basis is called compound interest.

If P dollars are lent at r percent for each interest period, the amount for the first period,

$A_1 = P + Pr = P(1 + r)$.

This amount bears interest for the second period,

$A_2 = P(1 + r) + P(1 + r) = P(1 + r)^2$.

If we accumulate the amount of an investment at the end of a number of periods, then the amount, A, can be shown as follows,

$A_1 = P(1 + r)$
$A_2 = P(1 + r)^2$
$A_3 = P(1 + r)^3$
$A_4 = P(1 + r)^4$
$A_k = P(1 + r)^k$.

We can conclude that the compound interest formula is $A = P(1 + r)^k$.

When interest is compounded n times per year at rate r per year, then the rate per period is r/n and the number of period in t year is nt. The formula then becomes

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Suppose the interest is compounded continuously. Then n increases without limit, thus, $nt \to \infty$. If the principal, P, is equal to 1, the amount, A has the following limit as $nt \to \infty$.

$$A = \lim_{nt\to\infty} \left(1 + \frac{1}{nt}\right)^{nt} = e$$

That is, if \$1 is continuously compounded at rate of r for t years where $rt = 1$, the amount accumulated will be \$2.72.

Suppose the beginning principal is P, the annual interest rate is r, and the time in years is t. Then we can find the final amount, A, when interest is compounded continuously, by the following formula:

$A = Pe^{rt}$

If \$500 is invested at 6 percent compounded continuously for 40 years, the final amount is

$A = Pe^{rt}$
$A = 500e^{.06(40)}$
$A = 500e^{2.4}$
$A = \$5,511.50$

Suppose \$175 is deposited in a saving account where the interest rate is 9½ percent compounded continuously. When will the original deposit be doubled?

$$A = Pe^{rt}$$
$$350 = 175\, e^{.095t}$$
$$\ln 2 = \ln e^{.095t}$$
$$\frac{\ln 2}{.095} = \frac{.095t}{095}$$
$$\frac{.6931}{.095} = t$$
$$t = 7.3 \text{ years}$$

For a certain strain of bacteria, $k = 0.584$ when t is measured in hours. In how many hours will 4 bacteria increase to 2,500 bacteria?

The general formula for growth and decay is

$$y = ne^{kt}$$
$$2{,}500 = 4e^{.584t}$$
$$625 = e^{.584t}$$
$$\ln 625 = .584t$$
$$\frac{\ln 625}{.584} = t$$
$$= \frac{\ln (6.25 \times 10^2)}{.584}$$
$$= \frac{\ln 6.25 \times (2 \ln 10)}{.584}$$
$$= \frac{1.8326 + 2\,(2.3026)}{.584} \approx 11 \textit{ hours}$$

Another of e's interesting properties is that the infinite series can be easily converted into a continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ldots}}}}}}$$

Bibliography

Hartkopf, R. *Math Without Tears.* New York: Emerson Books, 1981.

Johnsonbaugh, R. F. "Another Proof of an Estimate For *e*." *American Mathematical Monthly* (November 1974).

Olds, C. D. "The Simple Continued Fraction Expansion of *e*." *American Mathematical Monthly* (November 1970).

Porter, G. J. "An Alternative to the Integral Test for Infinite Series." *American Mathematical Monthly* (January 1981).

Rubel, L., and K. Stolarsky. "Subseries of the Power for e^x." *American Mathematical Monthly* (May 1980).

Yunker, L. E., G. D. Vannatta and F. J. Crosswhite. *Advanced Mathematical Concepts.* Bell & Howell, 1981.

Student to his parents:
"I got an underwater mark on that last test."
"What kind of grade is that?"
"Below 'C' level."

—Bob Phillips, *More Good Clean Jokes*