The Principle of Mathematical Induction: Its Power in Proving Conjunctive Propositions

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With the development and advance of the computer, the interest in one of its conceptual cohorts, discrete mathematics, has grown. Perhaps because of the new relevance of this branch of mathematics, the principle of mathematical induction, commonly referred to in textbooks simply as induction and abbreviated hereafter in this article as PMI, has become an increasingly important component of college-level mathematics courses and is, at the same time, making its way back into the high school curriculum. Apart from its inherent appeal there are, important pedagogical reasons for teachers to deepen their acquaintance with this principle.

The PMI is an apparently modest method of proving propositions about discrete numbers such as counting numbers or integers. Its apparent modesty is belied by its applicability to conjectures that at first seem beyond its scope. A case in point is its applicability to theorems about rational numbers, for example $\frac{d}{dx}(x^r) = rx^{r-1}$ (Beaver 1993).[1] There are many other apparently unlikely targets for the PMI including the focus of this article, "conjunctive conjectures," that is, conjectures consisting of the conjunction of two or more simpler conjectures.

The PMI—A Description and Explanation

In its simplest form, the PMI may be stated as follows:

A proposition $P(n)$ is true for all integers $n \geq m$ (where m is a given integer, often 1) if the following two conditions hold:

1) the proposition $P(m)$ is true and
2) if the proposition $P(k)$ is true for any integer $k \geq m$, then so is the proposition $P(k + 1)$.

The PMI can be demonstrated to follow logically from the induction axiom of the set of integers. However, one can accept it on an intuitive basis without recourse to that axiom if one reasons as follows. Restricting the argument to the case where $m = 1$, we may trace the implications of the PMI as follows. Condition 1) guarantees that $P(1)$ is true. Condition 2), referred to as the induction hypothesis, then guarantees that if $P(1)$ is true, so is $P(2)$; if $P(2)$ is true, so is $P(3)$; if $P(3)$ is true, so is $P(4)$; and so on ad infinitum. Thus $P(n)$ must be true for n any counting number.

A Simple Example of the PMI

Before examining the applicability of the PMI to conjunctive propositions, let us consider a classic, more straightforward example. Consider the series of odd counting numbers $1 + 3 + 5 + 7 + 9 \ldots + (2n - 1) + \ldots$. Using the conventional S_n to represent the sum of the first n terms of the series, it is easy to determine that $S_1 = 1$, $S_2 = 4$, $S_3 = 9$, $S_4 = 16$, $S_5 = 25$ and so on. These observations lead to the conjecture that $S_n = n^2$ or that $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$ for all $n \in N$, where N represents the natural or counting numbers. Intuitively, one seems justified in making this conjecture with some confidence. However, in the world of mathematics, the fact that this kind of intuitively based confidence has sometimes been ill-founded justifies the insistence that at some point such confidence be bolstered by proof. So let us proceed to see how the PMI can be applied to prove this conjecture. To do this, we formulate the conjecture into a proposition as follows:

Let $P(n)$ be the proposition that $1 + 3 + 5 + 7 + 9 \ldots (2n - 1) = n^2$ for all integers $n \geq 1$. Then

1) $P(1)$ is true since $1 = 1^2$.
2) Suppose that $P(k)$ is true for arbitrary $k \in N$, that is that

$$1 + 3 + 5 + 7 + 9 \ldots + (2k - 1) = k^2. \quad ①$$

We need to show that given assumption ① then $P(k + 1)$ is also true or that

$$1 + 3 + 5 + 7 + 9 \ldots + (2k - 1) + (2k + 1) = (k + 1)^2. \quad ②$$

[$2k + 1$ is the $(k + 1)^{st}$ term obtained by putting $n = k + 1$ in $2n - 1$.]

We begin by assuming that ① holds and add $2k + 1$ to both sides to obtain

$1 + 3 + 5 + 7 + 9 + \ldots + (2k - 1) - (2k + 1) = k^2 + (2k + 1)$

$\Rightarrow$ $1 + 3 + 5 + 7 + 9 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$ or ②.

Thus if ① is true, then ② is also true, that is $P(k)$ implies $P(k + 1)$.

3) According to the PMI, since 1) and 2) both hold [$P(1)$ is true, and if $P(k)$ is true then $P(k + 1)$ is also true], $P(n)$ holds for all $n \in N$.

The PMI and Conjunctive Propositions

In some cases a conjecture made with confidence, when subjected to the rigor of the PMI, particularly part 2), finds its validity to be hostage to a "missing piece." The missing piece may be a second pattern, separate from the one on which the conjecture was based, but one that may be placed in conjunction with the first to form an overarching compound proposition whose validity can be confirmed using the PMI. Such propositions will be referred to as conjunctive propositions.

Two examples growing out of rather different problems will serve to illustrate the missing piece phenomenon. The first, based on a simple game involving counters, includes the formulation of a conjunctive proposition based on three observed patterns but leaves the proof for the reader to explore. The second example, based on the Fibonacci Sequence, includes a conjunctive proposition and its proof.

Example 1

The first example concerns a counter game for two players, A and B. There are a given number of counters in play at the beginning of the game. Players A and B alternate moves with A making the first move. Each move consists of removing one or two counters. The person forced to remove the last counter loses the game. If each player is motivated to win and plays with complete foreknowledge,

i) determine who wins when the game begins with
 1 counter,
 2 counters,
 3 counters,
 4 counters,
 5 counters,
 6 counters,
 7 counters,
 8 counters or
 9 counters;

ii) make a conjecture about who wins that can be applied to any initial number of counters.

One way to explore this problem is through a tree diagram (Laufer 1984, 188–91). Such a diagram, by tracing all of the possibilities, can reveal the best possible way to play such a game. The method used here is to summarize the observations relating to i) in a chart, and use the chart to try to discover patterns to be incorporated in our conjecture as follows.

	Number of Counters	**Who Wins**	**Explanation**
a)	1	B	A is compelled to remove the one and only counter and B wins. (No motivation or skill needed.)
b)	2	A	A can take 1 counter compelling B to take the last one. A wins.
c)	3	A	A can take 2 counters compelling B to take the last one. A wins.
d)	4	B	If A takes 1 counter there are 3 counters when B first moves and, by c), B wins. If A takes 2 counters there are 2 counters when B first moves and, by b), B wins.
e)	5	A	A can take 1 counter leaving 4 counters when B first moves. By d), A wins.
f)	6	A	A can take 2 counters leaving 4 counters when B first moves. By d), A wins.
g)	7	B	If A takes 1 counter there are 6 counters when B first moves and, by f), B wins. If A takes 2 counters there are 5 counters when B first moves and, by e), B wins.
h)	8	A	A can take 1 counter leaving 7 when B first moves. By g), A wins.
i)	9	A	A can take 2 counters leaving 7 when B first moves. By g), A wins.

The patterns of wins appears to be BAABAABAA . . . . From the first nine cases we can conjecture that the general pattern of wins is as follows:

Number of Counters	Who Wins
1	B
2	A
3	A
4	B
5	A
6	A
.	.
.	.
.	.
$3n + 1$	B
$3n + 2$	A
$3n + 3$	A

This conjecture may be formulated into a proposition consisting of the conjunction of three statements as follows:

Let $P(n)$, $n \geq 0$, $n \in Z$ [Z denotes the integers], be the proposition that

i) B wins if there are $3n + 1$ counters,
ii) A wins if there are $3n + 2$ counters and
iii) A wins if there are $3n + 3$ counters.

We have already shown that $P(0)$ is true, that is B wins if there is 1 counter, and A wins if there are either 2 or 3 counters. Further, it is possible to show that if $P(k)$ is true for any $k \in Z$, $k \geq 0$, then $P(k + 1)$ must also be true. In this case $P(k)$ is the proposition

i) B wins if there are $3k + 1$ counters,
ii) A wins if there are $3k + 2$ counters and
iii) A wins if there are $3k + 3$ counters.

Putting $n = k + 1$ in $P(n)$ it follows that $P(k + 1)$ is the proposition

i) B wins if there are $3k + 4$ counters,
ii) A wins if there are $3k + 5$ counters and
iii) A wins if there are $3k + 6$ counters.

The reader is left to explore the problem of showing that $P(k + 1)$ follows from $P(k)$, and may conclude in the process that it may not be possible to prove any one of i), ii) or iii) of the proposition $P(n)$ independently of the others. But, with persistence, one can prove them in conjunction. The proof involves the recursive kind of reasoning invoked in the explanations of cases d) to i) in the chart above.

1 counter
2 counters
3 counters . . . .
. . . .who wins?

Example 2

The second example comes from the Fibonacci Sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55 . . .

This sequence has recursive definition: $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_{n-1} + F_n$. The Fibonacci Sequence is well known because it arises in many often unexpected contexts. Another of its attractions is the many patterns associated with it, some more obvious than others. This example focuses on two patterns that may not be so obvious, one taken from one of the problems in the centre calendar of *Mathematics Teacher* (1993). The two patterns are

i) $F_1^2 + F_2^2 = 1^2 + 1^2 = 2 = F_3$
$F_2^2 + F_3^2 = 1^2 + 2^2 = 1 + 4 = 5 = F_5$
$F_3^2 + F_4^2 = 2^2 + 3^2 = 4 + 9 = 13 = F_7$
. . .
. . .
. . .
$F_{n-1}^2 + F_n^2 = F_{2n-1}$, $n \geq 2$, $n \in N$ (a conjecture only at this point).

ii) $F_1F_2 + F_2F_3 = (1 \times 1) + (1 \times 2) = 1 + 2 = 3 = F_4$
$F_2F_3 + F_3F_4 = (1 \times 2) + (2 \times 3) = 2 + 6 = 8 = F_6$
$F_3F_4 + F_4F_5 = (2 \times 3) + (3 \times 5) = 6 + 15 = 21 = F_8$
. . .
. . .
. . .
$F_{n-1}F_n + F_nF_{n+1} = F_{2n}$, $n \geq 2$, $n \in N$ (a conjecture only at this point).

If we try to prove either of conjectures i) or ii) by itself, we find that we run into the "missing piece" phenomenon referred to earlier. In fact, as I discovered in a failed attempt to prove conjecture ii) on its own, each is the other's missing piece. Following on this lead, let $P(n)$ be the conjunctive proposition that for any counting number n greater than 1:

$F_{n-1}^2 + F_n^2 = F_{2n-1}$ and $F_{n-1}F_n + F_nF_{n+1} = F_{2n}$.

The proof of $P(n)$ by the PMI is as follows:

1) $P(2)$ is true since $F_{2-1}^2 + F_2^2 = F_{2\times2-1}$ and $F_{2-1}F_2 + F_2F_{2+1} = F_{2\times2}$ ①
That is, $F_1^2 + F_2^2 = F_3$ and $F_1F_2 + F_2F_3 = F_4$.
2) Assume that $P(k)$ is true for arbitrary $k \in N$, $k \geq 2$, that is, that ②
$F_{k-1}^2 + F_k^2 = F_{2k-1}$ and $F_{k-1}F_k + F_kF_{k+1} = F_{2k}$.

Then we need to show that $P(k + 1)$ is also true, that is that

$F_k^2 + F_{k+1}^2 = F_{2k+1}$ and $F_kF_{k+1} + F_{k+1}F_{k+2} = F_{2k+2}$.

One can show that ② follows from ① using the definition of the Fibonacci Sequence, the commutative and associative laws of addition and multiplication, and the distributive law of multiplication over addition as follows:

a) Consider $F_k^2 + F_{k+1}^2$ of ②

$F_k^2 + F_{k+1}^2 = F_k^2 + (F_{k-1} + F_k) F_{k+1}$	Def'n of Fib. Seq.
$\Rightarrow F_k^2 + F_{k+1}^2 = F_k^2 + F_{k-1} F_{k+1} + F_k F_{k+1}$	Dist. law
$\Rightarrow F_k^2 + F_{k+1}^2 = F_k^2 + F_{k-1} (F_{k-1} + F_k) + F_k F_{k+1}$	Def'n of Fib. Seq.
$\Rightarrow F_k^2 + F_{k+1}^2 = [F_{k-1}^2 + F_k^2] + \{F_{k-1} F_k + F_k F_{k+1}\}$	Comm., Assoc., and Dist. laws
$\Rightarrow F_k^2 + F_{k+1}^2 = F_{2k-1} + F_{2k}$	Assumption ①
$\Rightarrow F_k^2 + F_{k+1}^2 = F_{2k+1}$	Def'n of Fib. Seq.

This verifies that the first half of ② follows from ① .

$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k [F_{k-1} + F_k] + F_{k+1} [F_k + F_{k+1}]$	Def'n of Fib. Seq.
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k F_{k-1} + F_k^2 + F_k F_{k+1} + F_{k+1}^2$	Dist. and Comm. laws
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k^2 + F_{k+1}^2 + [F_{k-1} F_k + F_k F_{k+1}]$	Comm. and Assoc. laws
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k^2 + F_{k+1}^2 + F_{2k}$	Assumption ①
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_{2k+1} + F_{2k}$	Argument a), above
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_{2k+2}$	Def'n of Fib. Seq.

This verifies that the second half of ② follows from ① .Thus ② follows from ①, that is P(*k*) implies P(*k* + 1).

3) Since 1) and 2) both hold, by the PMI, the proposition P(*n*) is true for all $n \in N$, $n \geq 2$.

The Strong Principle of Mathematical Induction

Further examination of Example 2, above, can lead to a more encompassing proposition than P(*n*) that can be easily proved using a different form of the PMI as follows. One may notice that there are other patterns in the Fibonacci Sequence similar to i) and ii). For example, it can be shown that

$F_{n-1}F_{n+1} + F_nF_{n+2} = F_{2n+1}$.

These patterns may compel one to suspect that there is a more general pattern of which each of these is but a particular case. In fact, there is. This pattern may be formulated into the proposition P(*m*) as follows:

For $n, m \in N$, $n \geq 2$, $m \geq 0$, $F_{n-1}F_{n+m-1} + F_nF_{n+m} = F_{2n+m-1}$.

Here P(0) is the proposition $F_{n-1}^2 + F_n^2 = F_{2n-1}$, or conjecture i) of the last example, and P(1) is the proposition $F_{n-1}F_n + F_nF_{n+1} = F_{2n}$, or conjecture ii). The proposition P(*m*) can be easily proved using the Strong Principle of Mathematical Induction (SPMI). Many textbooks in discrete mathematics, including Roman (1989, 53), include formal descriptions of this principle. Suffice it to say here that, with respect to this example, the SPMI could be applied by showing that the following hold:

1) P(0) and P(1) are both true and
2) if P(*k* – 1) and P(*k*) are both true for $k \geq 1$, then P(*k* + 1) is also true.

This is a special case of the SPMI, but it should be clear without a full description of that principle that if both 1) and 2) hold, then P(*m*) holds for all $m \in$ n, $m \geq 0$. Assume, for example, that both 1) and 2) hold. Then if both P(0) and P(1) hold, by 2), P(2) also holds. Further, if both P(1) and P(2) hold, by 2), P(3) also holds. Further, if both P(1) and P(2) hold, by 2), P(3) also holds. And so on, ad infinitum. The proof of 1) is the same as that of P(*n*) in example 2, above; the proof of 2) is left to the reader.

Benefits of Familiarity with Conjunctive Propositions

Examples 1 and 2, and the generalization of Example 2 in the previous section, illustrate the applicability of the PMI (or its variant, the SPMI) to compound propositions, specifically conjunctive propositions. There are many problems in which integers may be called into play as indices or counters where such propositions are the most appropriate generalizations of observed patterns. Being sensitive to the need for considering the option of such propositions, and aware of the applicability of the PMI to the proof, can enhance one's ability to establish that patterns conjectured on the basis of a limited number of cases apply in general.

Conclusion

The principle of mathematical induction deserves the increasing attention that it appears to be receiving in our schools and colleges. It boasts applicability to a wide variety of problems involving either explicit or implicit integral indices and continues to challenge the mind because of the mental agility required in dealing with that variety. It has many affinities with the inductive processes that are vital components in the operation of programming of computers. For example, the logic of a program loop is analogous to that of the principle of mathematical induction. Consequently, the PMI has become a vital tool in proving program correctness, including verifying the legitimacy of program loops and demonstrating the validity of recursive algorithms. But perhaps its inherent beauty—particularly its ability to entice the mind and to lead it on a gratifying logical excursion—remains the greatest pedagogical appeal of this principle.

Note

1. Beaver proves this theorem by applying the PMI to the following cases in sequence:

i) $y = x^n$, $n \in N$;

ii) $y = x^{-n} = \frac{1}{x^n}$, $n \in N$, using the quotient rule;

iii) $y = x^{1/n}$, $n \in N$, using $x^{1/(n+1)} = \frac{x^{1/n}}{\left[x^{1/(n+1)}\right]^{1/n}}$ and the quotient rule;

iv) $y = x^{b/n} = \left[x^{1/n}\right]^b$, $n, b \in N$, using the chain rule; and

v) $y = x^{-b/n} = \left[\frac{1}{x^{1/n}}\right]^b$, $n, b \in N$, using the quotient rule.

Bibliography

Barnier, W., and J. B. Chan. *Discrete Mathematics with Applications*. St. Paul, Minn.: West, 1989.

Beaver, J. Proof by Induction of the Power Rule for Rational Powers. Communication at Augustana University College, Camrose, Alberta, July 1993.

Laufer, H. B. *Discrete Mathematics and Modern Applied Algebra*. Boston: PWS, 1984.

Calendar Question #18. *Mathematics Teacher* 86, no. 1 (January 1993): 47, 49.

Roman, S. *An Introduction to Discrete Mathematics*. New York: Harcourt Brace Jovanovich, 1989.

Accountability

Accountability means giving an account of the use of your ability.
[Not a bad idea, actually.]

—Donald C. Mainprize, *ABCs for Educators*