Blueprints, the Division Algorithm, Consistent Systems and Augmented Matrices

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Suppose that you are trying to design a rectangular carpet (or poster or floor or . . .) and you have decided to follow the plans in a crumpled old blueprint. The blueprint clearly specifies that the area of the rectangular region is to be $6x^2 + 19x + 12$ and the length is $2x + 5$. It is difficult to read the specified width of the rectangle because the blueprint is badly smudged, but it seems that the width is $ax + b$, for some numbers a and b. What can you conclude?

The answer is that either you have misread the blueprint or the author of the blueprint made a mathematical error. Various forms of the explanation could be given in courses ranging from prealgebra to precalculus. Among the topics that would be reinforced or motivated are uniqueness of the remainder in the division algorithm for polynomials; the Factor Theorem; and the role of rank and determinants in using augmented matrices to test for inconsistency in systems of linear equations.

The most direct explanation has to do with expressing area as a product: $(ax + b)(2x + 5) = 6x^2 + 19x + 12$. Since division is the inverse operation of multiplication, $ax + b = (6x^2 + 19x + 12) \div (2x + 5)$. Carrying out this division leads to

$$\begin{array}{r|l} & 3x + 2 \\ 2x + 5 & 6x^2 + 19x + 12 \\ & 6x^2 + 15x \\ & \qquad 4x + 12 \\ & \qquad 4x + 10 \\ \hline & \qquad\qquad 2 \end{array}.$$

Notice that the quotient is $3x + 2$ and, more importantly, the remainder is 2. Now, if $ax + b = (6x^2 + 19x + 12) \div (2x + 5)$, the remainder in the above long division would be 0, by the uniqueness of the remainder in the division algorithm. (For a careful statement of this result, see Dobbs and Peterson 1993, 139, 143.) In particular, $(6x^2 + 19x + 12) \div (2x + 5)$ is not a polynomial, and we can conclude that no numbers a and b exist with the above property.

By slightly generalizing the above reasoning, we can discover part of the Factor Theorem (see Dobbs and Peterson 1993, 141 for a statement and proof of the general result). Actually, we will end up proving the special case in which the dividend is a quadratic polynomial. Consider the question whether a given linear polynomial, $cx + d$, is a factor of a given quadratic polynomial, $ex^2 + fx + g$. Assume, as in most of the interesting examples, that $c \neq 0$ and $d \neq 0$. We find the following string of equivalent statements:

$$\begin{array}{r|l} & ax + b \\ cx + d & ex^2 + fx + g \end{array} \quad \Leftrightarrow$$

$(ax + b)(cx + d) = ex^2 + fx + g \Leftrightarrow$ [equate corresponding coefficients]
$ac = e,\ ad + bc = f,\ bd = g$. By solving linear equations, we find, in particular, that $a = e/c$ and $b = g/d$. With these expressions for a and b fixed, the string of equivalences continues:

$$\Leftrightarrow \quad \left[\frac{f - \frac{e}{c}d}{c}\right] d = g \quad \Leftrightarrow \text{[rewrite algebraically]}$$

$f(d/c) - e(d/c)^2 = g \Leftrightarrow$ [rewrite algebraically]
$e(-d/c)^2 + f(-d/c) + g = 0$. Thus, we have shown that $cx + d$ is a factor of $ex^2 + fx + g$ if and only if $-d/c$ is a root of $ex^2 + fx + g$. Since $cx + d = c\,(x - (-\,d/c))$, this means that $x - (-\,d/c)$ is a factor of $ex^2 + fx + g$ if and only if $-d/c$ is a root of $ex^2 + fx + g$. At this point, it would be natural for an algebra class to conjecture the more general result that if r is any number, then