For each of these samples, calculate its mean and its sample variance (s^2). Two examples follow:

1. Consider the sample {1, 1, 1} where the mean is 1 and $s^2 = \dfrac{0^2 + 0^2 + 0^2}{3 - 1} = \dfrac{0}{2} = 0.$

   The same value of s^2 will result for {2, 2, 2,}; {3, 3, 3,} and {10, 10, 10}.

2. {1, 2, 10}

   The mean is $\dfrac{1 + 2 + 10}{3} = \dfrac{13}{3}$ and

$$s^2 = \frac{(1 - \frac{13}{3})^2 + (2 - \frac{13}{3})^2 + (10 - \frac{13}{3})^2}{(3 - 1)} = \frac{\frac{100}{9} + \frac{49}{9} + \frac{289}{9}}{2} = \frac{438}{18} = \frac{73}{3}$$

The same value of s^2 will result for the permutations of 1, 2 and 10, namely, {1, 10, 2}; {2, 1, 10}; {2, 10, 1}; {10, 1, 2}; {10, 2, 1}.

Table 2 reports the entire set of results. The answers are reported as fractions so that the rounding which would accompany decimal representations does not interfere with the results.

Table 2

Sample	Mean	s^2	Sample	Mean	s^2
1, 1, 1	1	0	3, 1, 1	5/3	4/3
1, 1, 2	4/3	1/3	3, 1, 2	2	1
1, 1, 3	5/3	4/3	3, 1, 3	7/3	4/3
1, 1, 10	4	27	3, 1, 10	14/3	67/3
1, 2, 1	4/3	1/3	3, 2, 1	2	1
1, 2, 2	5/3	1/3	3, 2, 2	7/3	1/3
1, 2, 3	2	1	3, 2, 3	8/3	1/3
1, 2, 10	13/3	73/3	3, 2, 10	5	19
1, 3, 1	5/3	4/3	3, 3, 1	7/3	4/3
1, 3, 2	2	1	3, 3, 2	8/3	1/3
1, 3, 3	7/3	4/3	3, 3, 3	3	0
1, 3, 10	14/3	67/3	3, 3, 10	16/3	49/3
1, 10, 1	4	27	3, 10, 1	14/3	67/3
1, 10, 2	13/3	73/3	3, 10, 2	5	19
1, 10, 3	14/3	67/3	3, 10, 3	16/3	49/3
1, 10, 10	7	27	3, 10, 10	23/3	49/3
2, 1, 1	4/3	1/3	10, 1, 1	4	27
2, 1, 2	5/3	1/3	10, 1, 2	13/3	73/3
2, 1, 3	2	1	10, 1, 3	14/3	67/3
2, 1, 10	13/3	73/3	10, 1, 10	7	27
2, 2, 1	5/3	1/3	10, 2, 1	13/3	73/3
2, 2, 2	2	0	10, 2, 2	14/3	64/3
2, 2, 3	7/3	1/3	10, 2, 3	5	19
2, 2, 10	14/3	64/3	10, 2, 10	22/3	64/3
2, 3, 1	2	1	10, 3, 1	14/3	67/3
2, 3, 2	7/3	1/3	10, 3, 2	5	19
2, 3, 3	8/3	1/3	10, 3, 3	16/3	49/3
2, 3, 10	5	19	10, 3, 10	23/3	49/3
2, 10, 1	13/3	73/3	10, 10, 1	7	27
2, 10, 2	14/3	64/3	10, 10, 2	22/3	64/3
2, 10, 3	5	19	10, 10, 3	23/3	49/3
2, 10, 10	22/3	64/3	10, 10, 10	10	0

Finally, find the mean of all 64 values of s^2:

$$\frac{\sum s^2}{64} = \frac{\frac{2400}{3}}{64} = \frac{800}{64} = 12.5 \;.$$

Observe that this is the same value as σ^2 for the original population. The $(n - 1)$ correction is "just right."

If a specific three-element sample is randomly selected from our population, P, its sample variance might be much less than the population variance or it might be much larger. However, the mean of all sample variances is exactly equal to the population variance that the sample variances are intended to estimate. Because of this, statisticians say that s^2 is an "unbiased estimator" for σ^2.

Also, compute the mean of all sample means:

$$\frac{\frac{768}{3}}{64} = 4.$$

In other words, the mean of all the sample means is the same as the mean of the original population, P. This is accomplished without "tinkering" with the denominator. For this reason, the same definition of the mean $\left(\frac{\text{sum}}{\text{size}}\right)$, is used for both samples and the population. Statisticians say that the sample mean is an unbiased estimator for the population mean.

Challenges

1. Suppose that the values of s^2 of Table 2 were calculated using a denominator of 3, which is the sample size. Show that the mean of all the values of s^2 is no longer equal to the value of σ^2.
2. Replicate the steps of this article with a different four-element population.
3. Replicate the steps of this article for other size populations and other size samples.

Substitute natural numbers for the variables *a* and *b* so that the following expression is correct:

$(a + a) + 3(b + b) = a^a + b^a$