Seven Mathematical Processes in the Protocol: Activities Give Them Life

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The Common Curriculum Framework for K–12 Mathematics: Western Canadian Protocol for Collaboration in Basic Education (Alberta Education 1995), also known as the Protocol, specifies seven different mathematical processes: communication, connections, estimation and mental mathematics, problem solving, reasoning, technology and visualization. The Protocol (Alberta Education 1995, 5) states that

> There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and to encourage lifelong learning in mathematics.

It is clear that these components are critical to mathematics instruction in two ways: (a) the components are to be integrated into the mathematics program and activities, therefore a subject of instruction themselves; and (b) the components represent the most important aims of the mathematics program in that they relate to lifelong learning as opposed to specific concepts, facts or generalizations. This article will attempt to address each of these seven mathematical processes in turn and to provide classroom examples designed for the upper elementary classroom.

Communication

It is not difficult to build a case for the importance of communication in the classroom. Teaching and learning are primarily acts of communication; at least it is fair to say they are highly dependent on language. Consider the unique difficulties associated with trying to teach an individual who speaks a language different from your own, or consider the challenges associated with teaching problem solving to one who does not read or write. Skemp (1987) puts forward an excellent case for defending the role of communication and language in the mathematics classroom. As teachers, our task becomes one of deciding what forms of communication are most important, deciding how those forms can be taught and deciding which activities most effectively integrate with these elements of communication.

Activity: Dice Game (Problem)

Objective: Conduct probability experiments and explain the results, using the vocabulary of probability (Alberta Education 1995, 58).

Problem: Jane is playing a dice game with Frank. There are two dice to pick from and Jane gets first choice. One die is a regular six-sided die; the other die also has six sides, but contains three ones and three sixes. The game is played by rolling your die six times and summing the values rolled. The player with the highest sum after five rolls wins. Which die should Jane choose if she wants to win?

Discussion: Problem-solving activities provide a wonderful context for encouraging discussion and the use of mathematical terms. One common type of activity associated with a problem such as this is the comprehension guide (Stiff 1986). A comprehension guide is essentially a list of questions which students answer prior to actually seeking a solution. Obviously, the comprehension guide serves to reinforce the first stage of problem solving (understand the problem) which students often skip or dismiss. Questions which may appear on a comprehension guide include the following:

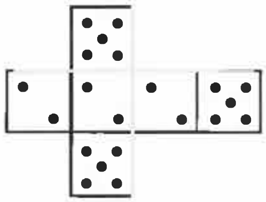
- In what ways are the two dice similar?
- In what ways do the two dice differ?
- How many times will each player roll his or her die?
- How is the winner for each game determined? Is it possible to have a tie?

Of course, questions such as these can be posed verbally and discussed in both large and small groups, but it is important for students to experiment with, and experience, both verbal and written forms of communication.

While Stiff (1986) talks about comprehension guides only as a means to reinforce the first stage of problem solving, comprehension guides can be extended to address the final stage (looking back) as well. Consider questions such as

- Is there another way to show which die Jane should choose?

- Assume the players will roll the die six times to play a game. How would this affect the die Jane should choose?
- Can you describe a rule which would help Jane decide which die she should choose?
- Assume the regular die was replaced with a die described by the net below. Which die should Jane choose now?



Many activities besides simple comprehension guides which emphasize communication skills. These activities are limited only by the imagination of the teacher. Consider short problem dramas (Matz and Leier 1992) where students act out a real-life situation which requires the application of some mathematical idea, process or principle. In a problem drama the actors freeze at some point during the presentation giving the audience time to find or compute a solution. The play is resumed to provide an answer to, and solution process for, the dramatized situation. Math journals also emphasize communication where students articulate daily their learnings, concerns or insights. Another option is the use of a math portfolio where students attempt to document their progress and select evidence of their mastery of concepts. The underlying purpose of these activities is to get students to articulate their thinking, and we can facilitate this by asking for information other than simple numerical answers.

Connections

There are many different types of connections which teachers must help students explore and develop during their study of mathematics. In order to develop a sense of mathematics in all of its facets students must develop connections between the following:

- Concepts and the physical world (applications to the world around them and to other disciplines)
- Concepts within mathematics
- Concepts and their representations (for example, manipulatives, models, diagrams, notations)
- Concepts and their related terms and algorithms

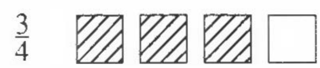
Each of these forms of connections individually represents one way to bring meaning to the learning of mathematics (or, more accurately, build understanding of mathematics). However, it is probably true that the greater the number of these connections which students hold, the more robust their learning and understanding of those concepts. In other words, by helping students build a variety and multitude of connections we help them build understanding.

Activity: From Fractions to Percents (Exploration)

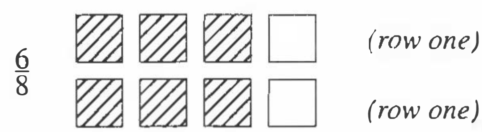
Objective: Demonstrate and explain the meaning of percentage concretely, pictorially and symbolically (Alberta Education 1995, 31).

Materials: Two-color counters

Description: In this exploration we are attempting to connect the concept of simple fractions to that of percents through the use of equivalent fractions. To start the activity, we may ask students to use their two-color counters to show a fraction of three-quarters:



The students are now asked to write the fraction and then add another row, to create:



The new fraction created is recorded and some discussion of how the new fraction (six-eighths) is similar to the first fraction (three-quarters) is conducted: the notion of equivalent fractions is emphasized. Students are asked to predict the next equivalent fraction, and, once a pattern is discovered, students are asked to generate the equivalent fraction that uses a total of 100 chips. Some students may wish to build the completed model.

Discussion: When working with models such as this, it is important to ask students to reflect the model in mathematical symbols at the same time the model is developed, and to ask students where each symbol within the notation can be found within the representation. For example, the mathematical equation reflected in the model above is

$$\frac{3}{4} \overset{\times 2}{\underset{\times 2}{=}} \frac{6}{8}$$

Students should be able to show what part of the model is represented by the 3, 4, 6 and 8, as well as 2.

Estimation and Mental Computation

Estimation is the ability to mentally approximate a value or measurement, while mental computation entails completing full computations without the aid of pencil, paper or computational aid (for example, calculator, computer and so on). Teaching estimation requires teaching skills such as rounding, and the notion of what is a reasonable, acceptable or tolerable estimate. Due to the need to address levels of acceptability in estimation activities, quick computations are often made using calculators or computers. These calculations are used to compare between the actual answer and the estimate; the use of the technological aid simply speeds feedback. Familiar estimation strategies include the front-end strategy, clustering, rounding, compatible numbers and special numbers (Reys 1986).

Unlike estimations, mental computations typically require exact answers. Teaching students to compute mentally involves teaching a variety of routines which can be carried out easily in the head. Many different mental computation strategies exist, including dropping common and trailing zeros, balancing, doubling, division by multiplying, thinking money and trading-off (Hope, Reys and Reys 1987).

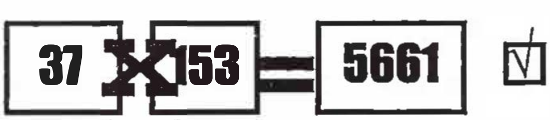
Estimation and mental computation strategies need to be consciously and carefully taught—-on average, students do not seem to develop them readily or independently. These skills should probably be addressed daily and along with the other six mathematical processes integrated into daily activities.

Activity: Target 5,000 (Calculator Game)

Objective: Estimate, mentally calculate, compute or verify, the product (three-digit by two-digit) and quotient (three-digit divided by one-digit) of whole numbers (Alberta Education 1995, 32).

Materials: Gameboard (see Figure 1), calculator, pencils

Rules: The first player selects a two-digit number and enters it into the first box on his or her opponent's side of the gameboard (in the example below, the first player entered a 37). The second player now enters a value in the second box (in the example the opponent entered a 153). The second player now takes the calculator and computes the product of the two numbers entered and records it on the score sheet (the product is 5,661). The players now reverse roles on the second side of the score sheet. The player whose product is closest to 5,000 on a given round scores a single point by checking the small box to the right of the equation. The player who collects the most points in five rounds wins.



Discussion: After playing the game a number of times, it may be worth talking to the students about strategy in the game. Students quickly learn that certain numbers are unwise to give to an opponent (for example, 10, 25 and 50) as a second multiplier can be easily identified which provides an exact product of 5,000. The teacher may find it interesting to discuss the different strategies which students employ in trying to find the best possible match. For example, students may implement a rounding strategy, and combine that strategy with a mental computation division process or even employ a "delete trailing zeros" strategy to simplify necessary computations. All the different strategies students employ are worth articulating, as undoubtedly a surprising variety will emerge.

The teacher may opt to extend and adapt Target 5,000 in a number of ways. For example, the game can be made more difficult by allowing students to enter decimal values or by changing the target number to 50,000. The game can be further adapted by asking the first player to provide his or her opponent with the first multiplicand *as well as* the target number—this adaptation significantly increases both the challenge and the range of mental computations which must be made.

Problem Solving

Problem solving can hardly be considered a new addition to this or any other program as it has been promoted as a focus of mathematics instruction for many years. However, the biggest barrier to classroom implementation remains the collection of curricularly relevant problems and techniques for integrating such problems into daily classroom activities. A further challenge plagues the classroom teacher in trying to collect a range of problems: problems should range not only in level of difficulty but in the problem-solving strategies and skills which these problems address. Problems should also range according to their degree of open-endedness as well. In other words, variety is key: we want our students to experience many problems, some of which may have a single correct answer, and some of which may have many different equally acceptable answers and solutions. These pedagogical challenges, though significant, are certainly worth meeting, as problem-solving abilities remain one of the most important and generalizable learning outcomes of the mathematics program.

Activity: Floor Tile Problem (Application)

Objective: Cover a surface using one or more tessellating shapes (Alberta Education 1995, 52).

Materials: Floor Handout (see Figure 2), calculator, pencils, colored pencils, pattern blocks and/or pattern block stickers or cutouts (pattern block shapes cut from appropriately colored construction paper)

Description: In this activity students are given a large rectangle which represents the floor of a room (see Figure 2) and a collection of pattern blocks which represent floor tiles. The students are informed that each hexagon tile costs 8¢, trapezoid tiles cost 5¢, blue parallelogram tiles cost 4¢, triangle tiles cost 3¢ and white parallelogram tiles cost 3¢ (these values are also found printed at the bottom of the handout—Figure 2). The students are given the challenge of covering the entire floor space with tiles creating the most pleasing design or pattern for the least cost.

Discussion: This application-style problem is highly open-ended. In this problem students can elect to cover the entire floor space with a single pattern block, or may choose to use several of any number of the six regular shapes. What makes the activity interesting is that (a) students employ their own preferences in color and shape, and (b) there are two variables which are to be considered in the overall product: esthetic value and cost. The activity can be easily extended by adding other criterion. For example, the following added condition changes the problem significantly: tiles can only be purchased in packages of five and once a package is opened all five tiles must be purchased.

Reasoning

The Protocol document does not actually define reasoning, but lists some of the related abilities, including the ability to make sense of mathematics and the ability to build or generalize mathematical ideas from past experience. We could describe reasoning as the mental ability to justify a given concept from prior knowledge (to re-generate an idea), or as the mental ability to construct and justify a new idea from present and past experience (to generate a *new* idea). In essence, reasoning has both generative and regenerative properties. It makes sense that this act of reasoning can only occur when one has control over his or her thought processes, implying one must possess both general cognitive processes (for example, the ability to sort, classify, generalize, curtail and so on) and related metacognitive processes (that is, an awareness of and ability to monitor/control the general cognitive processes). In general, when we say we wish students to be able to reason, we want them to be able to justify their ideas by relating them to prior knowledge and related concepts—to be able to identify the clues which substantiate the truth of a concept or idea.

It is not difficult then to see how closely related reasoning is to the notion of connections and language, both discussed above. To be able to reason, one must be able to establish relationships between present concepts and prior knowledge, and these relationships are simply the connections we hope students will develop. Further, evidence of reasoning is most likely observed in the act of communication where one tries to persuade or dissuade another of a given idea or concept. This should further reinforce to us the importance of both communication and the development of connections in the instruction of mathematics.

But, how do we address reasoning in the classroom? One of the most obvious methods would be to ask "why" questions:

- *Why* do you think this is true?
- *Why* don't you think this result is correct?
- *Why* do you think that strategy would be appropriate?

Why questions seem to seek justification and clarification and thus encourage reasoning. Ideally we hope students will begin to internalize the asking of such questions thus making the acts of justification and clarification learner characteristics. A second obvious method would be the use of logic puzzles and games. The challenge with such activities however is to ensure that they are curricularly relevant (that is, based on objectives drawn from appropriate curriculum documents).

Activity: Klondike (Game)

Objective: Place an object on a grid, using columns and rows (Alberta Education 1995, 52).

Materials: Klondike Gameboard (see Figure 3)—one per player, Klondike Leaderboard (see Figure 3), pencils

Description: This game is played on a 10 × 10 grid. The objective of the game is to find the 15 gold coins which are hidden on this grid using the clues given by the leader (usually the teacher). Players are divided into groups or teams. The teams take turns calling out a single location on the grid. If a gold coin is hidden in that location, the team scores a single point. If no coin is hidden in that location, the teacher informs the players of the number of gold coins which are hidden in any of the adjoining locations. For example, using the leaderboard found in Figure 3, if the team called out location "column 5, row 2" the teacher would inform them that there were two coins hidden somewhere within the eight locations surrounding the location called. A team may only call one location on a turn whether or not they find a coin. The team that finds the greatest number of coins once all coins have been located is the winning team.

Discussion: Students may need some coaching on how to use the gameboard itself and on how to use the results generated by other teams. It may be useful to make an overhead copy of the student gameboard and model the use of the elimination strategy in identifying locations where no gold coins are hidden. This game could be extended by using locations on a coordinate system rather than a grid and further extended by including more than the first quadrant on the gameboard.

Technology

The sheer availability alone of technological tools such as the calculator and computer demands that teachers find a place for them in classroom instruction. There are, of course, many opportunities in the mathematics classroom for the application of the calculator and computer, but caution is necessary. Our challenge is to look for ways to use the calculator and computer such that the application positively affects the learning outcomes of students. In other words, we want to use the calculator and computer in such a way that students gain experiences that they would not otherwise have. As well, calculators and computers can be used to provide a contrasting or alternative perspective to a concept. We want to use the calculator and computer in support of student thinking processes, not to replace them. In this sense, these tools provide many interesting options especially in the realm of problem solving, primarily (a) where limitations in student computational ability can be overcome and (b) where the use of these tools enables students to consider and explore a greater range of more complex data within problems.

Activity: Race to 100 (Problem or Game)

Objective: Verify solutions to addition and subtraction problems, using estimation and calculators (Alberta Education 1995, 34).

Materials: Calculator, paper, pencil

Description: This activity can be conducted as either a game (a race) or a problem-solving exercise. In this problem, each letter of the alphabet is assigned a value based upon its position within the alphabet. The letter A has a value of 1, the letter B has a value of 2, C has a value of 3 and so on (see the chart below). The value of a word is determined by the sum of the values of its letters. The word "mathematics" has a value of 112. Can you find a word that has value of exactly 100? To make this a game, have a race to find out who can be first to find such a word.

A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7
H = 8	I = 9	J = 10	K = 11	L = 12	M = 13	N = 14
O = 15	P = 16	Q = 17	R = 18	S = 19	T = 20	U = 21
V = 22	W = 23	X = 24	Y = 25	Z = 26		

Discussion: One can quickly see why working with a calculator makes solving this type of problem possible: the amount of information and the degree of guess and check which is likely to accompany it can be quite overwhelming for many students. This problem can be presented as a problem of the day, or can be used in an ongoing manner as students create a graffiti board or bulletin board display of the different words which have been found. The problem can be adapted to include such extensions as finding words with values for every number 1 through 100 (are any values impossible?). Other extensions include the following: What is the longest word you can find with a value of 100? What is the shortest word you can find with a value of 100? Who can find the word with a value of 100 which has the most vowels? Whose name has the greatest value?

Hint: Try computing the values for the days of the week.

Visualization

The Protocol document borrows from Armstrong (1993 in Alberta Education 1995, 11) when it defines visualization as "thinking in pictures and images and the ability to perceive, transform and recreate different aspects of the visual-spatial world." Clearly imagination and the ability to reconstruct concepts in a concrete and semi-abstract form is critical to visualization. We recognize visualization as an important mathematical process in that the mental pictures students create at least partially enable the transfer of learnings from one context to another and more generally, represent one mode of knowing or acquiring new knowledge. To help students develop visualization skills the teacher could explore the consistent use of models, pictures, graphs and diagrams in the classroom. Along with the introduction of these models, students should also be requested to create such representations themselves to explain and describe how they perceive the concepts they encounter.

Activity: Negotiation (Game)

Objective: Demonstrates concretely, pictorially and symbolically place-value concepts to give meaning to numbers up to 10,000 (Alberta Education 1995, 28).

Materials: Overhead set of base ten blocks (see Figure 4), paper, pencil, overhead projector

Description: To begin this game, all players are asked to secretly record a three-digit number which contains no zeros (for example, 187 or 251). These numbers are held privately and not shared with any other player until the conclusion of the game. Once numbers are recorded, players take turns adding any one block (cutout) to the overhead projector (for example, if the first block added is a flat block the value shown on the screen is 100). Alternatively, a player may opt to remove any one block on a turn. In order to remove a block, there must be at least one of that size block on the screen (that is, there is no regrouping). If at any time the screen displays a value equivalent to the number recorded secretly at the beginning of the game, the player who wrote that number is the winner.

Discussion: In this game, the value on the screen is never verbalized until the conclusion of the game. Students must construct the value through the visual representation constructed. To dramatically increase the challenge of the game the teacher may omit the overhead base ten blocks, thus forcing students to visualize (and construct a list of) the values built during the game. To further adapt the game, have students select four-digit rather than three-digit numbers.

Conclusion

The seven mathematical processes represent the most important learning outcomes of our mathematics curriculum. These processes can be encouraged and addressed directly through a conscientious effort to integrate them within activities designed to address the more specific learning objectives given for each grade level. It is necessary to know and understand these mathematical processes as we work to select and collect curricularly relevant activities which can be successfully introduced into the mathematics classroom.

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Figure 1: Gameboard for Target 5000

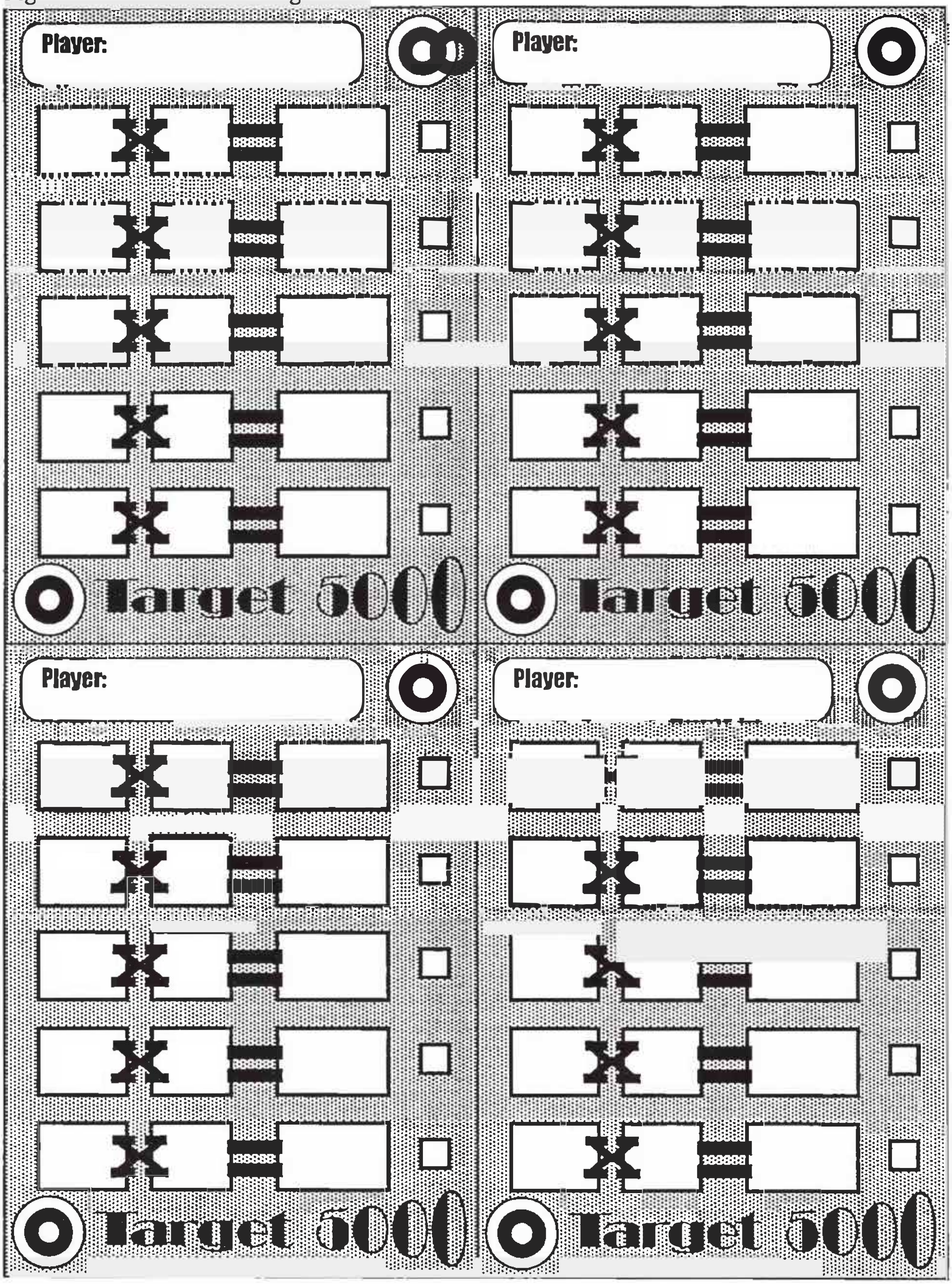


Figure 2: Floor Handout for Floor Tile Problem.

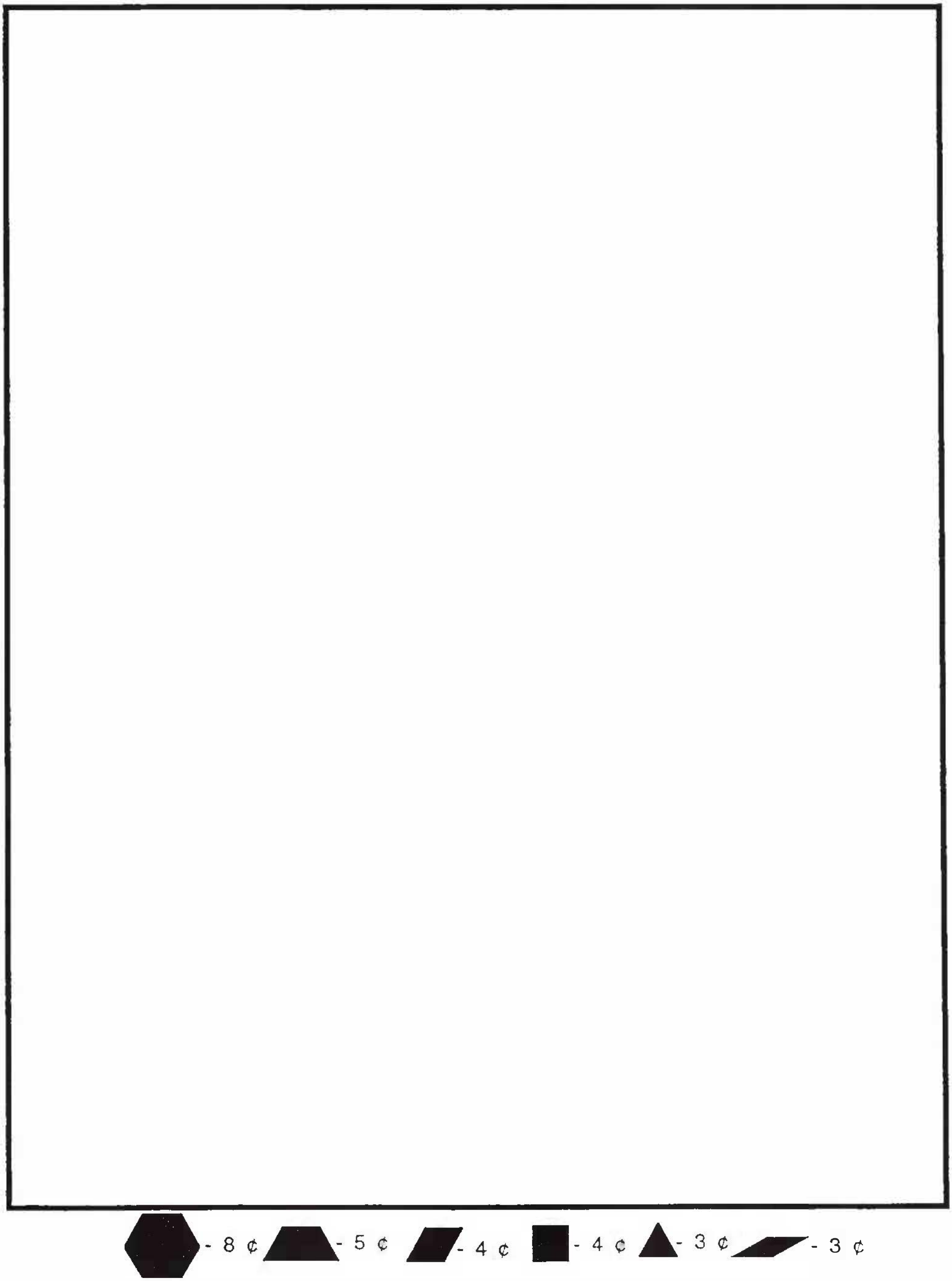
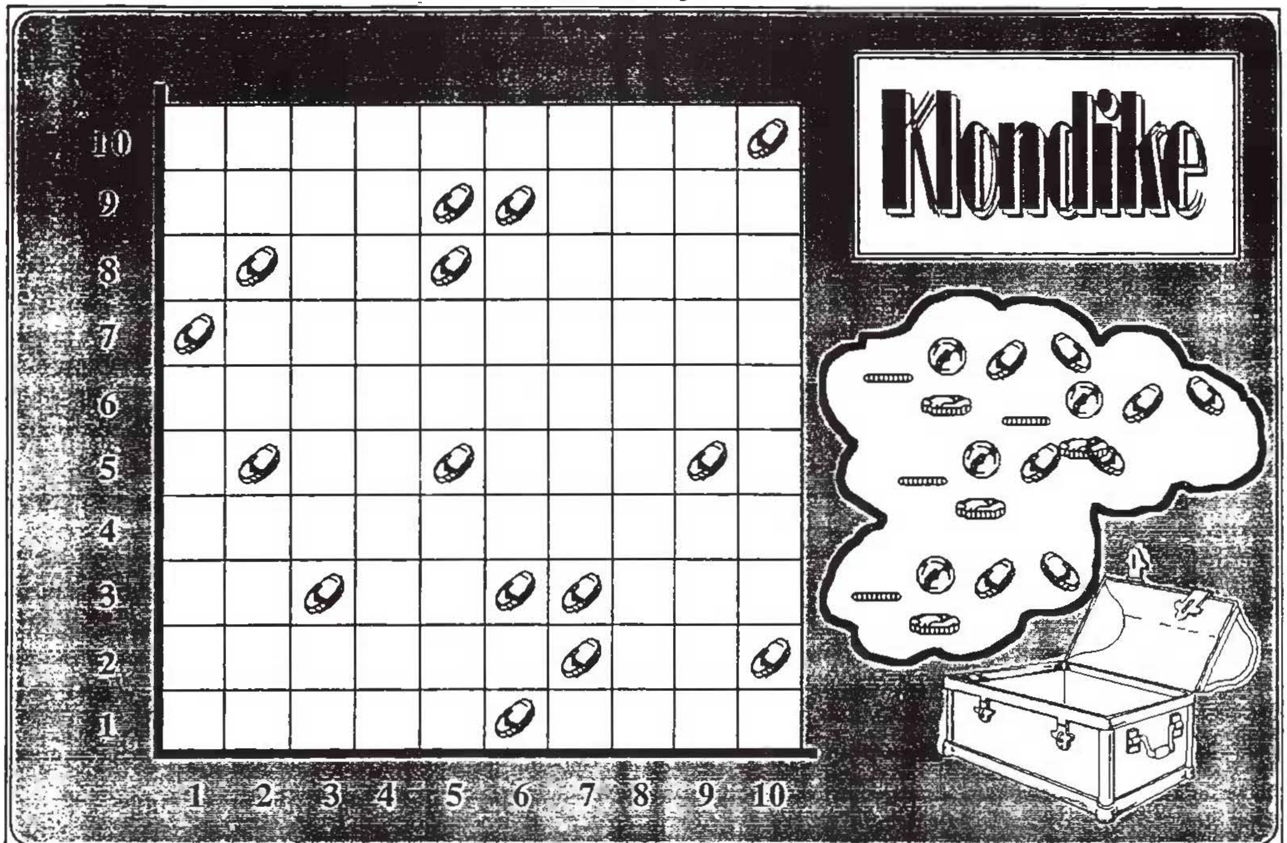


Figure 3: Leaderboard and Student gamecard for Klondike game.



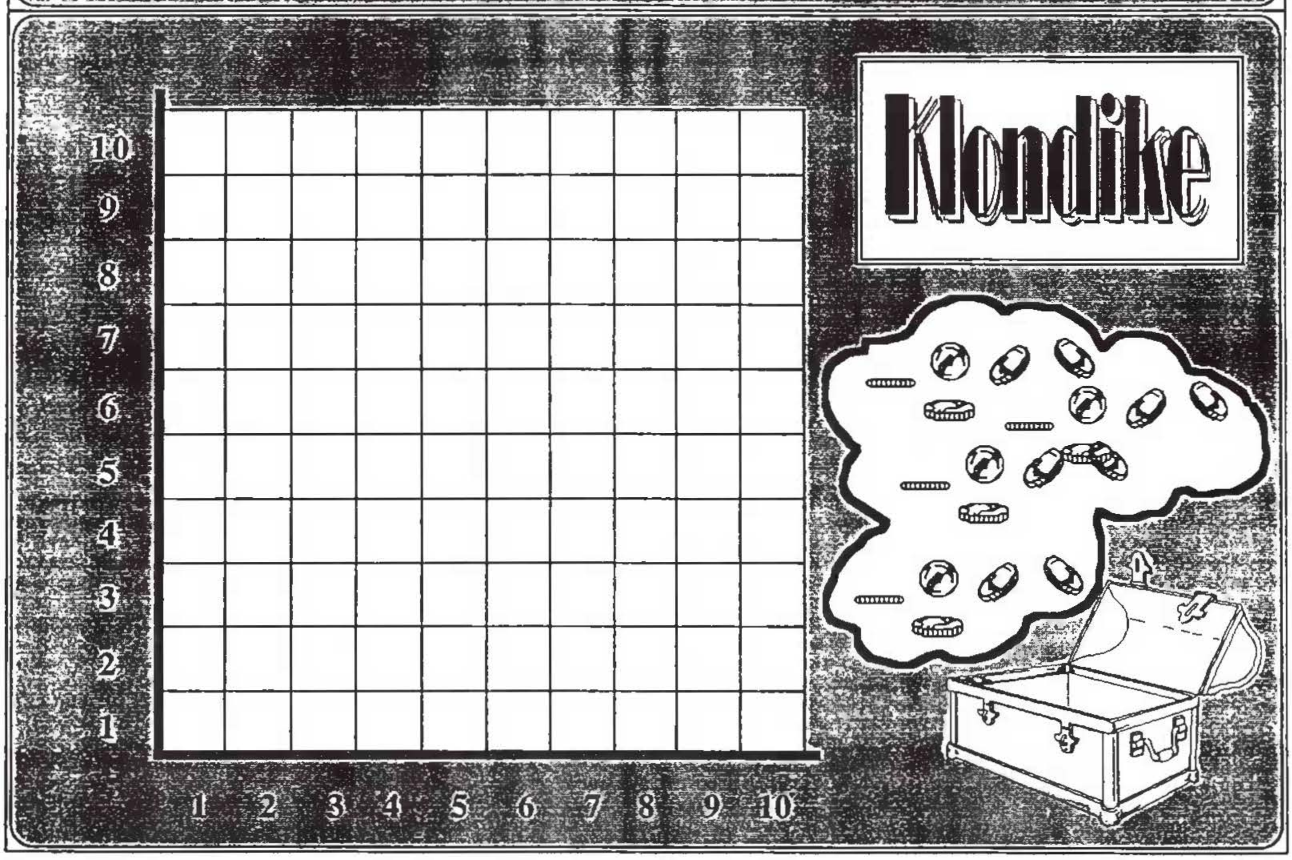


Figure 4: Base Ten Block cutouts for Negotiation (to be made into overhead transparency).