delta-k

JOURNAL OF THE MATHEMATICS COUNCIL OF THE ALBERTA TEACHERS' ASSOCIATION

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Volume 34, Number 1 May 1997

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FROM YOUR COUNCIL

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Probability

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GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. All contributors are encouraged to submit their manuscripts on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. Letters to the editor or reviews of curriculum materials are welcome.
7. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. Send manuscripts to Klaus Puhlmann, Editor, P.O. Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 34, Number 1 May 1997

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 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Klaus Puhlmann, P.O. Box 6482, Edson, Alberta T7E 1T9. EDITORIAL AND PRODUCTION SERVICES: Central Word Services staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in the Canadian Education Index. ISSN 0319-8367

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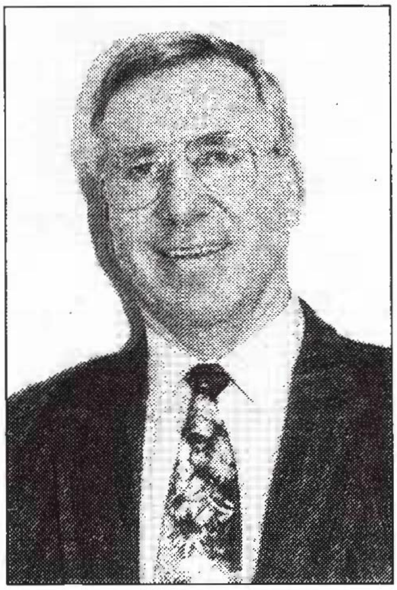
EDITORIAL



As I begin my term as editor, I would like to express my appreciation to Art Jorgensen and to all the editors before him, for their diligent work in assembling each issue of *delta-K*. Their hard work created a mathematics journal that is recognized by mathematics educators well beyond Alberta. It is my hope that I can build on this strong foundation.

As I set out to further enhance the quality of this journal and to develop it as an important vehicle of communication, I realize that its effectiveness can only be maintained or enhanced if the information it contains is relevant to our readers. It is not always easy to achieve relevancy for *all* readers, but we should certainly work toward that goal. However, with your help, it can become an indispensable resource for mathematics educators.

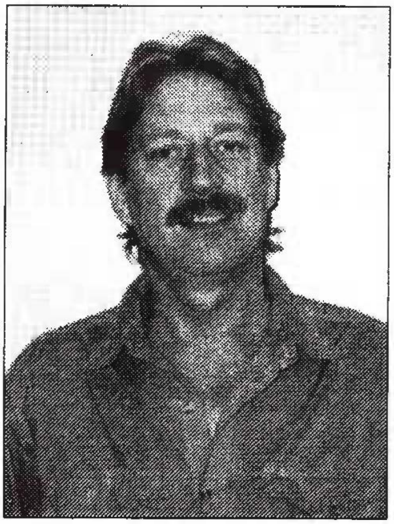
I know that many good things are happening in the mathematics classrooms of our schools, colleges and universities. It is important that they do not go unnoticed and that they are shared. It is in that spirit that I invite all of you—teachers, students, graduate students, university and college teachers—to submit items for inclusion.

delta-K must become a journal written by teachers for teachers. To achieve this, the table of contents has been revised to include additional sections to enhance your opportunities to participate. One such section called "Reader Reflections" allows you to reflect, react or comment on anything in the journal. "Student Corner" invites submissions from students. The section "Teaching Ideas" should simply overflow with your submissions because I know that many good ideas are out there. If you feel that resource reviews are important to be shared, then such a section could be established as well. Information from MCATA and, of course, more general—as well as recreational—mathematics articles will continue to be major sections in *delta-K*.

Please let me know how we can make *delta-K* an even better journal, one that facilitates conversation about mathematics education in Alberta and beyond. Remember—your submissions, comments and reactions are always welcome.

Klaus Puhlmann

President's Report 1995–96



This report was presented at the AGM, at the Capri Hotel in Red Deer, November 2, 1996.

It has been a pleasure to serve as president of MCATA over the past year. The main services and activities to our membership of 422 were the following: the annual conference in Lethbridge in September 1995, the resource fair in Calgary in May 1996, and publications of *delta-K* and the *Mathematics Council Newsletter*. MCATA continues to financially support the following: junior high school mathematics contests in Edmonton and Calgary, and the Alberta High School Mathematics Contest; teacher inservice projects; and teaching and learning materials.

The executive met five times during the year. We established and worked toward goals that we felt were imperative for the successful operation of the Council. For the consideration of the general membership, the executive has a revised constitution and executive schemata which we feel better reflect the current realities of this council at this time and into the future.

The challenges faced by the executive have been interesting, informative and enjoyable. The opportunity to work with this dedicated group of professionals has been personally rewarding.

A number of people have been influential in the success of the Council over the past year. My personal thanks to Wendy Richards, past president, and Dave Jeary, ATA staff advisor, who both provided needed direction and guidance when requested. Donna Chanasyk, secretary, and Doug Weisbeck, treasurer, have been totally committed to positions that offer very little in profile but without their work and dedication would render the workings of the Council somewhat ineffective in comparison.

Directors responsible for assignments from conferences, publications, issues and regionals to public relations—Cynthia Ballheim, Linda Brost, Daryl Chichak, Cindy Meagher, Betty Morris and Sandra Unrau—have unselfishly given of their time and talents to support the efforts of the executive. The publication editors, Art Jorgensen and Klaus Puhlmann, have provided expert publications for our membership.

By masterminding the successful "Math: Making Connections" conference, Graham Keogh demonstrated his ongoing dedication to the professional development of our colleagues. Margaret Marika, 1997 conference chair, is looking forward to similar successes which we are sure she will experience. Additional representatives and advisors—Dale Burnett, Mike Stone, Kay Melville and PEC Liaison Carol Henderson—have each contributed valuable input and guidance in our work over the past year.

Congratulations to Vice President Florence Glanfield, who was appointed to Chair of the Regional Services Committee with the National Council of Teachers of Mathematics (NCTM). Florence follows in a line of Canadians who have received the same honor. Two years ago, Richard Kopan, also a vice president, fulfilled the same responsibility with NCTM.

Your MCATA executive members attended a number of different conferences during this year. MCATA was represented at the NCTM annual conference and delegate assembly in San Diego in April by Betty Morris. As mentioned, Florence Glanfield serves as chair on the Regional Services Committee of the NCTM. Cindy Meagher and I attended the NCTM Canadian Leadership Conference in Montreal in July. Margaret Marika attended the ATA Summer Conference in Banff in August. Thanks to all the executive members who gave up their valuable holiday time to attend these conferences.

In closing, I wish to thank all members of the executive. We can all be very proud of what has been accomplished.

George Ditto

From the Education Director

Report from the NCTM Canadian Leadership Conference
Montreal, Quebec
July 3–6, 1996

Wednesday

A wine-and-cheese event was hosted by Groupe des responsables en mathématiques au secondaire (GRMS). The social setting allowed us to meet the other representatives from across Canada (and two people from the United States).

Thursday

The day began with introductions, followed by an activity in which we were to write an ad that would identify the qualities of a leader we would hire to chair our new mathematics group. This activity was concluded through a discussion of what we did in our provincial organizations to promote leadership, and offered words of wisdom to share with new associations.

We then shared successes and concerns of our individual organizations, thereby generating a list for the entire group. We proceeded to look more closely at the *Handbook for Affiliated Groups of NCTM* and reviewed various sections. We looked specifically at the NCTM Financial Services section which dealt with underwriting for conferences, joint membership drives and grant applications. The grant application section was particularly interesting and will continue to be useful to MCATA.

Friday

The morning began with groups actually writing grant proposals on a variety of topics. The proposals were exchanged and critiqued.

The remaining part of the day was used to review the roles and responsibilities of NCTM representatives. Suggestions on how to be involved with NCTM working committees were given.

The Ontario representative who handles all purchasing of NCTM materials from the catalogue explained his almost hassle-free system.

Saturday

The day was devoted to assessment. Groups were involved in a performance-based assessment task and discussed how achievement of the task should be assessed. Although agreement was not always reached, the opportunity allowed for great discussion of ideas.

The NCTM Assessment Standards were briefly discussed.

The representative from Vermont shared with us their approach to portfolio assessment.

General Information

- Copies of newsletters from each group were shared.
- Copies of the NCTM Affiliated Group Survey Response Form were shared. (I have all the copies if anyone would like to see them.)

Overall Comment

We had a jam-packed agenda to cover in a short time. The activities were all presented in a unique and entertaining way. I am now aware of the responsibilities of an NCTM representative and the benefits of being affiliated with the NCTM. The highlight of the conference was the wealth of infonnation I gained from colleagues around the country. Thank you for this marvelous experience.

Cindy Meagher

The Right Angle

Evaluation Branch

Mathematics 30 Diploma Examination Program

The following items are new for the 1996–97 school year:

1. There was an extra administration of the Mathematics 30 Diploma examination on November 8, 1996. Times, locations and so on, are published in the 1996–97 General Diploma Information Bulletin.
2. A Guide for students was published this fall in all the mathematics and science diploma examination subject areas.
3. The examination design for Mathematics 30 has changed for the 1996–97 school year in that the numerical-response questions are dispersed throughout the multiple-choice questions, placed according to content topic.

Please note that the Policy: Use of Scientific Calculators on Alberta Education Diploma Examinations is the same for all mathematics and science diploma examinations, and has not changed from the 1995–96 policy. A copy of this policy is found in the 1996–97 General Diploma Information Bulletin. It was inadvertently omitted from the Mathematics 30 and Physics 30 diploma bulletins during an electronic transfer printing problem.

Achievement Assessment Program

Information Bulletins reporting the results of the Grades 3, 6 and 9 June 1996 Achievement Assessments and providing information regarding the June 1997 assessments were sent to schools in November.

Development of the Achievement Tests will occur as they have for the last two years. Approximately two-thirds of the previous year's items are reused on the test and the rest are new.

In response to the changes in curriculum, the Grades 3 and 6 Achievement Tests will be blueprinted to the interim program of studies and will test outcomes that are common to the interim and protocol mathematics programs of studies.

The Grade 9 Achievement Test will be blueprinted to the general outcomes of the Protocol Program of Studies, June 1996. To assure fairness of the Grade 9 Achievement Test, extensive reviews and validations of the curriculum standards are being conducted.

Alberta Distance Learning Centre

Grade 7 Student Mathematics Package

A student package for Grade 7 Mathematics that matches the common curriculum (product number 311069) has been completed. The package includes seven modules and assignment booklets. There is a separate Learning Facilitator's Manual (product number 311035). Both may be ordered from the Learning Resources Distributing Centre (LRDC).

The materials for Mathematics 8 and Mathematics 9 are currently being developed and will be available for the 1997–98 school year.

Math 10 Beta CAI Program

The Math 10 Beta Computer Assisted Instruction (CAI) program is available for beta testing.

CAI mathematics programs are Computer Assisted courseware that provide self-paced interactive lessons containing instruction, examples, questions and self-administered tests. This courseware can be used on stand-alone or networked Macintosh microcomputers. The CAI programs address all student outcomes of the Alberta Mathematics Program of Studies. Some additional student outcomes are covered.

Beta software is prereleased software. Our CAI products go through three major releases: alpha, beta and final (production) version. The alpha release is internal developmental software. The beta release (or prerelease) is provided to selected sites for testing of the software. The production version is released after corrections and other enhancements have been made to the beta version.

If you are interested in becoming a Math 10 beta tester contact the Instructional Technology and Media Unit, Alberta Distance Learning Centre, Box 4000, Barrhead T7N 1P4.

Question Banks

Alberta Education provides question banks for many junior high and secondary school subjects for use with LXR TEST software. LXR TEST is a question storage and retrieval system available for both Macintosh and Windows. Using our question banks and LXR TEST, production of randomly generated or completely customized tests is both quick and easy. Teachers appreciate the full solutions included in our questions banks. The Scoring Edition of LXR TEST includes scoring and analyzing features.

Current question banks contain over 40,000 items. The following is a list of Mathematics Banks available including the approximate number of questions in each bank: Math 10 (2,616), Math 13 (1,735), Math 20 (1,786), Math 20–French (832), Math 23 (880), Math 30 (1,755), Math 31 (1,127) and Math 33 (1,198).

The Alberta Distance Learning Centre Question Bank CD-ROM is available from LRDC, 12360 142 Street NW, Edmonton T5L 4X9; phone 427-2767, fax 422-9750. The Buyer's Guide Number is 324153 (Macintosh) or 324161 (Windows). The cost to educators or individuals in Alberta and NWT is $35 plus GST.

Curriculum Standards Branch

An excellent publication that teachers should use with parents entitled "Working Together in Mathematics Education" is now available. The booklet provides an overview of the new mathematics curriculum and shows some of the knowledge, skills and attitudes students are expected to learn. It presents some ways parents and others can support student learning in mathematics. For more information, contact Dennis Belyk, acting assistant director, Math/Science Unit, Curriculum Standards Branch, Alberta Education, Devonian Building, West Tower, 11160 Jasper Avenue NW, Edmonton T5K 0L2; phone 422-3216 or fax 422-3745.

This, and other curriculum handbooks for parents, may be purchased at cost from the LRDC in packages of 20 for $15. It may be viewed at and downloaded from the Alberta Education Web site, (Parents section) at http://ednet.edc.gov.ab.ca.

Kay Melville

There is something very interesting about the numbers 12 and 60. Their product is 10 times as big as their sum: $12 \times 60 = 720$; $12 + 60 = 72$. Are there other natural number pairs for which this is true?

From the 1996 Conference Chair

Math: Making Connections

On November 1–3, 1996, a successful MCATA annual conference was held in Red Deer at the Capri Centre. On Friday, a preconference math symposium attracted 50 delegates to discuss math issues for the coming year. Conference delegates started to arrive Friday night and enjoyed connecting with old friends at a drop-in social. As delegates registered, they took in the opportunity to look over the "Bright Ideas" display which was located behind the registration table. The display left no doubt they were attending a math conference! A popular gathering point proved to be the large display board which showed the sessions that were being offered at the conference.

By Saturday morning, 385 delegates had registered for the conference and were enjoying the opening address by Pat Rogers from York University. Sessions continued throughout the day until 3:30 p.m. As well as attending some of the 60 sessions offered at the conference, covering K–12 and beyond, many delegates attended the "Make It— Take It" sessions on Saturday afternoon. Publishers and other displayers provided an excellent opportunity for delegates to obtain the latest products to take back to the classroom Monday morning. A number of delegates also visited Notre Dame High School in Red Deer on Saturday afternoon to see the latest in "high-tech schools" in Alberta.

Sessions that seemed to attract the most interest were those dealing with the changes in math education brought about by the Western Canadian Protocol and the new Math 33 program. Assessment issues attracted many delegates; the math-tech prep program that is being developed in central Alberta also attracted a lot of interest.

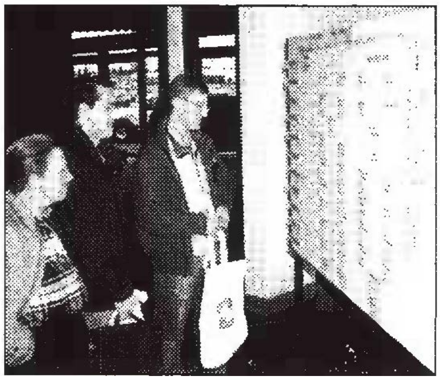
By Saturday night it was time to unwind. An enthusiastic group of delegates took in the comedy entertainment. Due to poor weather overnight some delegates left early, but most stayed for the Sunday sessions. The conference closed with a keynote speech by Katherine Heinrich from Simon Fraser University. And, due to the generous support of our sponsors and displayers, many delegates returned home with some extra gifts.

Graham Keogh

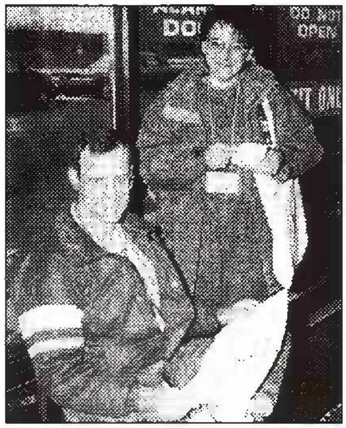
Graham Keogh, 1996 Conference Chair

Colleen Williamson, Program Chair

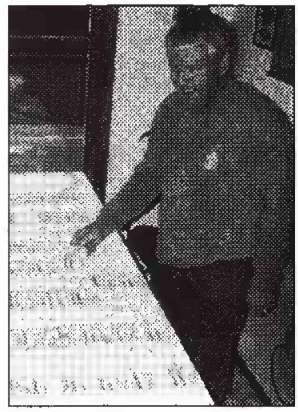
1996 Conference Photos



Participants from Fort McMurray: Claudette Guimond, Michael Jodoin and Derek Brown

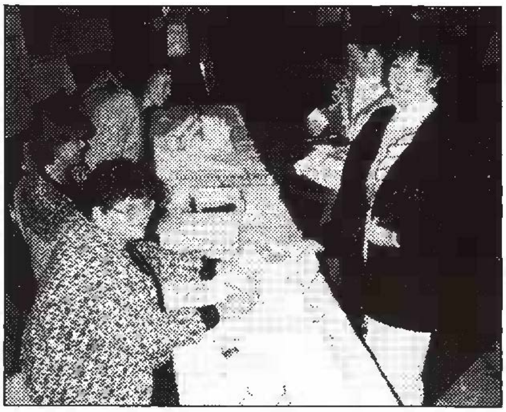


Larry McGovern (St. Michael School, Calgary) and Georgene Miko (Calgary)



Pat Cavanaugh preparing the display board.

Laurie Berezowski (Rimbey) registering with Carol Davis.

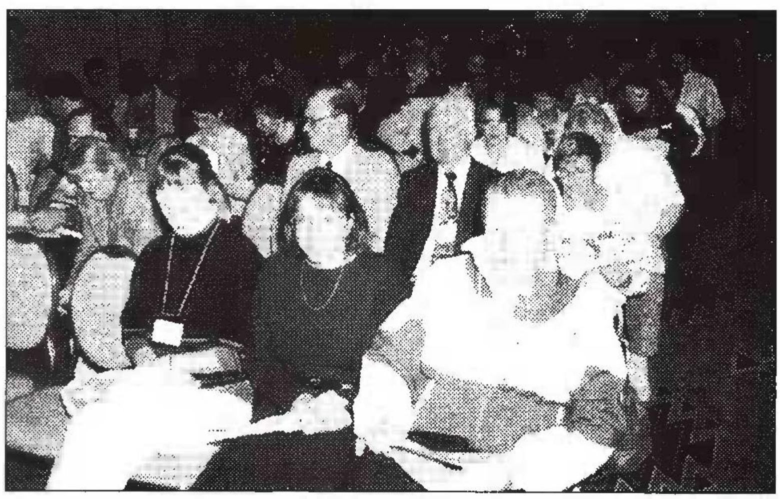


Anil Padayas (Spirit River) receives his registration materials.

Keynote Addresses

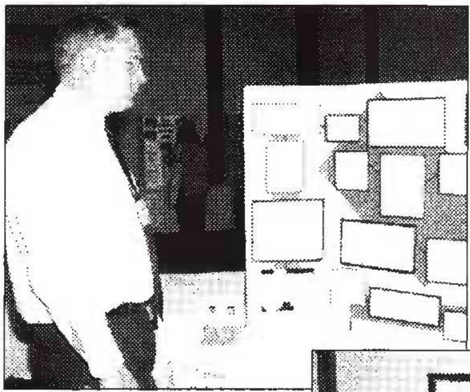
Keynote Speaker Pat Rogers

Keynote Speaker Katherine Heinrich



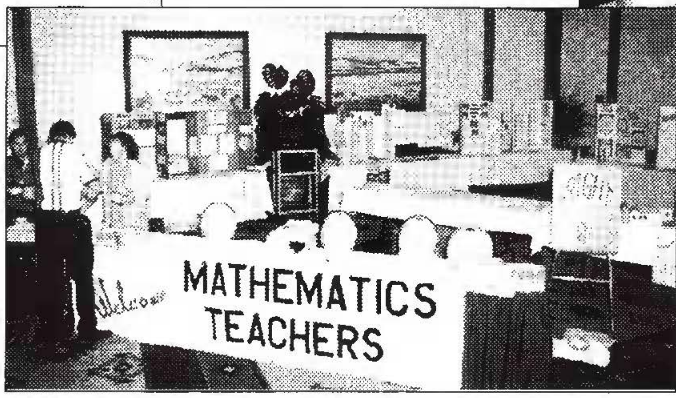
Audience at keynote—recognize anyone?

Bright Ideas

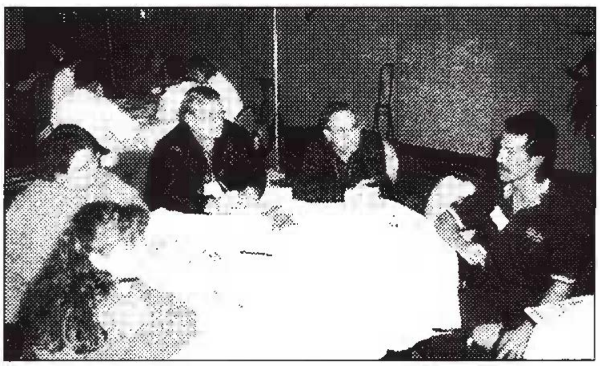


Roland Gauthier (E. W. Pratt High School, High Prairie)

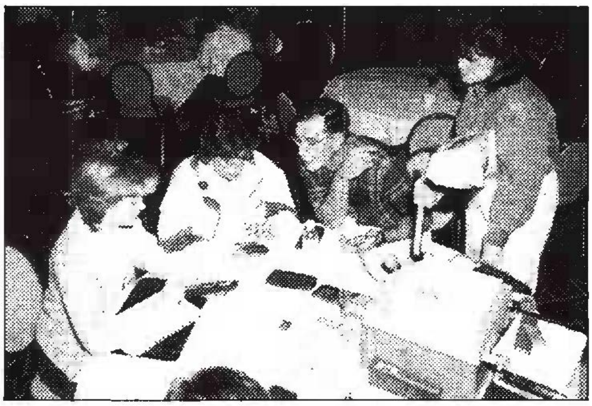
Joe Milne (Brownfield)



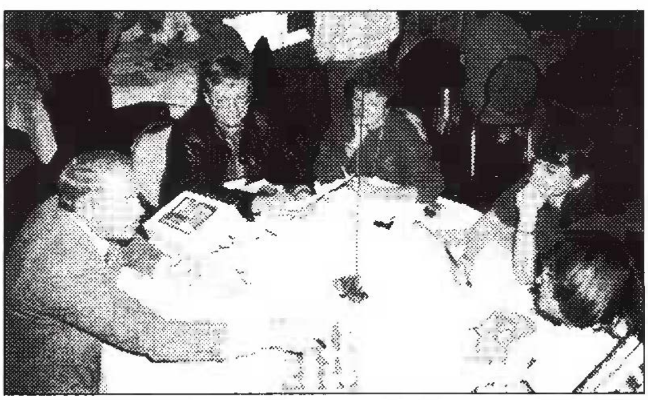
Math Fair: Make It—Take It



Gerry Varty discusses statistics analysis using a computer with interested group members.



Lisa Steel and Bob Hart demonstrate the use of Texas Instruments calculators.



John Morrow from Spectrum demonstrates Probability Kits.

Katie Pallos Haden shows off fraction blocks to her group.

1996 Outstanding Mathematics Educator Award: Evelyn Sawicki

Liz Donovan and Evelyn Sawicki

The following is the text of the award presentation at the annual conference in Red Deer, November 1996.

It is my pleasure to introduce to you this year's recipient of the Outstanding Mathematics Educator Award. As you may know, this award is bestowed upon those individuals who provide leadership and encouragement for the enhancement of teaching, learning and understanding mathematics.

This year's award-winner is Evelyn Sawicki. Evelyn is an outstanding individual and a knowledgeable and insightful leader in the area of mathematics. She is recognized by her peers as an exceptional mathematics resource person and it is most fitting that she is being recognized with this award. I would like to highlight Evelyn's career and her many accomplishments which have led up to this award.

I am told that Evelyn's interest and love of mathematics began in Kindergarten. It was as an ECS teacher that she became aware of the importance of math and the many opportunities for developing mathematical understanding even in very young children. Her interest in math led her to pursue a master's degree in mathematics education. She continued to teach at the elementary and junior high levels in the Calgary Catholic Board and she became involved in a wide range of committees. At the district level she has served on the elementary and junior high math committees, and was a member of the Holistic Math Committee. She worked in conjunction with the Calgary Catholic Board, the Calgary Board of Education and Alberta Education as a system mathematics specialist on the Education Quality Indicators Project.

In 1990, Evelyn was appointed to the position of mathematics consultant with the Calgary Catholic Board. During her term as consultant she showed exemplary leadership in establishing high standards for instruction, as well as organizing inservice support for implementation of the new mathematics curriculum.

She was then appointed to the position of supervisor of mathematics. At this level she has been responsible for the enormous task of developing, maintaining and improving program content and quality in all schools for Grades K–12.

Evelyn has made major presentations at NCTM conferences in Calgary, Edmonton and Toronto. She has also been a speaker at MCATA conferences and mini-conferences in Calgary, Medicine Hat and Pincher Creek.

Evelyn has been a member of the Alberta Education Elementary Mathematics Advisory Committee and she has had significant input into the Mathematics Common Curriculum Framework for Grades K–12.

As a district, we are very fortunate to have Evelyn as our supervisor. For myself, I am in awe of her energy and commitment to the field of mathematics. She has inspired me and made an incredible difference to my own teaching of math.

I would like to quote from some of her nomination letters for this award. Moira Martin describes Evelyn as a leader who demonstrates "initiative and vision, patience and understanding, knowledge and wisdom, commitment and focus." Barb Morrison says of Evelyn: "she has a genuine concern for the improvement of mathematics and this has gained her the respect of all colleagues with whom she has worked at the local and provincial levels."

This award is a very well-deserved honor and it is with pleasure and admiration that I would like to present you—Evelyn—with the Outstanding Mathematics Educator Award.

Liz Donovan

In this new section, we will share your points of view on teaching mathematics and your responses to articles. We appreciate the interest and value the views of those who write. The December 1995 issue of delta-K *(vol. 33, number 1) contained a student submission entitled "Linda's Trisection" (p. 18) in which a student, Linda Chiem, demonstrated the trisection of an angle using only a compass and straightedge. This prompted the following reaction from Professor Michael G. Stone.*

Examining the Impossible

Michael G. Stone

"Trisecting the angle does not compute."

—*Old Vulcan saying*

One of the most well-known impossibilities in mathematics is the trisection of an angle with compass and straightedge. In particular, it is impossible to trisect a 60° angle using only these tools. Roughly speaking, this is because the available operations will only allow us to construct from a unit length only all of those lengths which can be obtained by arithmetic operations and square roots. For a fascinating, yet simple, account of this and other mathematical impossibilities, see John Paulos' (1991) *Beyond Numeracy: Ruminations of a Numbers Man.* For a more detailed account, see Howard Eves' (1976) wonderful *An Introduction to the History of Mathematics.*

Linda Chiem's (December 1995) article, "Linda's Trisection," provides an excellent forum to promote classroom discussion of what is meant by the impossibility of such a construction. Certainly, some angles can be trisected easily (for example, 90°), although not by the method here! But not all angles can be trisected, and the construction given here, in particular, fails to do so. Perhaps the easiest way to do this is to add to the figure used in Linda's Trisection of the angle with centre B and radius BM. Then consider the triangles BD′M, BMN and BNE′, where D′ and E′ are the points where BD and BE meet the circle. If the angle were truly trisected by the given construction, these angles would all be congruent (side/angle/side = side/angle/side). However, they are clearly not congruent in general, as a little experimentation with very large angles will reveal.

Can you find the angles for which Linda's Trisection will not work? (Hint: What is the sum of the angles in triangle BMD′? Note that triangle MD′D is similar to BM′D.)

Don't be discouraged, Linda, by finding a flaw in your proof. Every working mathematician has had similar experiences! Each time we discover an error in our reasoning there is an opportunity to learn something new which strengthens our intuition. Here there is something to be learned about the way that arc and span are related. To extend this discussion to an analysis of a proof that you cannot [sic] bisect an angle see Eric Chandler's article in *Fallacies, Flaws, and Flim Flam*, an issue of *College Mathematics Journal* (1995) edited by Ed Barbeau. For more about capitalizing on errors to turn these opportunities into learning experiences, read "Capitalizing on Errors as 'Springboards to Inquiry' " by Raffaella Borasi.

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Borasi, R. "Capitalizing on Errors as 'Springboards to Inquiry.' " *Journal for Research in Mathematics Education* 25, no. 2 (1994): 166–208.

Chandler, E. "The Impossibility of Angle Bisection." *Fallacies, Flaws and Flim Flam*, edited by E. Barbeau. *The College Mathematics Journal* 26, no. 4 (September 1994): 302.

Chiem, L. "Linda's Trisection." *delta-K* 33, no. 1 (December 1995): 18.

Eves, H. *An Introduction to the History of Mathematics*. 4th ed. New York: Holt, Rinehart & Winston, 1976.

Paulos, J. A. *Beyond Numeracy: Ruminations of a Numbers Man*. New York: Knopf, 1991.

Editor's note: The trisection of an angle is also discussed in D. E. Smith's History of Mathematics *(New York: Dover: 1958, pp. 297–300). The problem of trisecting any angle with straightedge and compass alone was proved impossible by P. L. Wantzel in 1847. Any angle can be trisected, however, in several ways, for instance, by the use of a protractor, the limaçon of Pascal (that is, Etienne Pascal, the father of Blaise Pascal), the conchoid of Nicodemes or the trisectrix of Maclaurin.*

STUDENT CORNER

Mathematics as communication is an important curriculum standard, hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas, and to explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternate solution to a familiar problem or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of The Alberta Teachers' Association to publish "_____" in one of its publications. Parental permission is required if the student is under age 18. The student author must sign this statement, indicate the student's grade level, and provide an address and telephone number.

The following work, entitled "Polynomial Functions Restaurant," has been submitted by Colin Szasz, Grade 12 student at Paul Kane High School in St. Albert.

Polynomial Functions Restaurant

Colin Szasz

Appetizers

- *Ordered Pairs* . . . delicious little devils available in (1, 2), (3,10) or any relation you like.
- *Domains and Ranges* . . . a great mixture of *x*s, *y*s, <s, >s, {s and }s.
- *Zeros* . . . they'll intercept everyone's axis with their great taste.

Main Courses

- *Parabola Pot Pie* . . . so delicious that everyone "curves" them and can't eat just one.
- *Integral Polynomials* . . . no little, fractional pieces here, just full-size meaty chunks in an *x* and *y* sauce.
- *Algorithms* . . . freshly caught from the sea, those P(*x*)s, D(*x*)s, Q(*x*)s and Rs will divide up your appetite till it's gone!
- *Synthetic Division* . . . much better than the real kind, with *x* coefficients of 1 only, giving it a real appeal.
- *Potential Zeros* . . . just like our zeros, except more satisfying and you don't know what's inside till you try them out!

Beverages

- *"Roots" Beer* . . . in regular or lite, it'll eliminate your *y* values with its great taste!
- *Remainder Theorem Serum* . . . a thick, thirst-quenching drink that goes great with every equation, especially our Algorithms.

Desserts

- *Functions* . . . super-sweet treats in the flavors of quadratic, cubic, quartic or quintic.
- *Degrees*. . . cold ice cream in many flavors to include all your *x* terms.
- *Factor Theorem Cake* . . . it'll prove that you really are a factor.

The NCTM has three publications which have influenced—and will continue to influence—K–12 mathematics education in Alberta schools. The documents offer well-thought-out perspectives on the teaching, learning and assessment of mathematics.

i. Curriculum and Evaluation Standards for School Mathematics *(1989)*
ii. Professional Standards for Teaching Mathematics *(1991)*
iii. Assessment Standards for School Mathematics *(1995)*

This year, every issue of delta-K *will devote a section to the NCTM Standards. In this issue, the focus for the elementary level will be on Curriculum Standard 4: Mathematical Connections. Mathematics should include opportunities to make connections through investigations, interplay among various mathematical topics and their applications. "Connecting Literature, Language, and Fractions" by Betty Conaway and Ruby Bostick Midkiff deals with that Standard. It also involves students modeling various ways of communicating their mathematical understanding of fractions.*

At the middle-school level, the focus is on Evaluation Standard 9: Mathematical Procedures. It is important that the assessment of students' knowledge of procedures should provide evidence that they can—among other things—recognize when a procedure is appropriate, give reasons for the steps in a procedure, and reliably and efficiently execute procedures. The article "Conceptual Understanding and Computational Skill in School Mathematics" by Thomas R. Scavo and Nora K. Conroy shows that procedural knowledge is intertwined with conceptual knowledge. Students must be encouraged to appreciate the nature and role of procedures in mathematics; that is, they should appreciate that procedures are created or generated as tools to meet specific needs in an efficient manner and thus can be extended or modified to fit new situations.

The focus of the article at the senior high school level, "Apply the Curriculum Standards with Project Questions," by Richard T. Edgerton, is on Evaluation Standard 4: Mathematical Power. The assessment of students' mathematical power can be done in the context of real-life projects. Project questions from different areas of study offer a rich context which yields information about a student's ability to apply his/her knowledge to solve problems within mathematics and in other disciplines; to use mathematical language to communicate ideas; to reason and analyze; as well as their knowledge and understanding of concepts and procedures. The assessment should examine the extent to which students have integrated and made sense of information, whether they can apply it to situations that require reasoning and creative thinking, and whether they can use mathematics to communicate their ideas.

Connecting Literature, Language, and Fractions

Betty Conaway and Ruby Bostick Midkiff

Because of the symbolic nature of fractions and the procedural operations required to manipulate fractions mathematically, the concept is often difficult for students in early grades to master (Van de Walle 1990). Perhaps this difficulty results in part from the numeral contradictions presented by fractions. Furthermore, fractions are part of a mathematical language that is often foreign to students until they develop a personal understanding. "Children's literature presents a natural way to connect language and mathematics" (Midkiff and Cramer 1993, 303) and furnishes a foundation on which an understanding of the concepts can be based. As students read, write and discuss real-life situations requiring the use of fractions, they develop personal meanings for the abstract concepts.

Numerical Contradictions

Elementary school students have had many experiences with numbers. They know that as numbers increase in size, the amount represented by that number also increases. However, the amounts represented by fractional numbers do not always have the same relationships as those expressed in whole numbers. For example, as denominators increase numerically, the portion represented by that fraction actually decreases in size when the numerator remains constant.

Another complication is that a fractional number is read from top to bottom, but the top number cannot be fully understood unless the bottom number is visualized first. For example, 1/4 is read as "one-fourth," but the "one" has meaning only when the segment "fourth" is visualized as a section of a whole or a part of a set. Reading this fraction as "one part of four" or "one of four equal parts" may be more meaningful for some students. Other children will benefit by referring to the numerator as the "counting number" and the denominator as "what is being counted" (Van de Walle 1990, 178). In other words, students must develop a concrete and personal understanding of fractions on the basis of knowledge of both terms (Cruikshank and Sheffield 1992).

Developing Personal Understandings

Students cannot develop a thorough understanding of the characteristics of fractions if they only listen to the teachers "tell" about the concepts. Children must discover the mathematical relationships themselves, creating this information from their own experiences visually, tactually and auditorily. Furthermore, opportunities to develop an understanding of fractions through writing should be offered so that students can express the characteristics of fractions in terms that are meaningful to them.

For example, after students have experienced fractions through the use of patterning-block manipulatives and selections of children's literature, ask them to write personal definitions for such fractions as 1/2. When students have worked with the manipulatives (tactile), seen illustrations in books (visual) and talked about the relationships (auditory), their definitions will contain specific examples instead of simple repetitions of the definitions stated by the teacher. These experiences enable students to see that the quantity represented by 1/2 changes according to the context. As a result, one-half may be half of the class (12 people), half of a pizza (four pieces) or two of four triangles.

One way to assist students in developing a schema for fractional concepts is to use children's literature. Almost all students in the early grades have developed a firm understanding of story grammar (Brown and Murphy 1975). They recognize that stories have a beginning, a middle and an end, and they know that stories communicate information (Whaley 1981). This existing schema for the story can be used to introduce new concepts in other content areas, especially mathematics. This instructional strategy not only builds background information for mathematical concepts but also enhances children's ability to communicate mathematically, which is one of the standards in the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989). Numerous children's books suitable for students in K–6 include specific mathematical concepts as part of the story. The stories "can help build bridges between the concrete and the abstract" (Slaughter 1993, 4). Books of this type also present opportunities for extension activities that focus on developing a thorough understanding of fractional concepts.

Fractions in Children's Literature

Students in Grades K–3 will enjoy reading *Eating Fractions* by Bruce McMillan (1991). This book introduces the fractional concepts of whole, halves, thirds and fourths using photographs of real children dividing and eating various foods. *Eating Fractions* is essentially a wordless picture book, because only the fraction words and the numerals are included in the text. Two children are pictured eating two halves of a banana and two halves of an ear of corn. Thirds are demonstrated as the children divide a yeast roll into three pieces and a gelatin salad into three portions. Fourths are illustrated using a small pizza and a strawberry pie. Each food is first pictured as a whole and then pictured after it has been segmented into fractional parts.

This book offers many opportunities for discovery-learning activities. Simple recipes for the foods pictured in the book are given in an appendix at the end of the book. Each recipe uses fractions to measure the ingredients. Students can measure the ingredients and later divide the cooked foods. Older students can calculate new quantities for the ingredients if the recipes were to be doubled, tripled or halved. Primary students enjoy working with foods that require no cooking and are "soft" or easy to divide, such as bananas, English muffins or peanut butter sandwiches. Ask the children to use plastic knives to divide the bananas into halves, the English muffins into thirds and the peanut butter sandwiches into fourths. Encourage them to compare the sizes of the portions and finally eat the foods.

An additional activity using raisins, small crackers or bite-sized cookies can demonstrate the parts of a set. During this lesson, the teacher should ask students to work individually or in small groups to arrange the foods in sets of various sizes on paper plates. Then the teacher leads a discussion of the way to represent various fractional parts of each set. The following is an example:

> Show me a set of five cookies on your plate. What number represents 1/5 of the cookies? Divide your set to show 1/5 of the cookies. How do you know that 1/5 of the cookies are represented this way? Divide your set to show 3/5 of the cookies. How do you know 3/5 of the cookies are represented this way?

The Doorbell Rang by Pat Hutchins (1986) can be used to extend the understanding of fractions. The book tells the story of Victoria and Sam as they prepare to share a dozen cookies. As the two children begin to divide the 12 cookies equally between themselves, the doorbell rings and two more children arrive, then two more children and finally six more come to visit. After each group arrives, the children decide how to divide the cookies evenly among those present. Although this book does not directly discuss fractional concepts, the story provides a real-life problem-solving situation using fractional portions of 12. After the students have listened to the teacher read the book and discussed ways to divide the 12 items, ask the students to divide 12 paper cookies, 12 real cookies or 12 pieces of popcorn. Extend this discussion by writing similar problems using a different number of children and cookies. For example, these two bags contain 12 cookies each. How many cookies do we have? (24) We have 12 students in our group. How many cookies will each student receive? (2) What fractional part of the cookies is that? (2/12) Then, depending on the grade level, explore the concept of simplifying or renaming.

Moira's Birthday by Robert Munsch (1987) incorporates the concept of dividing a whole into portions while working with large numbers. Moira invites all the students who attend her school in "grade 1, grade 2, grade 3, grade 4, grade 5, grade 6, a-a-n-n-d-d kindergarten" to her birthday party. Then she orders 200 pizzas and 200 birthday cakes. The story explains the complications resulting from the delivery of the food and dividing the food among all the children. Although very young children enjoy hearing this story, the situation described is an opportunity to discuss with middle-grade students how many children 200 pizzas would feed if each one received one piece and each pizza was divided into eighths. Apply the same situation to the classroom. How many pizzas would be needed to feed all the students in this class? In the whole school? If each student ate two pieces, how many students would 200 pizzas serve?

Using the same procedures, lead students in a discussion of how to divide the cakes to serve all those attending the party. What fractional portion of each cake would each student receive? Other applications for older students include determining the total cost of the pizzas and the cakes, as well as the cost for each guest. Extend this activity by asking students to measure the number of servings in a quart or two-litre container of punch and then determine how many containers would be needed to serve all of the students at Moira's party. Ask older students what fractional parts of a quart or a two-litre bottle each guest will receive.

Another book that includes fractions and food is *Tom Fox and the Apple Pie* by Clyde Watson (1972). Tom Fox is the youngest in a family of 14 little foxes. With Ma and Pa Fox, they live at the end of Mulberry Lane. Tom buys an apple pie at the fair. As he walks home, he imagines the pie being divided into 16 pieces to serve everyone in the family, and he concludes that each piece will be very small. Tom decides to wait until 8 of his brothers and sisters are outside counting stars, divide the pie into 8 pieces, and serve 1 piece to himself and 1 each to each of the remaining 7 family members. Again he concludes that these pieces will be too small to satisfy his appetite. Next he visualizes the pie cut into fourths, with one piece for Ma, one for Pa, one for Lou-Lou, his favorite sister, and one for himself. Tom finally concludes that he would much rather eat the whole pie himself, which he does. This book is one way to reinforce the concept that as the denominator decreases, the portion it represent becomes larger. Sample questions that lead students to visualize this concept follow:

- What happens to the pieces of the pie when they are changed from sixteenths to eighths? From eighths to fourths?
- What would happen if the pie were divided into halves?
- As the denominator decreases in these fractions—1/16, 1/8, 1/4, 1/2—what happens to each piece of pie?
- Match these fractions to pie-shaped pieces. Which piece would you rather have? Why?

Extension activities for this book include asking students to draw a pie on paper and divide it into 16 sections using one colored marker. Next ask students to use a different color to divide the pie into eight sections and then to use another color to divide the pie into four sections. Ask the students how many sixteenths are in each eighth and in each fourth.

Practising with fractional parts using patterning blocks or other manipulative counters also builds understanding. When patterning blocks are used, a hexagon can represent one whole. The trapezoid would then represent one-half; the parallelogram would represent one-third; and the triangles, one-sixth. Construct a variety of shapes to reinforce the concept that as the denominator of a fraction increases and the numerator stays the same, the portion the fraction represents becomes smaller. Ask students to cut one paper pie into four pieces, another into eight pieces and another into sixteen pieces and to compare the sections. Ask students to reassemble the pie using different numbers of pieces of each size. Another way to use the same type of activity would be to use circular pieces of felt. Use one color of felt for the whole pie. Use different colors of felt for halves, fourths, eighths and sixteenths. Students will compare the various fractional parts by arranging the halves, fourths, eighths and sixteenths on top of the whole.

The concept of one-half is presented in a variety of formats in *The Half-Birthday Party* by Charlotte Pomerantz (1984). Daniel's sister, Katie, who is six months old, has just learned to stand and Daniel wants to have a half-birthday party to celebrate. He invites his friend Lily, Mr. Bangs, Grandma, Mom and Dad. Each guest is to bring half of a gift and a whole story about the half present. Lily brings one slipper, or half of a pair. Mother brings one earring, or half of a pair. Mr. Bangs and Grandma each bring half of a birthday cake. Daniel shows Katie the half-moon. Ask older students to determine their own half-birthdays. Ask younger students to select a half present for Katie and write a whole story explaining their choice.

Other Connections in Literature

Other books that are easily integrated into the study of fractions can be found through such resources as *How to Use Children's Literature to Teach Mathematics* (Welchman-Tischler 1992); *Books You Can Count On* (Griffiths and Clyne 1988); and *Read Any Good Math Lately? Children's Books for Mathematical Learning, K–6* (Whitin and Wilde 1992). *Book Cooks* (Bruno 1991) is a collection of recipes and extension activities based on 35 different books. This resource gives teachers a wide variety of activities for teaching fractions in the primary grades through classroom cooking.

Reviewing classroom collections for a mathematical point of view yields another source of books related to the study of fractions. A few of these might include the following:

- *Caps for Sale* (Slobodkina 1968) is an entertaining story for children in the primary grades that tells of a peddler who sells caps and whose caps are stolen by monkeys while he is resting. Use the illustrations of caps in five different colors as the basis for a discussion of fractions.
- *The Toothpaste Millionaire* (Merrill 1972) is embedded with mathematical content suitable for students in the middle grades. For example, the story includes one middle passage that reads, "You will need 2 1/4 yards of 36-inch-wide nylon, which is 97 cents a yard at Vince's, which will come to $2.18 1/4 plus sales tax" (p. 12).
- *A Rainbow Balloon* (Lenssen 1992) introduces such concepts as *rise* and *fall*, and *one* and *many* while following a hot-air balloon. Ask primary students to draw a picture of the balloon in the story. Then ask students to draw a set of three balloons. Discuss how individual balloons in each set can be identified. Although this book is intended for younger audiences, it could also be used in the middle grades to integrate mathematics and science. Building and launching miniature hot-air balloons as a class project is one way to incorporate numerous mathematical concepts, including fractions, as well as to develop an understanding of the science concepts presented in the book.
- *Wilbur's Space Machine* (Balian 1990) tells the story of a couple who builds a space machine to get away from pollution and noise. Illustrations of balloons furnish excellent examples of parts of sets to discuss and identify with primary students.
- *Earrings!* (Viorst 1990) tells the humorous story of a young girl who wants to have her ears pierced. Girls of all ages will enjoy reading this story, which could serve as the foundation for numerous problems involving fractions for several grade levels.
- *Seven Little Hippos* (Thaler and Smath 1991) uses a predictable pattern rhyme to tell the story of seven little hippos who enjoy jumping on the bed. Each hippo falls off the bed one at a time until none are left. This story presents an opportunity to introduce primary children to the concept of the numerator as the "counting" part of the fraction and the denominator as the "number of things counted." After the students have listened to the story, give them a set of seven small plastic bears. As the story is read a second time, ask the children to arrange the set of seven into subsets that represent the hippos still on the bed and those that fell off the bed. Identify each of the fractions and discuss the changing relationship with the students.
- *Ten Little Rabbits* (Grossman and Long 1991) is a counting book for primary children. Read the story aloud to the children. As the book is read a second time, write the fraction representing the part of the set of rabbits counted on each page (1/10, 2/10, 3/10 and so on). Young children enjoy drawing pictures of rabbits to illustrate each fraction.

Conclusion

Effective and meaningful instruction in fractional computation requires students to have a thorough understanding of basic fractional concepts. The books described here are only a small sampling of children's literature that can assist in the formation of personal understandings of this topic. When students are actively engaged in activities that portray fractions as one-third of a yeast roll, one-half of a dozen cookies, one-eighth of a pizza, one-fourth of an apple pie and one-half of a birthday gift, they form concrete connections between abstract number concepts and real-life experiences. When students work with fractions in a variety of formats, including literature, communication and manipulatives, they are able to form personal definitions and thus able to develop the understandings and readiness necessary to master more complex fractional concepts

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Prove that one half of the perimeter of any triangle is always greater than any of its sides.

Conceptual Understanding and Computational Skill in School Mathematics

Thomas R. Scavo and Nora K. Conroy

Evaluation Standard 9 of the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) addresses the issue of mathematics procedures including, but not limited to, computational methods and algorithms. The standards document relates that

> a knowledge of procedures involves much more than simple execution. Students must know when to apply them, why they work, and how to verify that they give correct answers; they also must understand concepts underlying a procedure and the logic that justifies it. (NCTM 1989, 228)

It is easy to agree with these conclusions, particularly the view that conceptual understanding is as important as computational skill or manipulative ability (see Kulm 1994).

In contrast, the inaugural issue of *Mathematics Teaching in the Middle School* contains a short letter to the editor that tells of a trick for subtracting mixed fractions. Unfortunately, "Brian's Method" (Curtis 1994) appears to be the exact opposite of the recommendations of the *Curriculum and Evaluations Standards*. Not only that, but as far as we can tell, the usual method of subtracting mixed fractions requires exactly the same amount of computation as Brian's method—one addition and one subtraction, as opposed to two subtractions, respectively—and so the latter hardly seems justified, either computationally or conceptually. (See Sasser 1994 for some discussion of Brian's method and Howe 1995 for reactions and a rejoinder.)

Equipping students with quick tricks might give them the edge in mathematical competitions, but we fear it will also give them a misleading impression of the nature of mathematics and may even hinder their progress in subsequent courses. Mathematics is not a bag of tricks or even a list of formulas. It is a way of thinking, a thought process that we seek to cultivate in our students, no matter what their age.

Emphasizing the "why" over the "how" in a mathematics classroom is an admirable but sometimes difficult goal. A situation taken from our own classroom illustrates this point.

A common word problem in a first-year algebra course is the following:

> Two numbers are in the ratio of 2 to 5. One of the numbers is 21 more than the other. What are the two numbers?

A typical solution follows.

Solution 1

Let x and y denote the two unknown numbers. Since they are in the ratio of 2 to 5, we may write

(1) $$\frac{x}{y} = \frac{2}{5}$$

and since one is 21 more than the other, we also have

(2) $$y = x + 21.$$

Equation (2) *cannot* be written as $x = y + 21$, since $x < y$ by our choice of variables in (1). Substituting (2) into (1), we find that

$$\frac{x}{x+21} = \frac{2}{5} \Rightarrow 5x = 2x + 42$$

(3) $$\Rightarrow 3x = 42$$
$$\Rightarrow x = 14.$$

Finally, substituting (3) into (2) yields the solution

$$\begin{cases} x = 14 \\ y = 35, \end{cases}$$

which satisfies both (1) and (2).

We usually do not expect our students to produce such detailed solutions, but we at least want them to write down equations (1) and (2) and substitute one into the other. We also want them to get into the habit of checking their work.

So what do we do when a bright young student comes up with this surprising alternative solution?

Solution 2

$$5 - 2 = 3$$

$$\frac{21}{3} = 7$$

$$2 \times 7 = 14 \; (= x)$$

$$5 \times 7 = 35 \; (= y)$$

Not only does this series of sample computations give the correct answer, but *it gives the correct answer every time*, for any constants! Because the student checked his answer to make sure it was correct, and because we do not usually require a detailed explanation like that given in solution 1, we feel compelled to give him credit for a purely mechanical process that just happens to work—every time. So we sit down and try to figure out *why* solution 2 works, in hopes that we can return to this student, and perhaps to the whole class, with some kind of explanation.

Our first attempt to justify the unorthodox solution resulted in a complicated pair of algebraic identities that we could never hope to explain to our first-year-algebra students. For a long time, that solution was the best we could do. Then later we learned of a relatively simple justification of solution 2 that allowed us to refine our arguments such that we could finally explain them to our eighth graders. By that time, however, our students had graduated and gone on to high school. If they were still with us today, we would offer the following justification.

Justification of Solution 2

This method of solving the problem simultaneously justifies the unorthodox solution. First, let us generalize the problem. We want to solve the equations

$$(4)\qquad \frac{x}{y} = \frac{a}{b}, \qquad y - x = d$$

given constants a, b, and d with $b \neq 0$. From the first equation in (4) we have

$$(5)\qquad \frac{x}{y} = \frac{a}{b} \Rightarrow \frac{x}{a} = \frac{y}{b}\ (= c)$$

assuming that $a \neq 0$. The constant c corresponds to the "magic" number 7 in solution 2. From the right-hand side of (5) we see that

$$(6)\qquad x = ac, \qquad y = bc$$

for some unknown constant c. The equations in (6), together with the second equation of (4), yield

$$(7)\qquad y - x = d \Rightarrow bc - ac = d \Rightarrow (b - a)\,c = d \Rightarrow c = \frac{d}{b-a}$$

provided that $b - a \neq 0$. Substituting (7) into (6), we obtain

$$(8)\qquad x = a \times \frac{d}{b-a}, \qquad y = b \times \frac{d}{b-a},$$

which solves (4). For our particular problem,

$$(8)\qquad x = 2 \times \frac{21}{5-2}, \qquad y = 5 \times \frac{21}{5-2},$$

which matches exactly the sequence of steps in solution 2.

Did we miss a golden opportunity? Perhaps so, but we may have learned something in the process. A teacher may not know how to explain a student's unorthodox solution, which was our initial reaction to solution 2, or may choose not to reveal it, but in either situation the student must realize that the problem is not completely solved until a proof is found that justifies the method in all cases. The teacher will certainly want to find such a proof—by reading books, talking to colleagues or posting to the Internet—in a form that students can understand and appreciate.

We *want* our students to discover patterns in problems. This discovery is an important part of the mathematical process. Because few students feel the need to generalize and prove their findings, however, part of our job as teachers is to create an atmosphere that encourages abstraction and proof as well as experimentation and conjecture, a process even our first-year-algebra students can appreciate. For example, we can tell them the story of the student who discovered solution 2 and how a justification was found. We can also point them to the literature for other more spectacular examples, such as the discovery by a Grade 9 student reported in Morgan (1994). But most of all, we should encourage the mathematical process in our classrooms and be prepared to take advantage of any situation that arises.

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Apply the Curriculum Standards with Project Questions

Richard T. Edgerton

A goal of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) is facilitating "mathematical power" in students. The curriculum standards use problem solving, communication, reasoning and connections as organizing principles. One way to apply these principles in the classroom is with the use of "project questions."

A project question is a real-world problem on which a student or group of students work over the course of many days. Different from an exercise, a project question

- has multiple solution methods,
- has no obvious solution sequence,
- requires some hands-on data collection, and
- affords an opportunity for students to explore various topics within and outside mathematics.

I currently use project questions as one of many ways students may earn extra credit. When students unavoidably miss a quiz or think that their grade needs bolstering, we discuss the possibility of using a project question. After a question is agreed on through student-teacher dialogue, a contract is signed. (See Figure 1 for a sample contract.) I require all written submissions to be accompanied by the signature of the supervising parents or guardians and of all students who expect to collaborate on a question.

Project-Question Evaluation

Evaluation of project questions involves the examination of the *process* that students take in exploring the problem rather than just their product. Frequent discussions between teacher and student are important. I usually meet with students about three times while they work on their question. This approach helps the students stay on track and permits me to monitor their progress. I base credit for project questions on the knowledge students gain in the process rather than the correctness of their final product. I expect students to learn a great deal by working on a project question even if they never fully solve it. Because I check their progress and have them explain their results to me before their presentation, my students always earn full credit.

Figure 1
Sample Contract

"Project questions" are intended to give an opportunity to explore concepts of classwork by working on and, it is hoped, solving a real-world problem. The nature of the problem as well as the due date and point value are negotiated in advance.

To receive credit, the student must perform the bulk of the work on the problem himself or herself (or themselves if working in a group). No one outside the student's household who has been enrolled in college classes may help with the question. All persons helping with the question must be listed in the report, which is to be submitted as the answer. A written report must accompany an oral presentation of the findings on the due date. Your written report must include

- answers to the questions on the "Project-Question Assessment Sheet" (to be filled out after your presentation),
- your procedure for gathering data,
- your approach to analyzing the data,
- your conclusion,
- each person's contribution to the project,
- things you would do differently next time on a project of this kind and
- the way in which credit should be divided among the group members.

Student(s): ______________________
Question: ______________________
Point value: ______________________
Date given: ______________________
Date of formative discussion: __________
Date of first results:
Date of final draft: ______________
Date due: ______________________
Describe how you plan to present your results:

Signatures
Student(s): ______________________
Supervising parents or guardians: __________
Teacher:

The final step in working on a project question is the presentation of results. A short oral report is given to the class after which the class has the opportunity to ask questions concerning the project. A written report is submitted at the time of the oral report. The written report must include

(a) an outline of each participant's role in the solution process,
(b) outside resources (such as parents, friends and neighbors) used,
(c) conception and refinement of the question,
(d) data-gathering techniques,
(e) data-analysis techniques,
(f) the apparent solution and
(g) things the group would do differently on future questions.

The time of presentation is usually one of celebration for the solution team and the class. At this time I also have the presenter and students complete a short evaluation form regarding the presentation (see Figure 2). Both groups typically enjoy the presentations: they are a break from their usual class activities, they are informative and they demonstrate some usefulness for mathematics.

Figure 2
Project-Question Assessment Sheet

Name: ______________________
Date: ______________________
Project question: ______________________
Presenter(s): ______________________

1. What did you learn while observing this presentation?
2. Please circle the numeral that best represents your assessment of the presenter(s) and his/her/their performance.

The presenter(s) was (were):
1 2 3 4 5 6 7 8 9 10
poorly prepared — well prepared

The pace of the presentation was:
1 2 3 4 5 6 7 8 9 10
too slow/fast — good

The discussion was: 1 2 3 4 5 6 7 8 9 10
unclear — clear

The question was: 1 2 3 4 5 6 7 8 9 10
explained poorly — explained well

The answer was: 1 2 3 4 5 6 7 8 9 10
explained poorly — explained well

Do you want your responses to be kept confidential? Yes No

Bob's Question

Bob was enrolled in a first-year integrated mathematics class at a four-year high school. After observing another student's success with a project question, Bob asked for one. From previous discussions, I knew Bob enjoyed baseball, so I asked him if he wanted his question to relate to baseball in some way. He was delighted but had no idea how baseball and mathematics could mix. After a few dead ends with such topics as batting averages, we centred on properties of a baseball itself. I asked Bob if he knew at what angle to throw a baseball to make it travel the farthest. Bob had no idea but guessed that a maximum distance would result from throwing at an angle of elevation of 20 degrees. After further discussion, Bob agreed to make this investigation his project question. The initial discussions that resulted in Bob's questions took about 30 minutes.

Bob and I discussed how he might make a slingshot from surgical tubing to ensure uniformity of throws. To obtain a better estimate, we agreed that he should measure each "throw" at least three times. Bob signed a contract to answer the question for a value equal to that of a major quiz and agreed to complete the question within two weeks.

The next day, Bob returned with a completed table of data (see Table 1) and his answer. I noticed from his data table that he performed four trials instead of the suggested three, about which he said, "I figured that doing four would make a better average." Bob spent the next two weeks writing the results of his experiment. During his oral report, Bob described the study well and included two very descriptive graphs. The first graph (see Figure 3) plotted the ball's distance against the angle of the launch. This graph adequately showed the relationship that he found from the data. Since Bob's reports were handwritten, the tables and graphs herein have been modified for clarity. Only the basic appearance of the graphs and tables has been changed.

Bob's second graph was more interesting than the first. Although this tactic was not previously discussed, Bob graphed the relative flight distances along with the angle of the launch (see Figure 4). Rays were drawn at their respective launch angle to show the relative distances that the ball traveled. This clever view was an even more meaningful representation of the data because each ray represented two pieces of information. During Bob's oral presentation, I made a point of telling the class how his innovative graph clarifies the data of his experiment.

Table 1
Bob's Table of Horizontal Distance Traveled by a Baseball Launched at Different Angles

Degrees	**0**	**30**	**45**	**60**	**65**	**70**	**80**	**90**
Trial 1	9	16.8	19.7	22.8	28.2	24.8	9.3	1.7
Trial 2	8.7	17.6	18.6	23.3	27.8	25.3	9	2
Trial 3	9.2	17.3	19.2	23.8	28.3	25.7	8.7	1.5
Trial 4	8.6	17	19.4	23.7	28.7	26	9.4	2.2
Average	8.9	17.2	19.2	23.4	28.3	25.5	9.1	1.9

One fact that I did not tell the class about Bob's answer was that it was not significantly close to the answer derived by either ballistics or physics. Although I am not sure what the correct answer should have been, I am certain that neither discipline would support his answer of 65 degrees. I chose not to mention that the "correct" answer should be closer to 55 degrees (45 degrees if no air resistance were involved) because Bob's *process* was correct—he just needed more accurate instrumentation and a stronger sling to increase the distance. Bob's data also showed horizontal distance for both 0 degrees and 90 degrees, which could not have been true in reality. I felt that if I were to point out the weaknesses of his experiment, I would reduce the importance what he had done and reinforce the conventional notion of the existence of one correct answer known by the teacher. Bob learned how to test a hypothesis scientifically and how to report experimental findings. To me, his process was worth much more than being significantly close to an ideal solution.

Summary

The project question was a powerful application of the NCTM's curriculum standards for Bob. He had an enjoyable time working on his question and gained confidence in his mathematical understanding. I believe that his classmates also benefited from hearing about the process and seeing new ways to think about mathematics.

The autonomy that Bob enjoyed helped him to experience mathematics in a participatory, occasionally frustrating mode. Project questions do not lend themselves to the neatness of traditional approaches, but guidance from the teacher during the problem-solving process helps to keep students' frustration under control while still allowing students to work independently. In evaluating project questions, the process is more important than the product.

Figure 3
Bob's Graph of Travel of a Baseball Launched at Different Angles

(The line segments have been added only to make the relationship between the discrete data points clearer.)

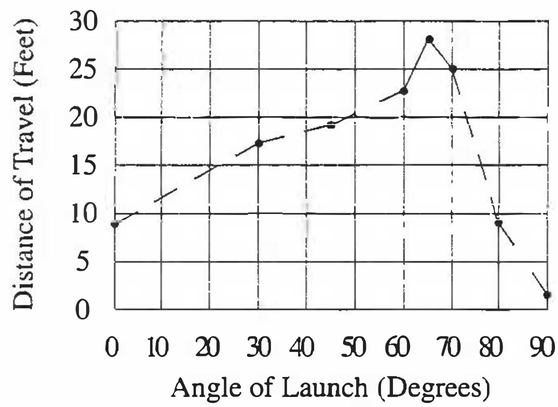
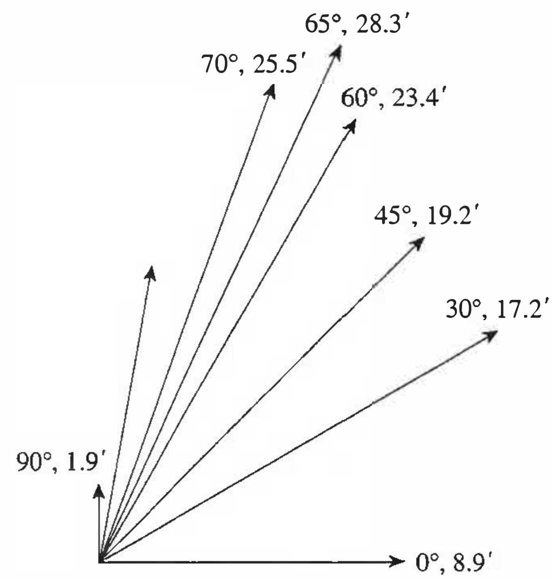


Figure 4
Bob's Graph of a Ball's Flight Relative to the Ground



Here are a few concluding thoughts about the use of project questions:

- Have students record their initial guess for a question, including hypothetical tables and graphs if possible.
- Start small by involving only a few students at a time.
- Be certain that students and parents know at the beginning the question and expectations for its solution.

- Be sure to detail the people who may be involved and to what extent.
- Indicate possible trivial solutions and ways of achieving the detail you want.
- Let other teachers in your school know your plans.
- Work to keep the lines of communication open among yourself, your students and their parents.

Reflecting on my initial use of project questions, I plan to increase the level of students' involvement in various ways. I will eventually have all students work on project questions as part of their coursework. I will have students define the problem and take more responsibility in writing the actual question.

Project questions give students opportunities to exercise their powers of reasoning, create critical mathematical connections, communicate mathematics with others and experience problem solving in a natural setting. Such questions are an ideal way to apply the NCTM's curriculum standards in the mathematics classroom.

Sample Project Questions

- What is the relationship between the wattage of light bulbs and their luminosity?
- What is the relationship between the length of a person's forearm and the length of his or her foot?
- How fast does hair grow?
- Which is steeper—the stairs in the school or the route to the summit of Mount Everest?
- If the Earth were a solid rock, into how many grains of sand would it be split?
- What is the next day for which an object's shadow is equal to the object's height at noon? (Note: not all latitudes will have a solution to this question; the latitude at which solutions begin is an interesting investigation, and the actual "critical latitude" is different from the mathematical result because of the refraction of light by air.)
- At what speed do raindrops fall?
- How much energy do you use each day?
- How much trash do you produce each day?
- How long will it take for a class computer to print each number from one to one million?
- Which variety of firewood heats most efficiently?
- How many BBs would be required to make a life-sized statue of yourself and how much would that statue weigh?
- How many names are in the white pages of the local telephone book?
- How many times will your heart beat during your lifetime?
- How unusual is it for a person your age to have exactly 28 natural teeth with no fillings?
- For how many hours would a person have to mow lawns to get a pile of grass clippings the same height as the world's tallest building?
- How tall is a stack of one million dollar bills?
- What is the area of your skin?
- How many sugar cubes would you need to make a scale model of your house?

Reference

NCTM. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: Author, 1989.

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A polygon with seven sides is called a hooligan.

NCTM Position Statement: Early Childhood Mathematics Education

The National Council of Teachers of Mathematics believes that early childhood mathematics education, for young children aged 3–8, should be developmentally appropriate. Developmentally appropriate instructional practices are those in which the mathematics learning environment takes into account the social, emotional, physical and intellectual needs of young children. Because young children actively construct knowledge, instruction should concentrate on facilitating learning through exploration and interaction with materials and people. In early childhood mathematics, how and when the curriculum is taught is as important as what is taught. Thus, endorsing a developmental philosophy for early childhood mathematics education suggests reorganizing classroom pratices around the total child rather than allowing materials and rigid timelines to dictate instruction. Furthermore, early childhood mathematics instruction should foster a positive environment, provide equal access for all children and account for cultural and ethnic diversity.

Therefore, the National Council of Teachers of Mathematics recommends developmentally appropriate mathematics instruction that has the following aims:

- Acknowledge and build on the children's accumulated knowledge by including children's experiences, languages and relevant, real-world contexts.
- Incorporate active and interactive learning. Children's understandings develop as they explore, investigate and discuss mathematical concepts. Physical and mental interactions with the environment, materials and other individuals give children opportunities to construct, modify and integrate ideas.
- Offer opportunities for children to develop and expand language acquisition, while structuring, restructuring and connecting mathematical understandings. Concepts should be repeatedly experienced through concrete, visual, verbal and pictorial formats. Gradually children should be encouraged to translate and record their experiences in more abstract representations.
- Be concept and problem-solving oriented. The classroom environment should provide for the regular study of mathematics, focusing on the development and integration of mathematical thinking, reasoning, understandings and relationships through concrete problem-solving experiences. Mathematical concepts should be integrated with other subject areas, making use of natural connections wherever they occur.
- Develop children's confidence in their mathematical abilities. Varied instructional strategies, meaningful child-related contexts and opportunities for active participation in the learning process encourage children to become capable mathematical thinkers and to believe in themselves as such.
- Include ongoing assessment. Teachers should make instructional decisions that are based on the progress of the children in their classroom. Children's progress is determined through the information obtained from the formal and informal assessment of each child's individual pattern of growth. Evaluation strategies such as observations, interviews and portfolios give evidence of children's thinking processes and their understanding of concepts.

The National Council of Teachers of Mathematics recommends that those who produce, select and purchase young children's mathematics curriculum materials support developmentally appropriate early childhood mathematics programs. Guidelines for early childhood mathematics encourage a child-centred approach to instruction. Preference should be given to mathematical learning environments that support active participation through observation, exploration, verbalization and hands-on experiences. The focus of instruction should be on the continuous development of mathematical processes and language through activities that gradually increase in difficulty, complexity and challenge as the children develop understanding and skills. Developmentally appropriate early childhood mathematics instruction should meet the needs of individual learners at different stages of readiness, by considering the influences of cultural backgrounds, prior experiences, learning styles and cognitive abilities.

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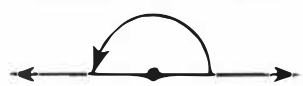
What's My Angle?

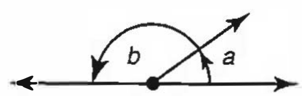
Jane Barnard

An angle is the union of two rays sharing only a common endpoint. Angles are frequently measured using degrees. The ancient Babylonians used a sexagesimal, or base 60, numeral system and assigned a measure of 360 degrees to one revolution of the circle. They divided a circular region into 6 sectors and subdivided each sector into 60 more sectors, giving 360 subdivisions. Each small subdivision is called a *degree*.

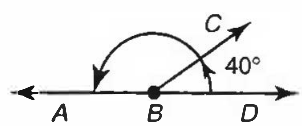
A *straight angle* is an angle formed by opposite rays and has a measure of 180 degrees. Two angles, and exactly two, are *supplementary* if they form a straight angle or the sum of their measures is 180 degrees.



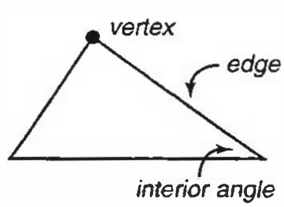
1. What is the sum of the measures of the two angles at the right?



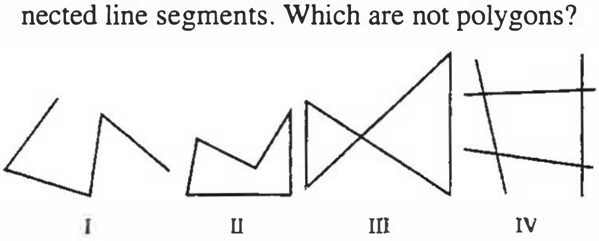
2. Angles *ABC* and *CBD* are supplementary. Find the measure of angle *ABC*, written $m \angle ABC$.



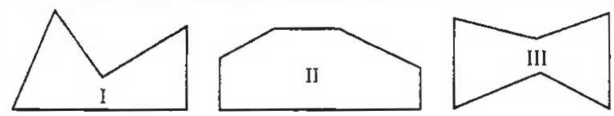
Polygons are composed of connected line segments, or *edges*, which meet only at endpoints and enclose a single portion of the plane, called the *interior* of the polygon. Some people call the edges sides. A *vertex* is a point where exactly two edges meet. The angle formed at each vertex and measured inside the polygon is an *interior angle*.



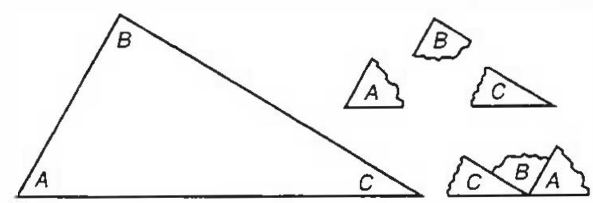
3. Each of the following shapes is composed of connected line segments. Which are not polygons?



4. A polygon is *convex* if every interior angle has a measure less than 180 degrees. Which of the following polygons are not *convex*? Such polygons are sometimes called concave.



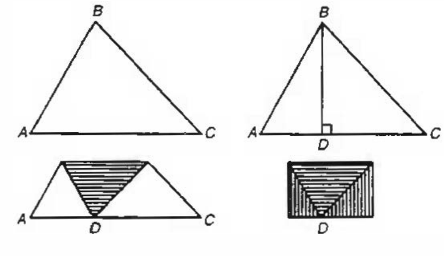
A triangle is a polygon having exactly three edges. Cut out a paper triangle like the one below and label the angles as shown. Tear off the corners of the triangle, that is, tear off $\angle A$, $\angle B$ and $\angle C$. Put the three angles together, as shown, so they share a common vertex.



5. What kind of angle is formed?
6. What is the degree measure of this angle?
7. What is the sum of the measures of $\angle A$, $\angle B$ and $\angle C$?
8. Recall the definition of supplementary angles. Are $\angle A$, $\angle B$ and $\angle C$ supplementary? Why or why not?
9. Repeat this activity using a different triangle. What do you observe?

Another way to look at the sum of the measures of the angles of a triangle is by paper folding. *Cut* a paper triangle like the one shown. Fold the triangle to create an altitude, the segment perpendicular to $\overline{AC}$ through the vertex *B*. Mark *D* as the point of intersection of the altitude and the base. Fold *B* down to *D*. Fold *A* and *C* over to *D*.

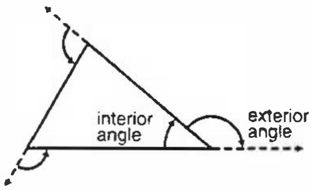
10. The three angles of the triangle fold to point *D* to form what kind of angle?



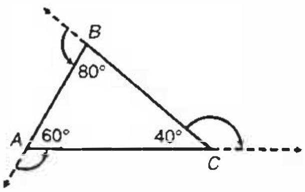
Exterior Angles

In addition to three interior angles, a triangle has exterior angles. To form one set of exterior angles, extend each edge beyond the vertex in one direction.

11. What kind of angle is formed by an interior angle of the triangle and its exterior angle? What is the sum of their measures?

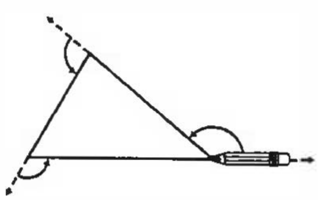


12. Find the measure of each exterior angle of triangle ABC. What is the sum of these measures?

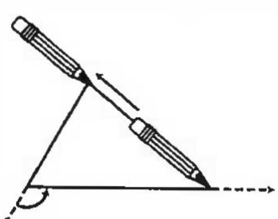


To check your answer, place a pencil flat on your paper along the extended edge as shown. Rotate the pencil counterclockwise about the vertex, turning it through the exterior angle until it lies along the edge of the triangle. Slide the pencil along the edge until its point lies on the next vertex. Rotate through the exterior angle again and slide along the edge of the triangle two more times. The pencil should come back to its original position.

13. Your pencil made a complete turn, or revolution, of how many degrees?

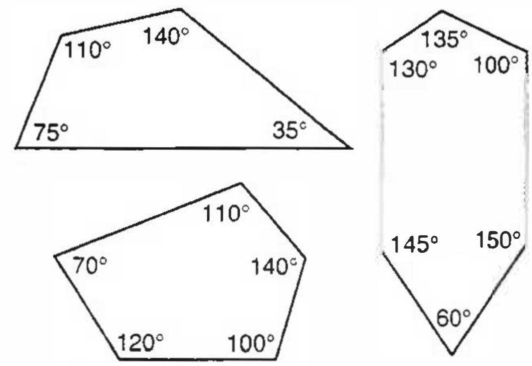


14. Find the measures of the exterior angles of the following convex polygons and use your findings to complete the chart. Use your information from the triangle in question 12.



15. For a convex dodecagon, which has 12 edges, what is the sum of the measures of the exterior angles?

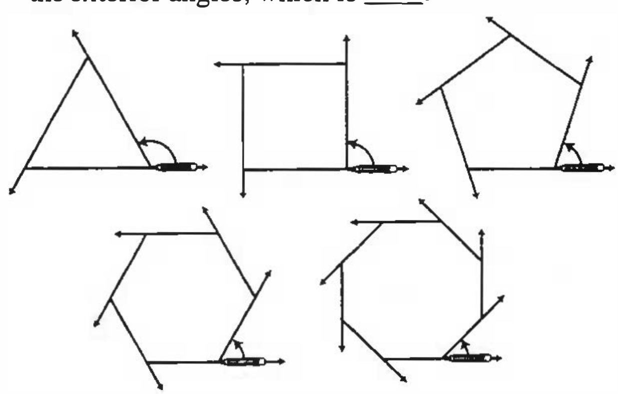
Polygon	Sum of Measures of Exterior Angles
Triangle	
Quadrilateral	
Pentagon	
Hexagon	



16. Complete the conjecture: In a convex polygon, the sum of the measures of the exterior angles is ____.

A polygon is *regular* if all edges are *congruent*, that is, have the same length, and all interior angles are *congruent*, that is, have the same measure.

17. For the following regular polygons, use the "pencil method" to verify the sum of the measures of the exterior angles, which is ____.



If interior angles of a regular polygon are congruent, then the exterior angles must also be congruent. An equilateral triangle has three congruent exterior angles, the sum of whose measures is 360 degrees. So each exterior angle has a measure of 360/3, or 120, degrees. Since interior and exterior angles are supplementary, the measure of each interior angle of the equilateral triangle is 180 – 120, or 60, degrees.

18. Using this method, find the measure of each exterior and each interior angle of these regular polygons in Table 1.

Table 1

Regular Polygon	No. of Edges	Measure of Each Exterior Angle	Measure of Each Interior Angle	Sum of the Measures of Interior Angles
Triangle	3	360°/3 = 120°	180°- 360°/3 = 180° - 120° = 60°	3 × 60° = 180°
Quadrilateral	4	360°/4 = 90°	180° - 360°/4 = 180° - 90° = 90°	4 × 90° = 360°
Pentagon	5			
Hexagon	6			
Heptagon	7	360°/7 = 51 3/7°	180° – 360°/7 = 180° – 51 3/7° = 128 4/7°	7 × 128 4/7° = 900°
Octagon	8			
Nonagon	9			
Decagon	10			
Dodecagon	12			
Icosagon	20			
n-gon	*n*			

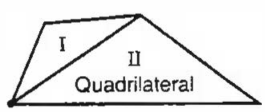
19.If a regular polygon has 30 edges, what is the measure of each exterior angle? Of each interior angle?

20.If the measure of an interior angle of a regular polygon is 170 degrees, how many edges does the polygon have?

21.Describe how to find the measure of each interior angle of a regular polygon with *n* edges, usually called a regular *n*-gon.

22. Write a generalization to accompany this explanation: The measure of each interior angle of a regular *n*-gon is ________________________ .

Triangulation



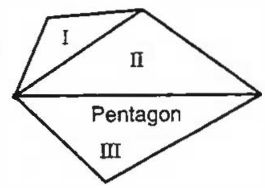
Another way to think about interior angles of any convex polygon is to look at triangles. Every such polygonal region can be *triangulated*, that is, it can be separated into triangular regions by drawing all the diagonals from any one vertex. For example, consider the quadrilateral shown, which is separated into two triangular regions.

23. The sum of the measures of the angles in triangle I is ________________________ .

24. The sum of the measures of the angles in triangle II is ________________________ .

25. The sum of the measures of the angles in the quadrilateral is the sum of the measures of the angles in triangles I and II, which is ____° + ____° = 2 × ____ °.

26. Triangulating the pentagon gives ____ triangular regions (see diagram that follows).



27. The sum of the measures of the angles in the pentagon is the sum of the measures of the angles in triangles I, II and II, which is ____° + ____° + ____° = 3 × ____°.

28.Draw diagonals from the given vertex to triangulate the following polygons. Complete the chart, then look for a pattern.

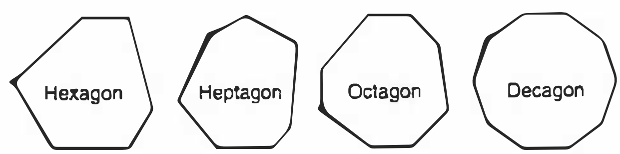


Table 2

Polygons	No. of Edges	No. of Triangles	Sum of Measures of Interior Angles
Triangle	3	1	180° × 1 = 180°
Quadrilateral	4	2	180° × 2 = 360°
Pentagon	5		
Hexagon	6		
Heptagon	7		
Octagon	8		
Decagon	10		
Icositetragon	24		
n-gon	*n*		

29. Using the pattern you discovered, find the measure of one interior angle of a regular *n*-gon:____.

30. You have found two generalizations for determining the measure of one interior angle of a regular *n*-gon. Algebraically, show that 180 - 360/*n*, which is a generalization of the exterior-angle method in question 18, is equal to (180 (*n* - 2))/*n*, which is the generalization of the triangulation method.

Can you . . .

- count all the diagonals of a regular triangle, quadrilateral, pentagon, . . . , n-gon?
- relate interior and exterior angle relationships to triangular numbers?
- establish a relationship between an exterior angle of a quadrilateral and its three nonadjacent interior angles?
- determine the number of edges of a regular polygon if you know the measure of either an interior or an exterior angle?
- determine the maximum number of intersection points of the diagonals of a convex n-gon?
- draw the inscribed and circumscribed circles for the regular polygons?
- describe a means of finding out when two convex polygons are similar? Two triangles are similar if their corresponding angles are congruent.
- determine the ratio of the interior angles of a triangle if you know the ratio of the exterior angles?

Did you know . . .

- that pilots give the direction in which the airplane is traveling by using degrees measured clockwise from north?
- that celestial navigation is a means of using the angle of elevation of certain stars to determine one's position on water?
- that Eratosthenes (275–195 B.C.) approximated the circumference of Earth by measuring lengths of shadows and sizes of angles?
- that certain regular polygons can be constructed by compass and straightedge and others cannot?
- that R. Buckminster Fuller used regular polygons to create his geodesic dome?
- that Aristarchus of Samos (310–230 B.C.) approximated the distance from Earth to the sun by using angle measures?
- that angle measure is essential in accurately portraying land descriptions?
- that a polygon with 11 edges is a hendecagon; with 13, a tridecagon or triskaidecagon (note that "kai" is Greek for "and"); with 14, a tetradecagon or tetrakaidecagon; and with 15, a pentadecagon or pentakaidecagon?

Mathematical Content

- Geometry

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Answers

1. 180 degrees.
2. 140 degrees.
3. I, III, IV.
4. I, III.
5. Straight angle.
6. 180 degrees.
7. 180 degrees.
8. No; three angles are used and the definition requires "exactly two."
9. The three angles form a straight angle.
10. Straight angle.
11. Straight; 180 degrees.
12. 120 degrees, 100 degrees, 140 degrees; 360 degrees.
13. 360 degrees.
14.

Polygon	Sum of Measures of Exterior Angles
Triangle	120 + 100 + 140 = 360
Quadrilateral	145 + 40 + 70 + 105 = 360
Pentagon	80 + 40 + 70 + 110 + 60 = 360
Hexagon	30 + 80 + 45 + 50 + 35 + 120 = 360

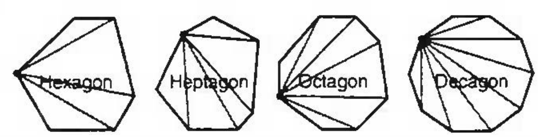
15. 360 degrees.
16. Always 360 degrees.
17. 360 degrees.

18.

No. of Edges	Measure of Each Exterior Angle	Measure of Each Interior Angle	Sum of the Measures of Interior Angles
5	$\frac{360°}{5} = 72°$	$180° - \frac{360°}{5} = 180° - 72° = 108°$	$5 \times 108° = 540°$
6	$\frac{360°}{6} = 60°$	$180° - \frac{360°}{6} = 180° - 60° = 120°$	$6 \times 120° = 720°$
8	$\frac{360°}{8} = 45°$	$180° - \frac{360°}{8} = 180° - 45° = 135°$	$8 \times 135° = 1080°$
9	$\frac{360°}{9} = 40°$	$180° - \frac{360°}{9} = 180° - 40° = 140°$	$9 \times 140° = 1260°$
10	$\frac{360°}{10} = 36°$	$180° - \frac{360°}{10} = 180° - 36° = 144°$	$10 \times 144° = 1440°$
12	$\frac{360°}{12} = 30°$	$180° - \frac{360°}{12} = 180° - 30° = 150°$	$12 \times 150° = 1800°$
20	$\frac{360°}{20} = 18°$	$180° - \frac{360°}{20} = 180° - 18° = 162°$	$20 \times 162° = 3240°$
n	$\frac{360°}{n}$	$180° - \frac{360°}{n}$	$n\left[180° - \frac{360°}{n}\right] = 180n - 360°$

19. 360°/30 = 12 degrees; 178 degrees.
20. 36 edges.
21. Find the measure of an exterior angle by dividing 360 degrees by the number of edges, and then subtract this result from 180 degrees.
22. 180 - (360°/n).
23. 180 degrees
24. 180 degrees.
25. $180° + 180° = 2 \times 180°$.
26. Three
27. $180° + 180° + 180° = 3 \times 180°$.

28.



Polygons	No. of Edges	No. of Triangles	Sum of Measures of Interior Angles
Pentagon	5	3	$180° \times 3 = 540°$
Hexagon	6	4	$180° \times 4 = 720°$
Heptagon	7	5	$180° \times 5 = 900°$
Octagon	8	6	$180° \times 6 = 1080°$
Decagon	10	8	$180° \times 8 = 1440°$
Icositetragon	24	22	$180° \times 22 = 3960°$
n-gon	n	$n - 2$	$180° \times (n - 2)$

29. $(180° \times (n - 2))/n$.

30. $\frac{180° \times (n - 2)}{n} = \frac{180n° - 360°}{n} = \frac{180n°}{n} - \frac{360°}{n} = 180° - \frac{360°}{n}$

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Calendar Math

Arthur Jorgensen

Here are 31 math exercises, one for each day of the month.

1. Find 3 consecutive numbers whose sum is 36.
2. A floor measures 3 m by 4 m. How many tiles that measure 10 cm on a side will be required to cover the floor?
3. Susan drives from Calgary to Edmonton at a rate of 100 km/h. She drives back at a rate of 110 km/h. What was her average speed?
4. Find the average of 151 whole numbers from 1–150, inclusive.
5. Cyprian leaves Vancouver on a flight that leaves at 7 a.m. to fly to Toronto. The flight takes 4 hours. What time will it be when he arrives in Toronto?
6. For every two balloons I buy at the regular price, I get a third balloon for a penny. I spent 45 cents for nine balloons. Find the regular price of a balloon.
7. Find the remainder when 230,060,145,717 is divided by 9.
8. If $a = 4$, $b = 5$, $c = 6$ and $d = 7$, find the value of $ab + bc + cd - da$.
9. A cat weights 2/3 of its weight plus 2 kg. Find the weight of the cat.
10. A small stadium has 23 seats in the first row, 21 seats in the second row, 19 seats in the third row and follows a similar pattern until it has 1 seat in the last row. How many seats are there in the stadium?
11. Lucy has 3 more pairs of slacks than she has dresses and 3 more blouses than she has pairs of slacks. Altogether she has 18 pieces of clothing. How many pairs of slacks, blouses and dresses does she have?
12. Jill's birthday is on a Tuesday. Lori's birthday is 10 days later. What day of the week does Lori's birthday fall?
13. Tom wants to create a play area for his dog. He has 100 m of fencing. What is the largest area he can make with this much fence?
14. Hotdogs cost $1.50 each and hamburgers cost $2 each. If Martha spends $19 for 11 articles of either hotdogs or hamburgers, how many of each does she buy?
15. Find a pattern and determine the next number in this series: 77, 49, 36, 18, ___.
16. If a field is 15 m by 50 m, how many metres will you save by running diagonally across the field than by running along the two sides?
17. Is 3/5 of 49 more or less than 25? Why?
18. Two fractions have a sum greater than zero but less than one. What are some other statements you can make about these two fractions?
19. Thurston gave Helen $2 more than she already had. After receiving Thurston's gift Helen had $26. How much did Thurston give Helen?
20. The town library charges a fine for each overdue library book. The fine is $0.25 plus $0.11 per day. Susan was fined $0.80. How many days overdue was Susan's book?
21. The eight-digit number 79*A*12504 is divisible by 6. What are the possible values of *A*?
22. These numbers are gumbos: 147; 63; 448; 6,370. None of these numbers are gumbos: 111; 37; 4,533. Which of these numbers are gumbos: 731; 980; 84; 1,111; 364?
23. I have $1.19 but I can't give change for a dollar, a quarter, a dime or a nickel. There are 11 coins. What coins do I have?
24. On one of the questions in a math test Michael divided by 9, rather than multiplying by 9. He got an answer of 18. What was the correct answer?
25. Jason has five friends at his birthday. To be polite, each person shakes hands with everyone else. How many handshakes will there be? (This is an excellent problem to solve by actual demonstration.)
26. Use the four numerals 3, 5, 6, 8, to form two-digit numbers, so that when they are multiplied you will get the largest product.
27. Cheryl will be 21 years old, three years from today, October 16, 1996. In what year was she born?
28. Name all the whole numbers that can replace the question mark so that the following expression has a value between 9 and 18. $2 + 3 \times ? =$ ___ .
29. If you write all the numbers from 1–100 how many times do you write the digit 5? Estimate first. Who in the class had the closest estimate?
30. Make a bar graph demonstrating the months of the year when each of the students in the class were born.

31. Put the appropriate signs between each of the following numbers so as to get an answer of 9.
 8 4 2 5 = 9

Many of these exercises were taken from or modified after problems in various issues of *Mathematics Teaching in the Middle School*, a publication of the NCTM. This is an excellent journal for teachers teaching students mathematics in the middle years.

Teachers can modify the difficulty of many of these exercises with minor revisions.

Answers

1. 11, 12, 13
2. 1,200 tiles
3. 105 km/h
4. 76
5. 2 p.m.
6. 7 cents
7. 0
8. 64
9. 6 kg
10. 144 seats
11. 3 dresses, 6 slacks, 9 blouses
12. Friday
13. A circle with an area of 793.8m^2
14. 9 hotdogs, 5 hamburgers
15. 9
16. 12.8 m (Use theorem of Pythagoras.)
17. More, because 49 is close to 50 and 3/5 is more than 1/2. One half of 50 is 25.
18. The average of the fractions has to be greater than zero but less than one-half. Both fractions could be less than one-half. They could have the same or different denominators.
19. $14
20. 5 days
21. 2, 8
22. 980, 84, 364
23. 4 pennies, 4 dimes, 3 quarters
24. 1,458
25. 10
26. 83 × 65 = 5,395
27. 1978
28. 3, 4, 5
29. 20
30. Draw the graph
31. (4 × 3) + 2 - 5 = 9

Measurability

One problem with grades is that they don't measure anything.

—Donald C. Mainprize, *ABCs for Educators*

Some Interesting Facts About Euler's Number

Sandra M. Pulver

Euler's number e, the base of natural logarithms, has various unusual properties which seem to be possessed by actions occurring in nature. It has an intimate relationship with natural phenomena such as radioactive decay, exponential growth and so on.

If a certain amount of material would produce an equal amount of the same material over a period of time, and the newly created material would produce still more new material at the same rate, then at the end of the period there would be approximately 2.7 times the original material. It can be shown that rate of growth involves the number, e.

Assuming that we start with one unit of this material and take a time halfway through some designated period, the original material will have increased to 1.5 units. For the second half of the period, there would be 1.5 pounds of the material producing still more material. At the end of the period, there would be a total of 2.25 pounds. As we divide the period into smaller parts, we take more and more account of the increase due to the newly created material itself.

If we are to find the exact amount of material at the end of the period, we must divide the period into an infinite number of infinitely small periods and add them together. The sum of this infinite series is numerically equivalent to e which is approximately 2.71828 . . .

The binomial formula can be used to derive the series for e as follows. Let b be any variable and apply the binomial formula to $(1 + b)^n$.

$$(1+b)^n = 1+nb + \frac{n(n-1)}{2\,!} b^2 + \frac{n(n-1)(n-2)}{3\,!} b^3 + \ldots$$

Now let k be a variable such that $b = 1/k$ and let x be a variable such that $kx = n$.

$$\left(1+\frac{1}{k}\right)^{kx} = 1+kx\left(\frac{1}{k}\right) + \frac{kx(kx-1)}{2\,!}\left(\frac{1}{k}\right)^2 + \frac{kx(kx-1)(kx-2)}{3\,!}\left(\frac{1}{k}\right)^3 + \ldots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{k}\right)}{2\,!} + \frac{x\left(x-\frac{1}{k}\right)\left(x-\frac{2}{k}\right)}{3\,!} = \ldots$$

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{kx} = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \frac{x^5}{5\,!} + \ldots$$

Now, let $x = 1$. Then

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k} = 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \frac{1}{5\,!} + \ldots$$

is defined as e.

Euler's number, e, can therefore be defined as the sum of the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n\,!}\,.$$

If x is a variable, then e^x can be approximated using the following:

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots$$

We can find e yet another way. We know that a given function $y = f(x)$ can be approximated by a sequence of polynomials $f_n(x)$ of the form

$$f_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

in which

$$f(0) = a_0,$$

$$f'(0) = a_1,$$

$$\frac{f''(0)}{2} = a_2,$$

.

.

.

$$\frac{f^{(n)}(0)}{n\,!} = a_n$$

We would like to represent the function $f(x) = e^x$ near $x = 0$ by such a sequence to get the general polynomial of $f(x) = e^x$.

For the given function, the derivatives at $x = 0$ are

$$f(0) = e^0 = 1$$

$$f'(0) = 1$$

.

.

.

$$f^{(n)}(0) = 1$$

Therefore

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

If $f(x) = e^x$, then at $x = 1$

$$f_n(x) = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}$$

and the original series,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is called the power series of e.

Often money is lent with the understanding that when earnings accumulate they are to be added to the original investment at specified times and thus become part of a new principal. Interest accrued on this basis is called compound interest.

If P dollars are lent at r percent for each interest period, the amount for the first period,

$$A_1 = P + Pr = P(1 + r).$$

This amount bears interest for the second period,

$$A_2 = P(1 + r) + P(1 + r) = P(1 + r)^2.$$

If we accumulate the amount of an investment at the end of a number of periods, then the amount, A, can be shown as follows,

$$A_1 = P(1 + r)$$
$$A_2 = P(1 + r)^2$$
$$A_3 = P(1 + r)^3$$
$$A_4 = P(1 + r)^4$$
$$A_k = P(1 + r)^k.$$

We can conclude that the compound interest formula is $A = P(1 + r)^k$.

When interest is compounded n times per year at rate r per year, then the rate per period is r/n and the number of period in t year is nt. The formula then becomes

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Suppose the interest is compounded continuously. Then n increases without limit, thus, $nt \to \infty$. If the principal, P, is equal to 1, the amount, A has the following limit as $nt \to \infty$.

$$A = \lim_{nt\to\infty} \left(1 + \frac{1}{nt}\right)^{nt} = e$$

That is, if \$1 is continuously compounded at rate of r for t years where $rt = 1$, the amount accumulated will be \$2.72.

Suppose the beginning principal is P, the annual interest rate is r, and the time in years is t. Then we can find the final amount, A, when interest is compounded continuously, by the following formula:

$$A = Pe^{rt}$$

If \$500 is invested at 6 percent compounded continuously for 40 years, the final amount is

$$A = Pe^{rt}$$
$$A = 500e^{.06(40)}$$
$$A = 500e^{2.4}$$
$$A = \$5{,}511.50$$

Suppose \$175 is deposited in a saving account where the interest rate is 9½ percent compounded continuously. When will the original deposit be doubled?

$$A = Pe^{rt}$$
$$350 = 175\, e^{.095t}$$
$$ln\ 2 = ln\ e^{.095t}$$
$$\frac{ln\ 2}{.095} = \frac{.095t}{095}$$
$$\frac{.6931}{.095} = t$$
$$t = 7.3 \text{ years}$$

For a certain strain of bacteria, $k = 0.584$ when t is measured in hours. In how many hours will 4 bacteria increase to 2,500 bacteria?

The general formula for growth and decay is

$$y = ne^{kt}$$
$$2{,}500 = 4e^{.584t}$$
$$625 = e^{.584t}$$
$$ln\ 625 = .584t$$
$$\frac{ln\ 625}{.584} = t$$
$$= \frac{ln\ (6.25 \times 10^2)}{.584}$$
$$= \frac{ln\ 6.25 \times (2\ ln\ 10)}{.584}$$
$$= \frac{1.8326 + 2\ (2.3026)}{.584} \approx 11 \textit{ hours}$$

Another of e's interesting properties is that the infinite series can be easily converted into a continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ldots}}}}}}$$

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Student to his parents:
"I got an underwater mark on that last test."
"What kind of grade is that?"
"Below 'C' level."

—Bob Phillips, *More Good Clean Jokes*

The Probability of Winning and Losing at Craps and Roulette

Sandra M. Pulver

One of the most popular betting games around seems to be the game of craps. We can use this to our advantage in the classroom by teaching something in which our students will have an interest.

Craps is a challenging game because it involves finding the sum of an infinite geometric series. It is played by having the player throw two dice. If a sum of 7 or 11 appears, the player automatically wins. The player loses immediately if the total is 2, 3 or 12. But if the total is any one of the remaining six possible sums (a 4, 5, 6, 8, 9 or 10), the player neither wins nor loses on the first throw, but continues to roll the dice until he/she either duplicates the first throw or gets a sum of 7.

The total shown by the dice on the player's first throw is called his/her point. If that player throws his point next, he wins. If he throws a 7 next, he will lose.

It is well known that when craps is played in gambling houses, the house never throws the dice. We may therefore feel reasonably certain that the odds are against the player who is throwing the dice.

The set of all possibilities of getting one of the numbers is as follows:

2	3	4	5	6	7	8	9	10	11	12
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)					
	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)				
		(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)			
			(4,1)	(4,2)	(4,3)	(4,4)	(4.5)	(4,6)		
				(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	
					(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

We then obtain the following probabilities for the various throws:

Total of Throw	Probability
2 or 12	1/36
3 or 11	2/36 or 1/18
4 or 10	3/36 or 1/12
5 or 9	4/36 or 1/9
6 or 8	5/36
7	6/36 or 1/6

We see at once that the probability that the player throwing the dice will win on the first throw (getting a 7 or 11) is

$6/36 + 2/36 = 8/36 = 2/9.$

The probability that he/she will definitely lose (that is, get a 2, 3 or 12) is

$1/36 + 2/36 + 1/36 = 4/36 = 1/9.$

The chance that he/she will either win or lose on the first throw is then

$2/9 + 1/9$ or $3/9 = 1/3.$

Therefore, the probability that the first throw will not be decisive is 2/3.

Suppose the shooter's point is 4, for example, an event with probability 1/12. Then on the second roll the conditional probability of rolling another 4 is 1/12, the conditional probability of losing immediately by rolling a 7 is 1/6, and thus the probability of no decision on the second roll is 3/4. Hence the probability of winning, given that the shooter's point is 4 is

$$\frac{1}{12} + \frac{3}{4} \times \frac{1}{12} + \left(\frac{3}{4}\right)^2 \times \frac{1}{12} + \left(\frac{3}{4}\right)^3 \times \frac{1}{12} + \dots = \frac{1}{12}\left(\frac{1}{1-\frac{3}{4}}\right) = \frac{1}{3},$$

the sum of the probabilities of winning on the 2nd, 3rd, 4th, . . . rolls of the dice.

Similarly, the probability of winning, given that the shooter's point is 5 is found to be

$$\frac{1}{9} + \frac{13}{18} \times \frac{1}{9} + \left(\frac{13}{18}\right)^2 \times \frac{1}{9} + \left(\frac{13}{18}\right)^3 \times \frac{1}{9} + \dots = \frac{1}{9}\left(\frac{1}{1-\frac{13}{18}}\right) = \frac{2}{5},$$

and the probability of winning, given that the shooter's point is 6 is 5/11.

So the total probability of the shooter's winning is

P(7 or 11 on the 1st roll) +
P(4 on 1st roll) × P(win | 4 is point) +
P(5 on 1st roll) × P(win | 5 is point) +
P(6 on 1st roll) × P(win | 6 is point) +
P(8 on 1st roll) × P(win | 8 is point) +
P(9 on 1st roll) × P(win | 9 is point) +
P(10 on 1st roll) × P(win is point) =

$$\frac{2}{9} + \frac{1}{12} \times \frac{1}{3} + \frac{1}{9} \times \frac{2}{5} + \frac{5}{36} \times \frac{5}{11} + \frac{5}{36} \times \frac{5}{11} + \frac{1}{9} \times \frac{2}{5} + \frac{1}{12} \times \frac{1}{3} = \frac{244}{495}.$$

The probability of the shooter's losing is therefore

$$1 - \frac{244}{495} = \frac{251}{495}.$$

The game is only very slightly unfavorable to the shooter, with the expected payoff for an even-money bet of $1 being

$$\$1 \times \frac{244}{495} + \$(-1)\frac{251}{495} = -\$.014\overline{14}$$

—a loss of less than 1½ cents per game. The reason so much money changes hands in a craps game, of course, is that many large bets may be made for or against the shooter and the game usually requires less than a minute to reach a decision.

• • • • • • •

The probability of calculating the odds in the game of roulette is also suitable for a student of elementary probability and will provide another interesting application for the classroom.

The roulette wheel contains 38 slots, of which two are numbered 0 and 00, and the rest are numbered from 1 through 36. Half of the numbers 1 through 36, (not including 0 and 00), are red, and the other half black.

The player can bet on red, black, odd numbers, even numbers or any of the 38 numbers.

If the player wins on red, black, odd or even, the house plays even (the amount of money the player bet). If the player wins on one of the 38 numbers, the house pays 36 times the money wagered.

The probability of winning on even is 18/38, because half of the 36 numbers are even. The probability of winning on odd is also the same. The same holds true for betting on red or black, because the 0 and 00 are not colored.

The probability of winning when betting on one of the numbers is 1/38, and this is where the "house" gets its odds because the house pays only 36 times the amount bet, not 38.

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A student of statistics defines a census taker:
"A census taker is a man who goes from house to house increasing the population."

The Principle of Mathematical Induction: Its Power in Proving Conjunctive Propositions

Murray Lauber

With the development and advance of the computer, the interest in one of its conceptual cohorts, discrete mathematics, has grown. Perhaps because of the new relevance of this branch of mathematics, the principle of mathematical induction, commonly referred to in textbooks simply as induction and abbreviated hereafter in this article as PMI, has become an increasingly important component of college-level mathematics courses and is, at the same time, making its way back into the high school curriculum. Apart from its inherent appeal there are, important pedagogical reasons for teachers to deepen their acquaintance with this principle.

The PMI is an apparently modest method of proving propositions about discrete numbers such as counting numbers or integers. Its apparent modesty is belied by its applicability to conjectures that at first seem beyond its scope. A case in point is its applicability to theorems about rational numbers, for example $\frac{d}{dx}(x^r) = rx^{r-1}$ (Beaver 1993).[1] There are many other apparently unlikely targets for the PMI including the focus of this article, "conjunctive conjectures," that is, conjectures consisting of the conjunction of two or more simpler conjectures.

The PMI—A Description and Explanation

In its simplest form, the PMI may be stated as follows:

A proposition $P(n)$ is true for all integers $n \geq m$ (where m is a given integer, often 1) if the following two conditions hold:

1) the proposition $P(m)$ is true and
2) if the proposition $P(k)$ is true for any integer $k \geq m$, then so is the proposition $P(k + 1)$.

The PMI can be demonstrated to follow logically from the induction axiom of the set of integers. However, one can accept it on an intuitive basis without recourse to that axiom if one reasons as follows. Restricting the argument to the case where $m = 1$, we may trace the implications of the PMI as follows. Condition 1) guarantees that P(1) is true. Condition 2), referred to as the induction hypothesis, then guarantees that if P(1) is true, so is P(2); if P(2) is true, so is P(3); if P(3) is true, so is P(4); and so on ad infinitum. Thus $P(n)$ must be true for n any counting number.

A Simple Example of the PMI

Before examining the applicability of the PMI to conjunctive propositions, let us consider a classic, more straightforward example. Consider the series of odd counting numbers $1 + 3 + 5 + 7 + 9 \ldots + (2n - 1) + \ldots$. Using the conventional S_n to represent the sum of the first n terms of the series, it is easy to determine that $S_1 = 1$, $S_2 = 4$, $S_3 = 9$, $S_4 = 16$, $S_5 = 25$ and so on. These observations lead to the conjecture that $S_n = n^2$ or that $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$ for all $n \in N$, where N represents the natural or counting numbers. Intuitively, one seems justified in making this conjecture with some confidence. However, in the world of mathematics, the fact that this kind of intuitively based confidence has sometimes been ill-founded justifies the insistence that at some point such confidence be bolstered by proof. So let us proceed to see how the PMI can be applied to prove this conjecture. To do this, we formulate the conjecture into a proposition as follows:

Let $P(n)$ be the proposition that $1 + 3 + 5 + 7 + 9 \ldots (2n - 1) = n^2$ for all integers $n \geq 1$. Then

1) P(1) is true since $1 = 1^2$.
2) Suppose that $P(k)$ is true for arbitrary $k \in N$, that is that

$$1 + 3 + 5 + 7 + 9 \ldots + (2k - 1) = k^2. \quad ①$$

We need to show that given assumption ① then $P(k + 1)$ is also true or that

$$1 + 3 + 5 + 7 + 9 \ldots + (2k - 1) + (2k + 1) = (k + 1)^2. \quad ②$$

[$2k + 1$ is the $(k + 1)^{st}$ term obtained by putting $n = k + 1$ in $2n - 1$.]

We begin by assuming that ① holds and add $2k + 1$ to both sides to obtain

$1 + 3 + 5 + 7 + 9 + \ldots + (2k - 1) - (2k + 1) = k^2 + (2k + 1)$

$\Rightarrow \quad 1 + 3 + 5 + 7 + 9 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$ or ②.

Thus if ① is true, then ② is also true, that is $P(k)$ implies $P(k + 1)$.

3) According to the PMI, since 1) and 2) both hold [$P(1)$ is true, and if $P(k)$ is true then $P(k + 1)$ is also true], $P(n)$ holds for all $n \in N$.

The PMI and Conjunctive Propositions

In some cases a conjecture made with confidence, when subjected to the rigor of the PMI, particularly part 2), finds its validity to be hostage to a "missing piece." The missing piece may be a second pattern, separate from the one on which the conjecture was based, but one that may be placed in conjunction with the first to form an overarching compound proposition whose validity can be confirmed using the PMI. Such propositions will be referred to as conjunctive propositions.

Two examples growing out of rather different problems will serve to illustrate the missing piece phenomenon. The first, based on a simple game involving counters, includes the formulation of a conjunctive proposition based on three observed patterns but leaves the proof for the reader to explore. The second example, based on the Fibonacci Sequence, includes a conjunctive proposition and its proof.

Example 1

The first example concerns a counter game for two players, A and B. There are a given number of counters in play at the beginning of the game. Players A and B alternate moves with A making the first move. Each move consists of removing one or two counters. The person forced to remove the last counter loses the game. If each player is motivated to win and plays with complete foreknowledge,

i) determine who wins when the game begins with
1 counter,
2 counters,
3 counters,
4 counters,
5 counters,
6 counters,
7 counters,
8 counters or
9 counters;

ii) make a conjecture about who wins that can be applied to any initial number of counters.

One way to explore this problem is through a tree diagram (Laufer 1984, 188–91). Such a diagram, by tracing all of the possibilities, can reveal the best possible way to play such a game. The method used here is to summarize the observations relating to i) in a chart, and use the chart to try to discover patterns to be incorporated in our conjecture as follows.

	Number of Counters	**Who Wins**	**Explanation**
a)	1	B	A is compelled to remove the one and only counter and B wins. (No motivation or skill needed.)
b)	2	A	A can take 1 counter compelling B to take the last one. A wins.
c)	3	A	A can take 2 counters compelling B to take the last one. A wins.
d)	4	B	If A takes 1 counter there are 3 counters when B first moves and, by c), B wins. If A takes 2 counters there are 2 counters when B first moves and, by b), B wins.
e)	5	A	A can take 1 counter leaving 4 counters when B first moves. By d), A wins.
f)	6	A	A can take 2 counters leaving 4 counters when B first moves. By d), A wins.
g)	7	B	If A takes 1 counter there are 6 counters when B first moves and, by f), B wins. If A takes 2 counters there are 5 counters when B first moves and, by e), B wins.
h)	8	A	A can take 1 counter leaving 7 when B first moves. By g), A wins.
i)	9	A	A can take 2 counters leaving 7 when B first moves. By g), A wins.

The patterns of wins appears to be BAABAABAA . . . . From the first nine cases we can conjecture that the general pattern of wins is as follows:

Number of Counters	Who Wins
1	B
2	A
3	A
4	B
5	A
6	A
.	.
.	.
.	.
$3n + 1$	B
$3n + 2$	A
$3n + 3$	A

This conjecture may be formulated into a proposition consisting of the conjunction of three statements as follows:

Let $P(n)$, $n \geq 0$, $n \in Z$ [Z denotes the integers], be the proposition that

i) B wins if there are $3n + 1$ counters,
ii) A wins if there are $3n + 2$ counters and
iii) A wins if there are $3n + 3$ counters.

We have already shown that P(0) is true, that is B wins if there is 1 counter, and A wins if there are either 2 or 3 counters. Further, it is possible to show that if $P(k)$ is true for any $k \in Z$, $k \geq 0$, then $P(k + 1)$ must also be true. In this case $P(k)$ is the proposition

i) B wins if there are $3k + 1$ counters,
ii) A wins if there are $3k + 2$ counters and
iii) A wins if there are $3k + 3$ counters.

Putting $n = k + 1$ in $P(n)$ it follows that $P(k + 1)$ is the proposition

i) B wins if there are $3k + 4$ counters,
ii) A wins if there are $3k + 5$ counters and
iii) A wins if there are $3k + 6$ counters.

The reader is left to explore the problem of showing that $P(k + 1)$ follows from $P(k)$, and may conclude in the process that it may not be possible to prove any one of i), ii) or iii) of the proposition $P(n)$ independently of the others. But, with persistence, one can prove them in conjunction. The proof involves the recursive kind of reasoning invoked in the explanations of cases d) to i) in the chart above.

1 counter
2 counters
3 counters . . . .
. . . .who wins?

Example 2

The second example comes from the Fibonacci Sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55 . . .

This sequence has recursive definition: $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_{n-1} + F_n$. The Fibonacci Sequence is well known because it arises in many often unexpected contexts. Another of its attractions is the many patterns associated with it, some more obvious than others. This example focuses on two patterns that may not be so obvious, one taken from one of the problems in the centre calendar of *Mathematics Teacher* (1993). The two patterns are

i) $F_1^2 + F_2^2 = 1^2 + 1^2 = 2 = F_3$
$F_2^2 + F_3^2 = 1^2 + 2^2 = 1 + 4 = 5 = F_5$
$F_3^2 + F_4^2 = 2^2 + 3^2 = 4 + 9 = 13 = F_7$
. . .
. . .
. . .
$F_{n-1}^2 + F_n^2 = F_{2n-1}$, $n \geq 2$, $n \in N$ (a conjecture only at this point).

ii) $F_1F_2 + F_2F_3 = (1 \times 1) + (1 \times 2) = 1 + 2 = 3 = F_4$
$F_2F_3 + F_3F_4 = (1 \times 2) + (2 \times 3) = 2 + 6 = 8 = F_6$
$F_3F_4 + F_4F_5 = (2 \times 3) + (3 \times 5) = 6 + 15 = 21 = F_8$
. . .
. . .
. . .
$F_{n-1}F_n + F_nF_{n+1} = F_{2n}$, $n \geq 2$, $n \in N$ (a conjecture only at this point).

If we try to prove either of conjectures i) or ii) by itself, we find that we run into the "missing piece" phenomenon referred to earlier. In fact, as I discovered in a failed attempt to prove conjecture ii) on its own, each is the other's missing piece. Following on this lead, let $P(n)$ be the conjunctive proposition that for any counting number n greater than 1:

$F_{n-1}^2 + F_n^2 = F_{2n-1}$ and $F_{n-1}F_n + F_nF_{n+1} = F_{2n}$.

The proof of $P(n)$ by the PMI is as follows:

1) P(2) is true since $F_{2-1}^2 + F_2^2 = F_{2\times2-1}$ and $F_{2-1}F_2 + F_2F_{2+1} = F_{2\times2}$ ①
That is, $F_1^2 + F_2^2 = F_3$ and $F_1F_2 + F_2F_3 = F_4$.
2) Assume that $P(k)$ is true for arbitrary $k \in N$, $k \geq 2$, that is, that ②
$F_{k-1}^2 + F_k^2 = F_{2k-1}$ and $F_{k-1}F_k + F_kF_{k+1} = F_{2k}$.

Then we need to show that $P(k + 1)$ is also true, that is that

$F_k^2 + F_{k+1}^2 = F_{2k+1}$ and $F_kF_{k+1} + F_{k+1}F_{k+2} = F_{2k+2}$.

One can show that ② follows from ① using the definition of the Fibonacci Sequence, the commutative and associative laws of addition and multiplication, and the distributive law of multiplication over addition as follows:

a) Consider $F_k^2 + F_{k+1}^2$ of ②

$F_k^2 + F_{k+1}^2 = F_k^2 + (F_{k-1} + F_k) F_{k+1}$	Def'n of Fib. Seq.
$\Rightarrow F_k^2 + F_{k+1}^2 = F_k^2 + F_{k-1} F_{k+1} + F_k F_{k+1}$	Dist. law
$\Rightarrow F_k^2 + F_{k+1}^2 = F_k^2 + F_{k-1} (F_{k-1} + F_k) + F_k F_{k+1}$	Def'n of Fib. Seq.
$\Rightarrow F_k^2 + F_{k+1}^2 = [F_{k-1}^2 + F_k^2] + \{F_{k-1} F_k + F_k F_{k+1}]$	Comm., Assoc., and Dist. laws
$\Rightarrow F_k^2 + F_{k+1}^2 = F_{2k-1} + F_{2k}$	Assumption ①
$\Rightarrow F_k^2 + F_{k+1}^2 = F_{2k+1}$	Def'n of Fib. Seq.

This verifies that the first half of ② follows from ① .

$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k [F_{k-1} + F_k] + F_{k+1} [F_k + F_{k+1}]$	Def'n of Fib. Seq.
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k F_{k-1} + F_k^2 + F_k F_{k+1} + F_{k+1}^2$	Dist. and Comm. laws
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k^2 + F_{k+1}^2 + [F_{k-1} F_k + F_k F_{k+1}]$	Comm. and Assoc. laws
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_k^2 + F_{k+1}^2 + F_{2k}$	Assumption ①
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_{2k+1} + F_{2k}$	Argument a), above
$\Rightarrow F_k F_{k+1} + F_{k+1} F_{k+2} = F_{2k+2}$	Def'n of Fib. Seq.

This verifies that the second half of ② follows from ① .Thus ② follows from ①, that is P(*k*) implies P(*k* + 1).

3) Since 1) and 2) both hold, by the PMI, the proposition P(*n*) is true for all $n \in N$, $n \geq 2$.

The Strong Principle of Mathematical Induction

Further examination of Example 2, above, can lead to a more encompassing proposition than P(*n*) that can be easily proved using a different form of the PMI as follows. One may notice that there are other patterns in the Fibonacci Sequence similar to i) and ii). For example, it can be shown that

$F_{n-1}F_{n+1} + F_nF_{n+2} = F_{2n+1}$.

These patterns may compel one to suspect that there is a more general pattern of which each of these is but a particular case. In fact, there is. This pattern may be formulated into the proposition P(*m*) as follows:

For $n, m \in N$, $n \geq 2$, $m \geq 0$, $F_{n-1}F_{n+m-1} + F_nF_{n+m} = F_{2n+m-1}$.

Here P(0) is the proposition $F_{n-1}^2 + F_n^2 = F_{2n-1}$, or conjecture i) of the last example, and P(1) is the proposition $F_{n-1}F_n + F_nF_{n+1} = F_{2n}$, or conjecture ii). The proposition P(*m*) can be easily proved using the Strong Principle of Mathematical Induction (SPMI). Many textbooks in discrete mathematics, including Roman (1989, 53), include formal descriptions of this principle. Suffice it to say here that, with respect to this example, the SPMI could be applied by showing that the following hold:

1) P(0) and P(1) are both true and
2) if P(*k* – 1) and P(*k*) are both true for $k \geq 1$, then P(*k* + 1) is also true.

This is a special case of the SPMI, but it should be clear without a full description of that principle that if both 1) and 2) hold, then P(*m*) holds for all $m \in n$, $m \geq 0$. Assume, for example, that both 1) and 2) hold. Then if both P(0) and P(1) hold, by 2), P(2) also holds. Further, if both P(1) and P(2) hold, by 2), P(3) also holds. Further, if both P(1) and P(2) hold, by 2), P(3) also holds. And so on, ad infinitum. The proof of 1) is the same as that of P(*n*) in example 2, above; the proof of 2) is left to the reader.

Benefits of Familiarity with Conjunctive Propositions

Examples 1 and 2, and the generalization of Example 2 in the previous section, illustrate the applicability of the PMI (or its variant, the SPMI) to compound propositions, specifically conjunctive propositions. There are many problems in which integers may be called into play as indices or counters where such propositions are the most appropriate generalizations of observed patterns. Being sensitive to the need for considering the option of such propositions, and aware of the applicability of the PMI to the proof, can enhance one's ability to establish that patterns conjectured on the basis of a limited number of cases apply in general.

Conclusion

The principle of mathematical induction deserves the increasing attention that it appears to be receiving in our schools and colleges. It boasts applicability to a wide variety of problems involving either explicit or implicit integral indices and continues to challenge the mind because of the mental agility required in dealing with that variety. It has many affinities with the inductive processes that are vital components in the operation of programming of computers. For example, the logic of a program loop is analogous to that of the principle of mathematical induction. Consequently, the PMI has become a vital tool in proving program correctness, including verifying the legitimacy of program loops and demonstrating the validity of recursive algorithms. But perhaps its inherent beauty—particularly its ability to entice the mind and to lead it on a gratifying logical excursion—remains the greatest pedagogical appeal of this principle.

Note

1. Beaver proves this theorem by applying the PMI to the following cases in sequence:

i) $y = x^n$, $n \in N$;

ii) $y = x^{-n} = \frac{1}{x^n}$, $n \in N$, using the quotient rule;

iii) $y = x^{1/n}$, $n \in N$, using $x^{1/(n+1)} = \frac{x^{1/n}}{\left[x^{1/(n+1)}\right]^{1/n}}$ and the quotient rule;

iv) $y = x^{b/n} = \left[x^{1/n}\right]^b$, $n, b \in N$, using the chain rule; and

v) $y = x^{-b/n} = \left[\frac{1}{x^{1/n}}\right]^b$, $n, b \in N$, using the quotient rule.

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Accountability

Accountability means giving an account of the use of your ability.
[Not a bad idea, actually.]

—Donald C. Mainprize, *ABCs for Educators*

Blueprints, the Division Algorithm, Consistent Systems and Augmented Matrices

David E. Dobbs

Suppose that you are trying to design a rectangular carpet (or poster or floor or . . .) and you have decided to follow the plans in a crumpled old blueprint. The blueprint clearly specifies that the area of the rectangular region is to be $6x^2 + 19x + 12$ and the length is $2x + 5$. It is difficult to read the specified width of the rectangle because the blueprint is badly smudged, but it seems that the width is $ax + b$, for some numbers a and b. What can you conclude?

The answer is that either you have misread the blueprint or the author of the blueprint made a mathematical error. Various forms of the explanation could be given in courses ranging from prealgebra to precalculus. Among the topics that would be reinforced or motivated are uniqueness of the remainder in the division algorithm for polynomials; the Factor Theorem; and the role of rank and determinants in using augmented matrices to test for inconsistency in systems of linear equations.

The most direct explanation has to do with expressing area as a product: $(ax + b)(2x + 5) = 6x^2 + 19x + 12$. Since division is the inverse operation of multiplication, $ax + b = (6x^2 + 19x + 12) \div (2x + 5)$. Carrying out this division leads to

$$\begin{array}{r|l} & \quad 3x + 2 \\ 2x + 5 & 6x^2 + 19x + 12 \\ & 6x^2 + 15x \\ & \qquad\;\; 4x + 12 \\ & \qquad\;\; 4x + 10 \\ \cline{2-2} & \qquad\qquad\;\; 2 \end{array}$$

Notice that the quotient is $3x + 2$ and, more importantly, the remainder is 2. Now, if $ax + b = (6x^2 + 19x + 12) \div (2x + 5)$, the remainder in the above long division would be 0, by the uniqueness of the remainder in the division algorithm. (For a careful statement of this result, see Dobbs and Peterson 1993, 139, 143.) In particular, $(6x^2 + 19x + 12) \div (2x + 5)$ is not a polynomial, and we can conclude that no numbers a and b exist with the above property.

By slightly generalizing the above reasoning, we can discover part of the Factor Theorem (see Dobbs and Peterson 1993, 141 for a statement and proof of the general result). Actually, we will end up proving the special case in which the dividend is a quadratic polynomial. Consider the question whether a given linear polynomial, $cx + d$, is a factor of a given quadratic polynomial, $ex^2 + fx + g$. Assume, as in most of the interesting examples, that $c \neq 0$ and $d \neq 0$. We find the following string of equivalent statements:

$$\begin{array}{r|l} & ax + b \\ cx + d & ex^2 + fx + g \end{array} \quad \Leftrightarrow$$

$(ax + b)(cx + d) = ex^2 + fx + g \Leftrightarrow$ [equate corresponding coefficients]
$ac = e$, $ad + bc = f$, $bd = g$. By solving linear equations, we find, in particular, that $a = e/c$ and $b = g/d$. With these expressions for a and b fixed, the string of equivalences continues:

$$\Leftrightarrow \quad \left[\frac{f - \frac{e}{c}d}{c}\right] d = g \quad \Leftrightarrow \text{[rewrite algebraically]}$$

$f(d/c) - e(d/c)^2 = g \Leftrightarrow$ [rewrite algebraically]
$e(-d/c)^2 + f(-d/c) + g = 0$. Thus, we have shown that $cx + d$ is a factor of $ex^2 + fx + g$ if and only if $-d/c$ is a root of $ex^2 + fx + g$. Since $cx + d = c\,(x - (-\,d/c))$, this means that $x - (-\,d/c)$ is a factor of $ex^2 + fx + g$ if and only if $-d/c$ is a root of $ex^2 + fx + g$. At this point, it would be natural for an algebra class to conjecture the more general result that if r is any number, then

For each of these samples, calculate its mean and its sample variance (s^2). Two examples follow:

1. Consider the sample {1, 1, 1} where the mean is 1 and $s^2 = \frac{0^2 + 0^2 + 0^2}{3 - 1} = \frac{0}{2} = 0.$

   The same value of s^2 will result for {2, 2, 2,}; {3, 3, 3,} and {10, 10, 10}.

2. {1, 2, 10}

   The mean is $\frac{1 + 2 + 10}{3} = \frac{13}{3}$ and

$$s^2 = \frac{(1 - \frac{13}{3})^2 + (2 - \frac{13}{3})^2 + (10 - \frac{13}{3})^2}{(3 - 1)} = \frac{\frac{100}{9} + \frac{49}{9} + \frac{289}{9}}{2} = \frac{438}{18} = \frac{73}{3}$$

The same value of s^2 will result for the permutations of 1, 2 and 10, namely, {1, 10, 2}; {2, 1, 10}; {2, 10, 1}; {10, 1, 2}; {10, 2, 1}.

Table 2 reports the entire set of results. The answers are reported as fractions so that the rounding which would accompany decimal representations does not interfere with the results.

Table 2

Sample	Mean	s^2
1, 1, 1	1	0
1, 1, 2	4/3	1/3
1, 1, 3	5/3	4/3
1, 1, 10	4	27
1, 2, 1	4/3	1/3
1, 2, 2	5/3	1/3
1, 2, 3	2	1
1, 2, 10	13/3	73/3
1, 3, 1	5/3	4/3
1, 3, 2	2	1
1, 3, 3	7/3	4/3
1, 3, 10	14/3	67/3
1, 10, 1	4	27
1, 10, 2	13/3	73/3
1, 10, 3	14/3	67/3
1, 10, 10	7	27
2, 1, 1	4/3	1/3
2, 1, 2	5/3	1/3
2, 1, 3	2	1
2, 1, 10	13/3	73/3
2, 2, 1	5/3	1/3
2, 2, 2	2	0
2, 2, 3	7/3	1/3
2, 2, 10	14/3	64/3
2, 3, 1	2	1
2, 3, 2	7/3	1/3
2, 3, 3	8/3	1/3
2, 3, 10	5	19
2, 10, 1	13/3	73/3
2, 10, 2	14/3	64/3
2, 10, 3	5	19
2, 10, 10	22/3	64/3
3, 1, 1	5/3	4/3
3, 1, 2	2	1
3, 1, 3	7/3	4/3
3, 1, 10	14/3	67/3
3, 2, 1	2	1
3, 2, 2	7/3	1/3
3, 2, 3	8/3	1/3
3, 2, 10	5	19
3, 3, 1	7/3	4/3
3, 3, 2	8/3	1/3
3, 3, 3	3	0
3, 3, 10	16/3	49/3
3, 10, 1	14/3	67/3
3, 10, 2	5	19
3, 10, 3	16/3	49/3
3, 10, 10	23/3	49/3
10, 1, 1	4	27
10, 1, 2	13/3	73/3
10, 1, 3	14/3	67/3
10, 1, 10	7	27
10, 2, 1	13/3	73/3
10, 2, 2	14/3	64/3
10, 2, 3	5	19
10, 2, 10	22/3	64/3
10, 3, 1	14/3	67/3
10, 3, 2	5	19
10, 3, 3	16/3	49/3
10, 3, 10	23/3	49/3
10, 10, 1	7	27
10, 10, 2	22/3	64/3
10, 10, 3	23/3	49/3
10, 10, 10	10	0

Finally, find the mean of all 64 values of s^2:

$$\frac{\sum s^2}{64} = \frac{\frac{2400}{3}}{64} = \frac{800}{64} = 12.5 \;.$$

Observe that this is the same value as σ^2 for the original population. The $(n - 1)$ correction is "just right."

If a specific three-element sample is randomly selected from our population, P, its sample variance might be much less than the population variance or it might be much larger. However, the mean of all sample variances is exactly equal to the population variance that the sample variances are intended to estimate. Because of this, statisticians say that s^2 is an "unbiased estimator" for σ^2.

Also, compute the mean of all sample means:

$$\frac{\frac{768}{3}}{64} = 4.$$

In other words, the mean of all the sample means is the same as the mean of the original population, P. This is accomplished without "tinkering" with the denominator. For this reason, the same definition of the mean $\left(\frac{\text{sum}}{\text{size}}\right)$, is used for both samples and the population. Statisticians say that the sample mean is an unbiased estimator for the population mean.

Challenges

1. Suppose that the values of s^2 of Table 2 were calculated using a denominator of 3, which is the sample size. Show that the mean of all the values of s^2 is no longer equal to the value of σ^2.
2. Replicate the steps of this article with a different four-element population.
3. Replicate the steps of this article for other size populations and other size samples.

Substitute natural numbers for the variables *a* and *b* so that the following expression is correct:

$(a + a) + 3(b + b) = a^a + b^a$

Seven Mathematical Processes in the Protocol: Activities Give Them Life

A. Craig Loewen

The Common Curriculum Framework for K–12 Mathematics: Western Canadian Protocol for Collaboration in Basic Education (Alberta Education 1995), also known as the Protocol, specifies seven different mathematical processes: communication, connections, estimation and mental mathematics, problem solving, reasoning, technology and visualization. The Protocol (Alberta Education 1995, 5) states that

> There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and to encourage lifelong learning in mathematics.

It is clear that these components are critical to mathematics instruction in two ways: (a) the components are to be integrated into the mathematics program and activities, therefore a subject of instruction themselves; and (b) the components represent the most important aims of the mathematics program in that they relate to lifelong learning as opposed to specific concepts, facts or generalizations. This article will attempt to address each of these seven mathematical processes in turn and to provide classroom examples designed for the upper elementary classroom.

Communication

It is not difficult to build a case for the importance of communication in the classroom. Teaching and learning are primarily acts of communication; at least it is fair to say they are highly dependent on language. Consider the unique difficulties associated with trying to teach an individual who speaks a language different from your own, or consider the challenges associated with teaching problem solving to one who does not read or write. Skemp (1987) puts forward an excellent case for defending the role of communication and language in the mathematics classroom. As teachers, our task becomes one of deciding what forms of communication are most important, deciding how those forms can be taught and deciding which activities most effectively integrate with these elements of communication.

Activity: Dice Game (Problem)

Objective: Conduct probability experiments and explain the results, using the vocabulary of probability (Alberta Education 1995, 58).

Problem: Jane is playing a dice game with Frank. There are two dice to pick from and Jane gets first choice. One die is a regular six-sided die; the other die also has six sides, but contains three ones and three sixes. The game is played by rolling your die six times and summing the values rolled. The player with the highest sum after five rolls wins. Which die should Jane choose if she wants to win?

Discussion: Problem-solving activities provide a wonderful context for encouraging discussion and the use of mathematical terms. One common type of activity associated with a problem such as this is the comprehension guide (Stiff 1986). A comprehension guide is essentially a list of questions which students answer prior to actually seeking a solution. Obviously, the comprehension guide serves to reinforce the first stage of problem solving (understand the problem) which students often skip or dismiss. Questions which may appear on a comprehension guide include the following:

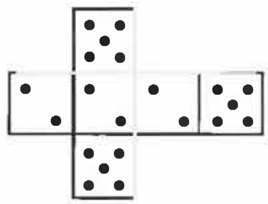
- In what ways are the two dice similar?
- In what ways do the two dice differ?
- How many times will each player roll his or her die?
- How is the winner for each game determined? Is it possible to have a tie?

Of course, questions such as these can be posed verbally and discussed in both large and small groups, but it is important for students to experiment with, and experience, both verbal and written forms of communication.

While Stiff (1986) talks about comprehension guides only as a means to reinforce the first stage of problem solving, comprehension guides can be extended to address the final stage (looking back) as well. Consider questions such as

- Is there another way to show which die Jane should choose?

- Assume the players will roll the die six times to play a game. How would this affect the die Jane should choose?
- Can you describe a rule which would help Jane decide which die she should choose?
- Assume the regular die was replaced with a die described by the net below. Which die should Jane choose now?



Many activities besides simple comprehension guides which emphasize communication skills. These activities are limited only by the imagination of the teacher. Consider short problem dramas (Matz and Leier 1992) where students act out a real-life situation which requires the application of some mathematical idea, process or principle. In a problem drama the actors freeze at some point during the presentation giving the audience time to find or compute a solution. The play is resumed to provide an answer to, and solution process for, the dramatized situation. Math journals also emphasize communication where students articulate daily their learnings, concerns or insights. Another option is the use of a math portfolio where students attempt to document their progress and select evidence of their mastery of concepts. The underlying purpose of these activities is to get students to articulate their thinking, and we can facilitate this by asking for information other than simple numerical answers.

Connections

There are many different types of connections which teachers must help students explore and develop during their study of mathematics. In order to develop a sense of mathematics in all of its facets students must develop connections between the following:

- Concepts and the physical world (applications to the world around them and to other disciplines)
- Concepts within mathematics
- Concepts and their representations (for example, manipulatives, models, diagrams, notations)
- Concepts and their related terms and algorithms

Each of these forms of connections individually represents one way to bring meaning to the learning of mathematics (or, more accurately, build understanding of mathematics). However, it is probably true that the greater the number of these connections which students hold, the more robust their learning and understanding of those concepts. In other words, by helping students build a variety and multitude of connections we help them build understanding.

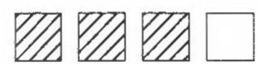
Activity: From Fractions to Percents (Exploration)

Objective: Demonstrate and explain the meaning of percentage concretely, pictorially and symbolically (Alberta Education 1995, 31).

Materials: Two-color counters

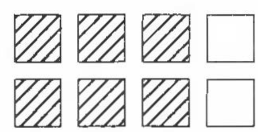
Description: In this exploration we are attempting to connect the concept of simple fractions to that of percents through the use of equivalent fractions. To start the activity, we may ask students to use their two-color counters to show a fraction of three-quarters:

$\frac{3}{4}$



The students are now asked to write the fraction and then add another row, to create:

$\frac{6}{8}$



(row one)

(row one)

The new fraction created is recorded and some discussion of how the new fraction (six-eighths) is similar to the first fraction (three-quarters) is conducted: the notion of equivalent fractions is emphasized. Students are asked to predict the next equivalent fraction, and, once a pattern is discovered, students are asked to generate the equivalent fraction that uses a total of 100 chips. Some students may wish to build the completed model.

Discussion: When working with models such as this, it is important to ask students to reflect the model in mathematical symbols at the same time the model is developed, and to ask students where each symbol within the notation can be found within the representation. For example, the mathematical equation reflected in the model above is

$$\frac{3}{4} \underset{\times 2}{\overset{\times 2}{=}} \frac{6}{8}$$

Students should be able to show what part of the model is represented by the 3, 4, 6 and 8, as well as 2.

Estimation and Mental Computation

Estimation is the ability to mentally approximate a value or measurement, while mental computation entails completing full computations without the aid of pencil, paper or computational aid (for example, calculator, computer and so on). Teaching estimation requires teaching skills such as rounding, and the notion of what is a reasonable, acceptable or tolerable estimate. Due to the need to address levels of acceptability in estimation activities, quick computations are often made using calculators or computers. These calculations are used to compare between the actual answer and the estimate; the use of the technological aid simply speeds feedback. Familiar estimation strategies include the front-end strategy, clustering, rounding, compatible numbers and special numbers (Reys 1986).

Unlike estimations, mental computations typically require exact answers. Teaching students to compute mentally involves teaching a variety of routines which can be carried out easily in the head. Many different mental computation strategies exist, including dropping common and trailing zeros, balancing, doubling, division by multiplying, thinking money and trading-off (Hope, Reys and Reys 1987).

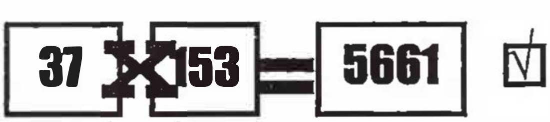
Estimation and mental computation strategies need to be consciously and carefully taught—on average, students do not seem to develop them readily or independently. These skills should probably be addressed daily and along with the other six mathematical processes integrated into daily activities.

Activity: Target 5,000 (Calculator Game)

Objective: Estimate, mentally calculate, compute or verify, the product (three-digit by two-digit) and quotient (three-digit divided by one-digit) of whole numbers (Alberta Education 1995, 32).

Materials: Gameboard (see Figure 1), calculator, pencils

Rules: The first player selects a two-digit number and enters it into the first box on his or her opponent's side of the gameboard (in the example below, the first player entered a 37). The second player now enters a value in the second box (in the example the opponent entered a 153). The second player now takes the calculator and computes the product of the two numbers entered and records it on the score sheet (the product is 5,661). The players now reverse roles on the second side of the score sheet. The player whose product is closest to 5,000 on a given round scores a single point by checking the small box to the right of the equation. The player who collects the most points in five rounds wins.



Discussion: After playing the game a number of times, it may be worth talking to the students about strategy in the game. Students quickly learn that certain numbers are unwise to give to an opponent (for example, 10, 25 and 50) as a second multiplier can be easily identified which provides an exact product of 5,000. The teacher may find it interesting to discuss the different strategies which students employ in trying to find the best possible match. For example, students may implement a rounding strategy, and combine that strategy with a mental computation division process or even employ a "delete trailing zeros" strategy to simplify necessary computations. All the different strategies students employ are worth articulating, as undoubtedly a surprising variety will emerge.

The teacher may opt to extend and adapt Target 5,000 in a number of ways. For example, the game can be made more difficult by allowing students to enter decimal values or by changing the target number to 50,000. The game can be further adapted by asking the first player to provide his or her opponent with the first multiplicand *as well as* the target number—this adaptation significantly increases both the challenge and the range of mental computations which must be made.

Problem Solving

Problem solving can hardly be considered a new addition to this or any other program as it has been promoted as a focus of mathematics instruction for many years. However, the biggest barrier to classroom implementation remains the collection of curricularly relevant problems and techniques for integrating such problems into daily classroom activities. A further challenge plagues the classroom teacher in trying to collect a range of problems: problems should range not only in level of difficulty but in the problem-solving strategies and skills which these problems address. Problems should also range according to their degree of open-endedness as well. In other words, variety is key: we want our students to experience many problems, some of which may have a single correct answer, and some of which may have many different equally acceptable answers and solutions. These pedagogical challenges, though significant, are certainly worth meeting, as problem-solving abilities remain one of the most important and generalizable learning outcomes of the mathematics program.

Activity: Floor Tile Problem (Application)

Objective: Cover a surface using one or more tessellating shapes (Alberta Education 1995, 52).

Materials: Floor Handout (see Figure 2), calculator, pencils, colored pencils, pattern blocks and/or pattern block stickers or cutouts (pattern block shapes cut from appropriately colored construction paper)

Description: In this activity students are given a large rectangle which represents the floor of a room (see Figure 2) and a collection of pattern blocks which represent floor tiles. The students are informed that each hexagon tile costs 8¢, trapezoid tiles cost 5¢, blue parallelogram tiles cost 4¢, triangle tiles cost 3¢ and white parallelogram tiles cost 3¢ (these values are also found printed at the bottom of the handout—Figure 2). The students are given the challenge of covering the entire floor space with tiles creating the most pleasing design or pattern for the least cost.

Discussion: This application-style problem is highly open-ended. In this problem students can elect to cover the entire floor space with a single pattern block, or may choose to use several of any number of the six regular shapes. What makes the activity interesting is that (a) students employ their own preferences in color and shape, and (b) there are two variables which are to be considered in the overall product: esthetic value and cost. The activity can be easily extended by adding other criterion. For example, the following added condition changes the problem significantly: tiles can only be purchased in packages of five and once a package is opened all five tiles must be purchased.

Reasoning

The Protocol document does not actually define reasoning, but lists some of the related abilities, including the ability to make sense of mathematics and the ability to build or generalize mathematical ideas from past experience. We could describe reasoning as the mental ability to justify a given concept from prior knowledge (to re-generate an idea), or as the mental ability to construct and justify a new idea from present and past experience (to generate a *new* idea). In essence, reasoning has both generative and regenerative properties. It makes sense that this act of reasoning can only occur when one has control over his or her thought processes, implying one must possess both general cognitive processes (for example, the ability to sort, classify, generalize, curtail and so on) and related metacognitive processes (that is, an awareness of and ability to monitor/control the general cognitive processes). In general, when we say we wish students to be able to reason, we want them to be able to justify their ideas by relating them to prior knowledge and related concepts—to be able to identify the clues which substantiate the truth of a concept or idea.

It is not difficult then to see how closely related reasoning is to the notion of connections and language, both discussed above. To be able to reason, one must be able to establish relationships between present concepts and prior knowledge, and these relationships are simply the connections we hope students will develop. Further, evidence of reasoning is most likely observed in the act of communication where one tries to persuade or dissuade another of a given idea or concept. This should further reinforce to us the importance of both communication and the development of connections in the instruction of mathematics.

But, how do we address reasoning in the classroom? One of the most obvious methods would be to ask "why" questions:

- *Why* do you think this is true?
- *Why* don't you think this result is correct?
- *Why* do you think that strategy would be appropriate?

Why questions seem to seek justification and clarification and thus encourage reasoning. Ideally we hope students will begin to internalize the asking of such questions thus making the acts of justification and clarification learner characteristics. A second obvious method would be the use of logic puzzles and games. The challenge with such activities however is to ensure that they are curricularly relevant (that is, based on objectives drawn from appropriate curriculum documents).

Activity: Klondike (Game)

Objective: Place an object on a grid, using columns and rows (Alberta Education 1995, 52).

Materials: Klondike Gameboard (see Figure 3)—one per player, Klondike Leaderboard (see Figure 3), pencils

Description: This game is played on a 10 × 10 grid. The objective of the game is to find the 15 gold coins which are hidden on this grid using the clues given by the leader (usually the teacher). Players are divided into groups or teams. The teams take turns calling out a single location on the grid. If a gold coin is hidden in that location, the team scores a single point. If no coin is hidden in that location, the teacher informs the players of the number of gold coins which are hidden in any of the adjoining locations. For example, using the leaderboard found in Figure 3, if the team called out location "column 5, row 2" the teacher would inform them that there were two coins hidden somewhere within the eight locations surrounding the location called. A team may only call one location on a turn whether or not they find a coin. The team that finds the greatest number of coins once all coins have been located is the winning team.

Discussion: Students may need some coaching on how to use the gameboard itself and on how to use the results generated by other teams. It may be useful to make an overhead copy of the student gameboard and model the use of the elimination strategy in identifying locations where no gold coins are hidden. This game could be extended by using locations on a coordinate system rather than a grid and further extended by including more than the first quadrant on the gameboard.

Technology

The sheer availability alone of technological tools such as the calculator and computer demands that teachers find a place for them in classroom instruction. There are, of course, many opportunities in the mathematics classroom for the application of the calculator and computer, but caution is necessary. Our challenge is to look for ways to use the calculator and computer such that the application positively affects the learning outcomes of students. In other words, we want to use the calculator and computer in such a way that students gain experiences that they would not otherwise have. As well, calculators and computers can be used to provide a contrasting or alternative perspective to a concept. We want to use the calculator and computer in support of student thinking processes, not to replace them. In this sense, these tools provide many interesting options especially in the realm of problem solving, primarily (a) where limitations in student computational ability can be overcome and (b) where the use of these tools enables students to consider and explore a greater range of more complex data within problems.

Activity: Race to 100 (Problem or Game)

Objective: Verify solutions to addition and subtraction problems, using estimation and calculators (Alberta Education 1995, 34).

Materials: Calculator, paper, pencil

Description: This activity can be conducted as either a game (a race) or a problem-solving exercise. In this problem, each letter of the alphabet is assigned a value based upon its position within the alphabet. The letter A has a value of 1, the letter B has a value of 2, C has a value of 3 and so on (see the chart below). The value of a word is determined by the sum of the values of its letters. The word "mathematics" has a value of 112. Can you find a word that has value of exactly 100? To make this a game, have a race to find out who can be first to find such a word.

A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7
H = 8	I = 9	J = 10	K = 11	L = 12	M = 13	N = 14
O = 15	P = 16	Q = 17	R = 18	S = 19	T = 20	U = 21
V = 22	W = 23	X = 24	Y = 25	Z = 26		

Discussion: One can quickly see why working with a calculator makes solving this type of problem possible: the amount of information and the degree of guess and check which is likely to accompany it can be quite overwhelming for many students. This problem can be presented as a problem of the day, or can be used in an ongoing manner as students create a graffiti board or bulletin board display of the different words which have been found. The problem can be adapted to include such extensions as finding words with values for every number 1 through 100 (are any values impossible?). Other extensions include the following: What is the longest word you can find with a value of 100? What is the shortest word you can find with a value of 100? Who can find the word with a value of 100 which has the most vowels? Whose name has the greatest value?

Hint: Try computing the values for the days of the week.

Visualization

The Protocol document borrows from Armstrong (1993 in Alberta Education 1995, 11) when it defines visualization as "thinking in pictures and images and the ability to perceive, transform and recreate different aspects of the visual-spatial world." Clearly imagination and the ability to reconstruct concepts in a concrete and semi-abstract form is critical to visualization. We recognize visualization as an important mathematical process in that the mental pictures students create at least partially enable the transfer of learnings from one context to another and more generally, represent one mode of knowing or acquiring new knowledge. To help students develop visualization skills the teacher could explore the consistent use of models, pictures, graphs and diagrams in the classroom. Along with the introduction of these models, students should also be requested to create such representations themselves to explain and describe how they perceive the concepts they encounter.

Activity: Negotiation (Game)

Objective: Demonstrates concretely, pictorially and symbolically place-value concepts to give meaning to numbers up to 10,000 (Alberta Education 1995, 28).

Materials: Overhead set of base ten blocks (see Figure 4), paper, pencil, overhead projector

Description: To begin this game, all players are asked to secretly record a three-digit number which contains no zeros (for example, 187 or 251). These numbers are held privately and not shared with any other player until the conclusion of the game. Once numbers are recorded, players take turns adding any one block (cutout) to the overhead projector (for example, if the first block added is a flat block the value shown on the screen is 100). Alternatively, a player may opt to remove any one block on a turn. In order to remove a block, there must be at least one of that size block on the screen (that is, there is no regrouping). If at any time the screen displays a value equivalent to the number recorded secretly at the beginning of the game, the player who wrote that number is the winner.

Discussion: In this game, the value on the screen is never verbalized until the conclusion of the game. Students must construct the value through the visual representation constructed. To dramatically increase the challenge of the game the teacher may omit the overhead base ten blocks, thus forcing students to visualize (and construct a list of) the values built during the game. To further adapt the game, have students select four-digit rather than three-digit numbers.

Conclusion

The seven mathematical processes represent the most important learning outcomes of our mathematics curriculum. These processes can be encouraged and addressed directly through a conscientious effort to integrate them within activities designed to address the more specific learning objectives given for each grade level. It is necessary to know and understand these mathematical processes as we work to select and collect curricularly relevant activities which can be successfully introduced into the mathematics classroom.

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Figure 1: Gameboard for Target 5000

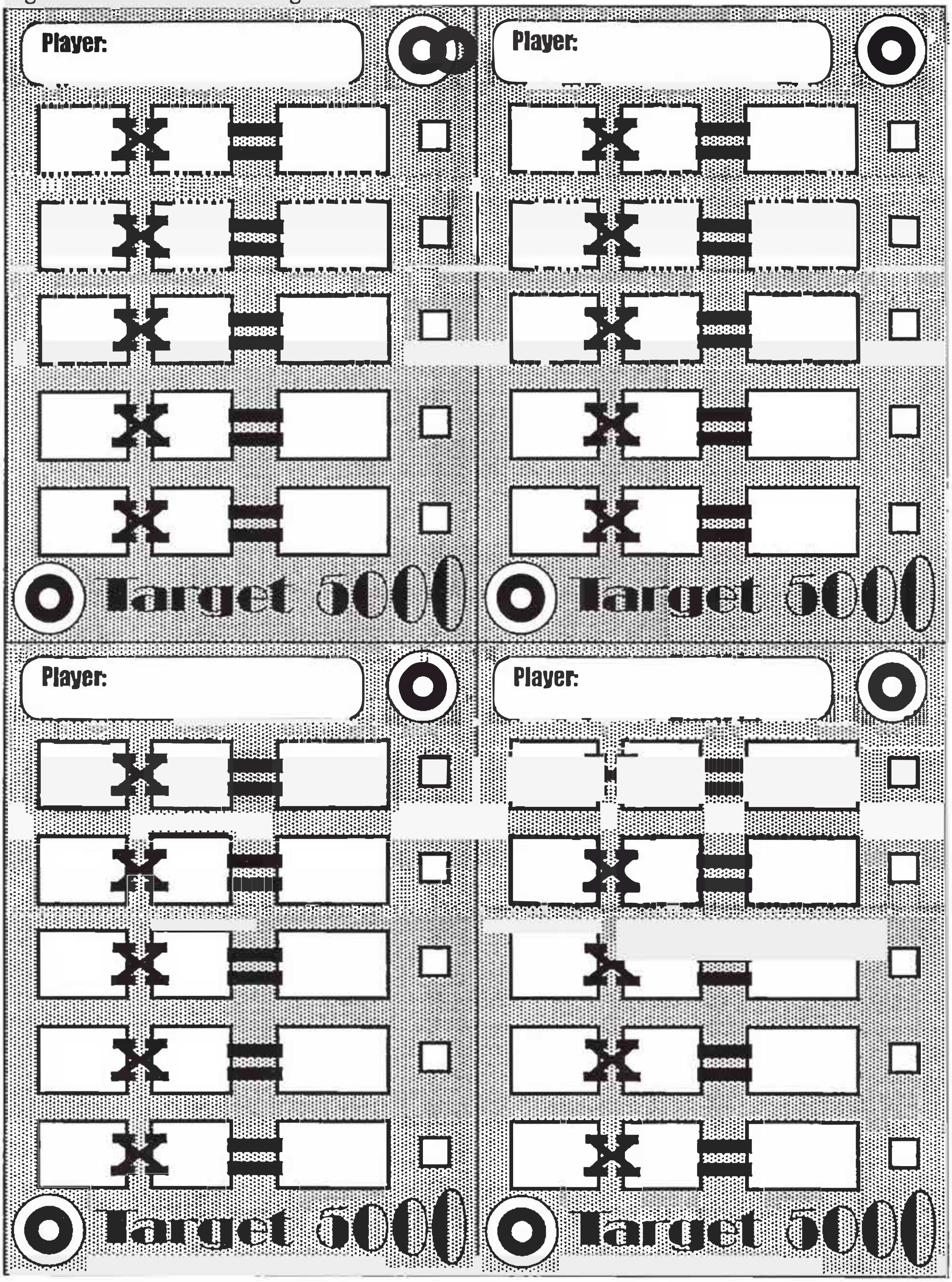


Figure 2: Floor Handout for Floor Tile Problem.

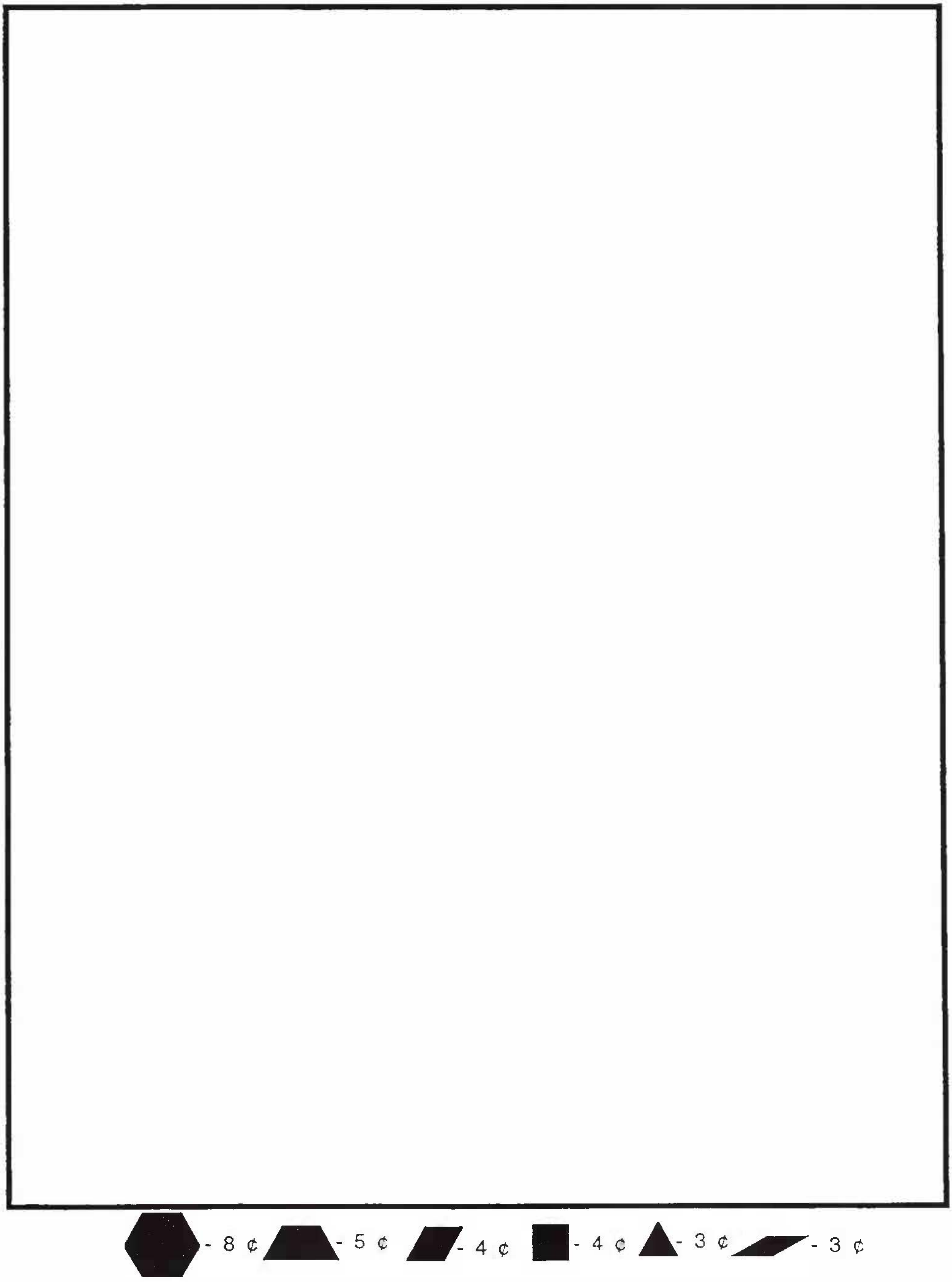
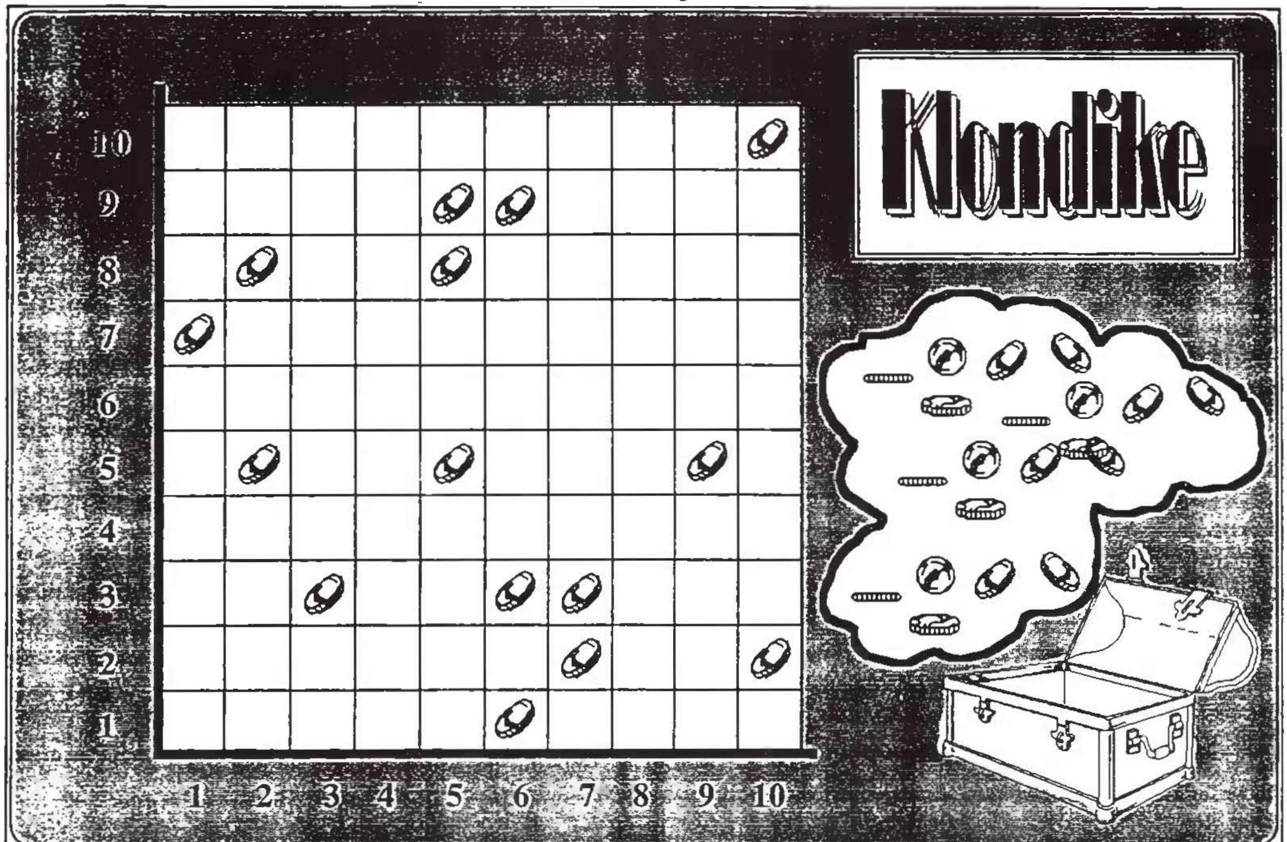


Figure 3: Leaderboard and Student gamecard for Klondike game.



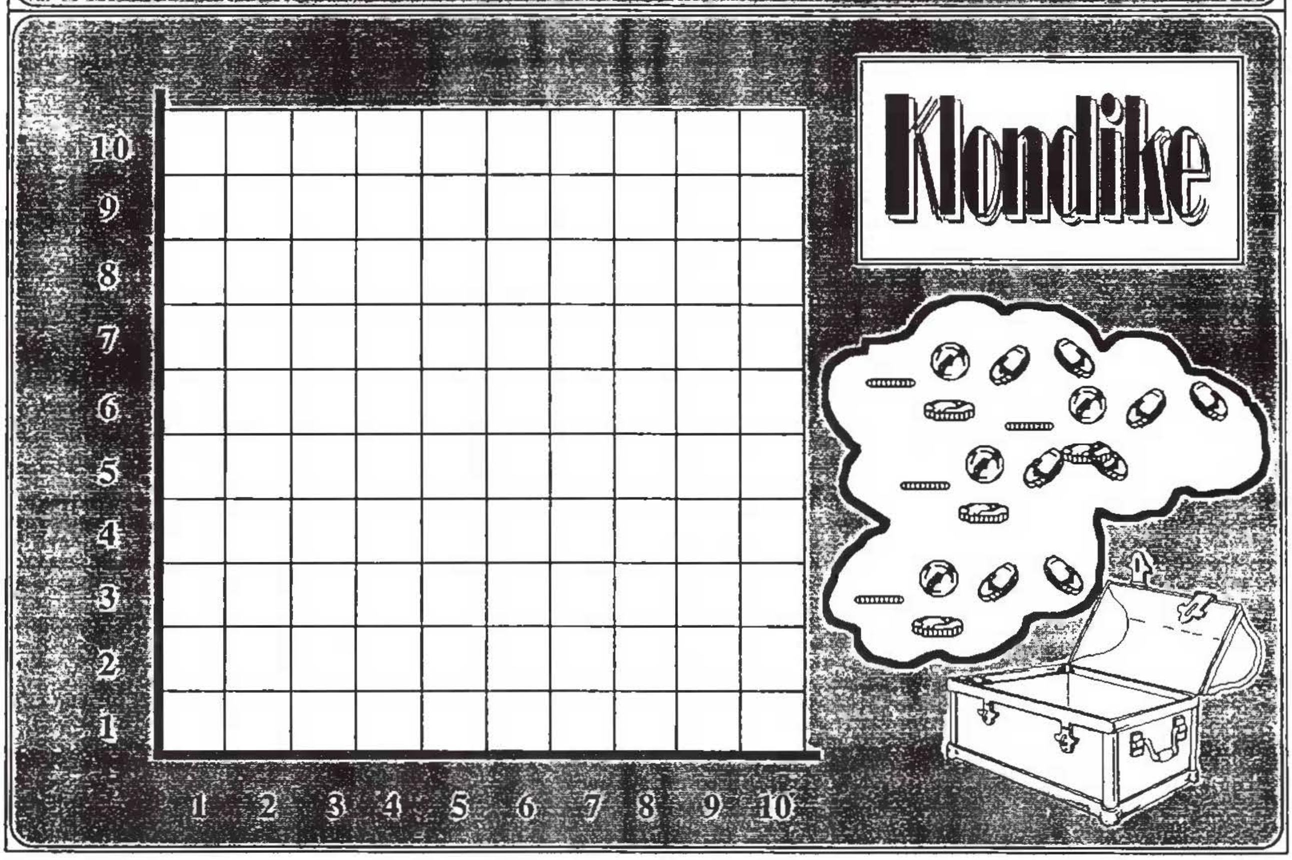
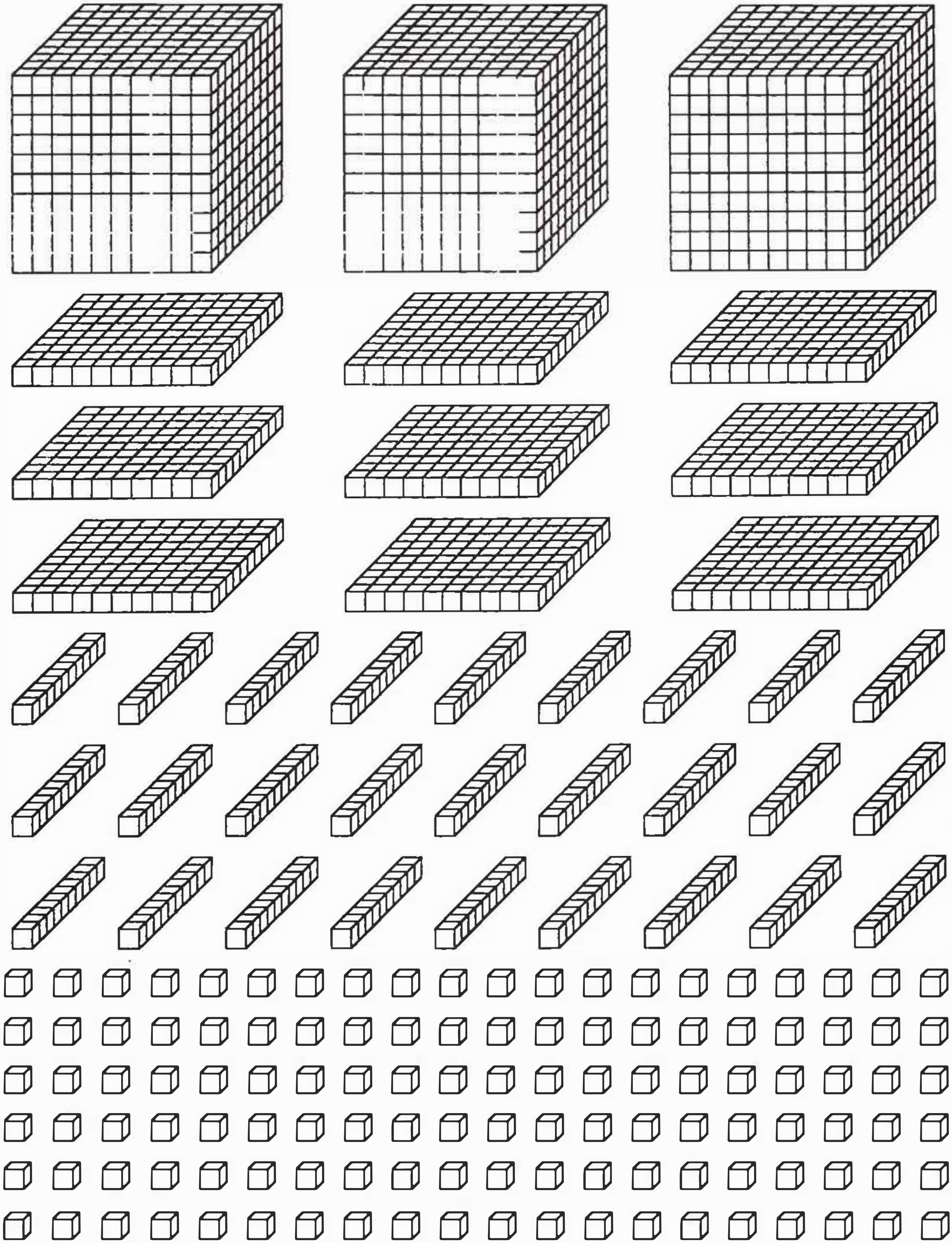


Figure 4: Base Ten Block cutouts for Negotiation (to be made into overhead transparency).



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