How Much Zooming Is Enough?

David E. Dobbs and John C. Peterson

The ability to use the "zoom in" and "zoom out" features of a graphing calculator enables us to focus on the important parts of a graph. As the examples given below show, the zooming features can be particularly useful when we are trying to solve equations or inequalities, as well as when trying to determine if two algebraic expressions define the same function. In viewing the graphs which arise from algebraic expressions, we need to know not only when to begin zooming but also when to stop. We can best make such practical decisions by using theory appropriately. To illustrate the procedure, we begin with five "zooming in" examples, and then give four "zooming out" examples.

Zooming In

Example 1 shows how zooming in on a "flat" part of a graph may reveal numerous x-intercepts and, thus, numerous zeros of the corresponding function.

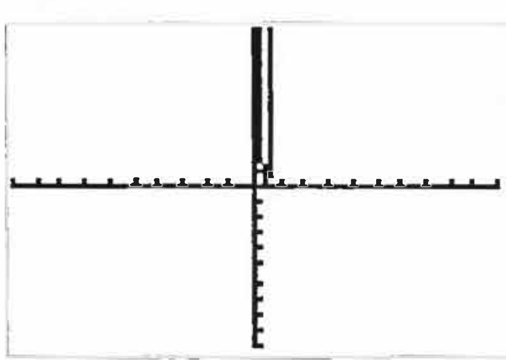
Example 1

Use a graphing calculator to solve the equation $1000x^4 - 1780x^3 + 1187.9x^2 - 352.262x + 39.1644 = 0$.

Solution

The real number solutions of the equation are the x-coordinates of the x-intercepts of the graph of $y = 1000x^4 - 1780x^3 + 1187.9x^2 - 352.262x + 39.1644$. Using the default setting of a TI-82 graphing calculator, we obtain the graph in Figure 1. How many times does this graph intersect the x-axis? The answer is not clear by inspecting Figure 1.

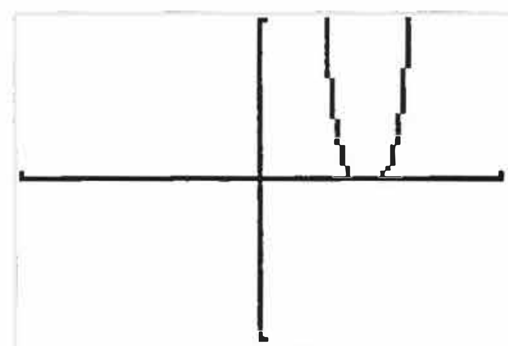
Figure 1



Xmin = −10 Xmax = 10 Xscl = 1
Ymin = −10 Ymax = 10 Yscl = 1

If you zoom in once (with both zoom factors set at 10), you find a graph like the one shown in Figure 2. Here, it looks as if there is a flat part of the graph which runs along an interval of the x-axis, but it is not clear whether the graph ever falls below the x-axis.

Figure 2

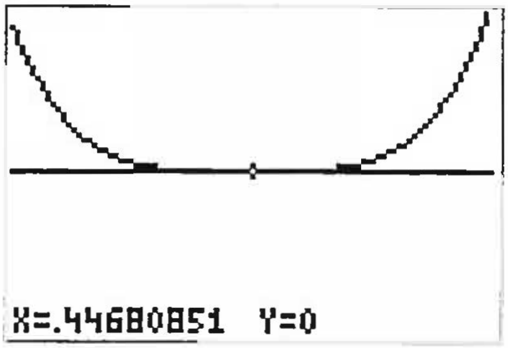


Xmin = −1 Xmax = 1 Xscl = 1
Ymin = −1 Ymax = 1 Yscl = 1

Tracing to a point near the middle of the flat part and then zooming in again, you get the graph in Figure 3. Using the trace feature again, you find a point on the graph with coordinates given by $x = 0.45106383$ and $y = -2.215 \times 10^{-6}$. Since this y-coordinate is negative, it seems as if the graph must cross the x-axis at least twice. (In fact it must, by applying the Intermediate Value Theorem for continuous functions.) Additional use of the trace feature on Figure 3 reveals enough sign changes of the y-values to indicate that the graph crosses the x-axis at least four times.

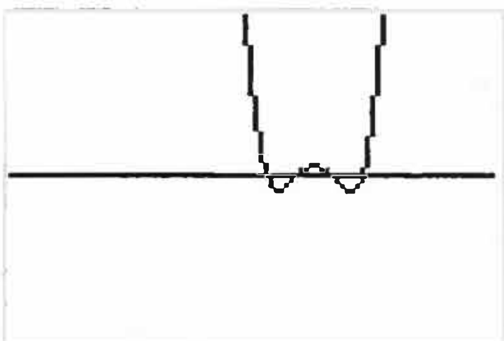
What would happen if you zoomed in and traced again? Would you find that the graph crosses the x-axis six (or possibly eight) times? How many x-intercepts are concealed within the "flat" parts of the graphs in Figures 1–3? How much zooming in is enough? The answer, in this case, depends on a theoretical result. According to one statement of the Fundamental Theorem of Algebra (Dobbs and Peterson 1993, 164–65), an nth degree polynomial has, counting multiplicities, exactly n complex roots. Thus, at least for this example, you have zoomed in enough. You have concluded that the given equation has four solutions and each of these is a real number. This can also be seen by inspecting Figure 4. (By the way, the

Figure 3



Xmin = −0.346808511 Xmax = 0.546808511 Xscl = 1
Ymin = −0.1 Ymax = 0.1 Yscl = 1

Figure 4



Xmin = −0.35 Xmax = 0.5 Xscl = 1
Ymin = −0.0001 Ymax = 0.0001 Yscl = 1

solutions are really rational numbers, namely $x = 0.43$, 0.44, 0.45 and 0.46.)

The method in Example 1 needs to be fine-tuned in case the underlying polynomial has a multiple root. Examples 2 and 3 show how the Factor Theorem, together with zooming in, deals with such situations.

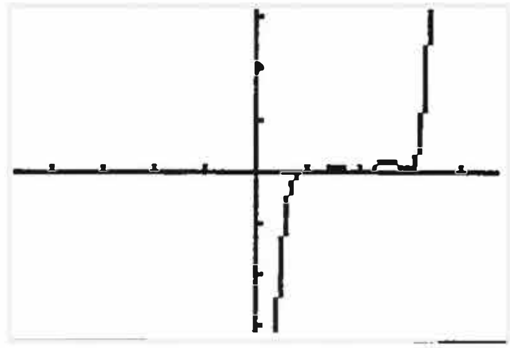
Example 2

Using a graphing calculator to solve the equation $x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996 = 0$.

Solution

As in Example 1, we first try to find all the real number solutions, by investigating the x-intercepts of the graph of $y = x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996$. An initial view of this graph is shown in Figure 5. As you can see, the graph seems to have flat parts on the x-axis near $x = 1$ and $x = 2$; also, it appears that the graph narrowly misses hitting the x-axis near $x = 3$.

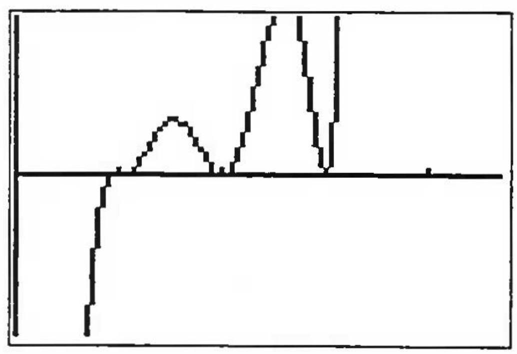
Figure 5



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = −3.1 Ymax = 3.1 Yscl = 1

By changing the window settings, you can obtain the graph in Figure 6. Although this view of the graph does not resolve the question, it does confirm our impression that we need to zoom in near $x = 1$, $x = 2$ and $x = 3$. By zooming repeatedly, you can check that there are solutions at $x = 1$, $x = 2$, $x = 2.99$ and $x = 3.01$. No matter how much more you zoom in (or out), you will not find any evidence of additional real number solutions.

Figure 6



Xmin = 0 Xmax = 4.7 Xscl = 1
Ymin = −0.2 Ymax = 0.2 Yscl = 1

According to the Fundamental Theorem of Algebra, the given seventh-degree polynomial has seven roots. If we have found all the real solutions (1, 2, 2.99, 3.01), you might suppose that three nonreal complex solutions remain to be found. However, according to the Conjugate Root Theorem (Dobbs and Peterson 1993, 168), nonreal complex zeros of a polynomial with real coefficients come in complete conjugate pairs, although 3 is not an even number! What's wrong?

In fact, there are no nonreal complex solutions in this example. We actually already have all seven solutions—all that is needed is a more careful application of the Fundamental Theorem of Algebra and the Linear Factor Theorem (Dobbs and Peterson 1993, 165). The point is that nth degree polynomials have exactly n roots if the roots are counted according to their multiplicities. In this example, 1 is a root with multiplicity three, 2 has multiplicity two, and 2.99 and 3.01 each have multiplicity one. Thus, we have found the even solutions, since $3 + 2 + 1 + 1 = 7$.

The above conclusions about multiplicity could possibly be conjectured by studying Figure 6, although the differences in behavior near $x = 2$ and near $x = 3$ may not be geometrically evident from that figure. But these differences can be determined algebraically by using the Factor Theorem, as follows. Let's consider $x = 1$. By substitution, you can check that $x = 1$ satisfied the given equation. So, by the Factor Theorem (Dobbs and Peterson 1993, 141), $x - 1$ is a factor of $x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996$. By division, you find the quotient, $x^6 - 12x^5 + 57.9999x^4 - 143.9994x^3 + 192.9987x^2 - 131.9988x + 35.9996$. Next, by substitution, you can check that $x = 1$ is a root of this sixth-degree polynomial. So, by the Factor Theorem, $x = 1$ is a root of the given seventh-degree polynomial of multiplicity at least two. Continuing in this way, you find that $x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996 = (x - 1)^3(x^4 - 10x^3 + 36.9999x^2 - 59.9996x + 35.9996)$. By calculation, you can check that $x = 1$ is not a root of $x^4 - 10x^3 + 36.9999x^2 - 59.9996x + 35.9996$. Thus, the process of successive divisions stops, and 1 is indeed a root with multiplicity three. The other assertions are verified similarly.

Example 3

Use a graphing calculator to solve the equation $x^6 + 2.989x^5 + 2.96699x^4 + 0.96697x^3 - 0.01103x^2 - 0.00001x = 0$.

Solution

This example is somewhat similar to Example 2. It turns out that -1 is a root of multiplicity three, while -0.001, 0 and 0.01 are each roots with multiplicity one. Thus, the only solutions (real or complex) are $x = -1, -0.001, 0$ and 0.01.

In the next example, matters become somewhat complex.

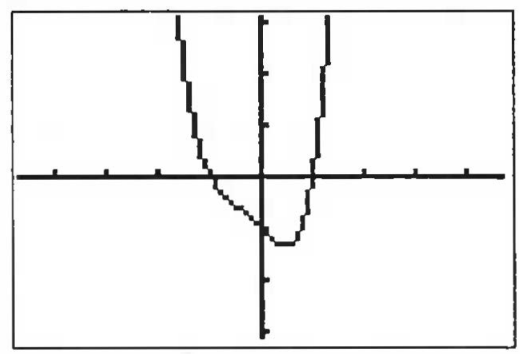
Example 4

Use a graphing calculator to solve the equation $x^4 + x^3 - x - 1 = 0$.

Solution

The graph of $y = x^4 + x^3 - x - 1$ in Figure 7 suggests that there are solutions near $x = -1$ and $x = 1$. By substitution, it is easy to verify that $x = -1$ and $x = 1$ are indeed solutions. Using division, as in Examples 2 and 3, you can check that $r_1 = -1$ and $r_2 = 1$ are each roots of multiplicity one. So, by the Fundamental Theorem of Algebra, two roots are still missing. The two missing roots—let's call them r_3 and r_4—satisfy $x^4 + x^3 - x - 1 = (x + 1)(x - 1)(x - r_3)(x - r_4)$, according to the Linear Factor Theorem. By division, r_3 and r_4 are the roots of $\frac{x^4 + x^3 - x - 1}{(x + 1)(x - 1)} = x^2 + x + 1$. Hence, by the Quadratic Formula, r_3 and r_4 are given by $\frac{-1 \pm \sqrt{3}i}{2}$. No amount of inspection of the graph in Figure 7 would reveal these nonreal complex solutions. In summary, the solution set of the given equation is $\left\{-1, 1, \frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2}\right\}$.

Figure 7



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = −3.1 Ymax = 3.1 Yscl = 1

The analysis in Examples 1–4 was possible in part because each of the functions being graphed was a polynomial. As Example 5 shows, "zooming in" can be applied to solve equations involving nonpolynomial functions, even though the algebraic theory of such functions is more complicated than that of polynomials.

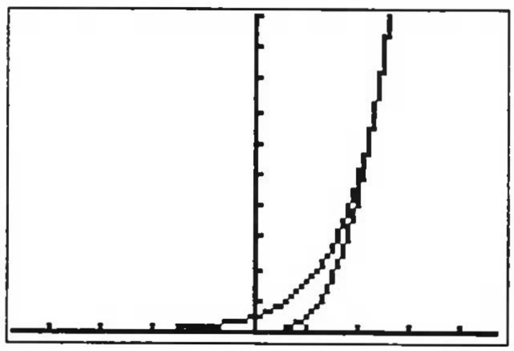
Example 5

Using a graphing calculator to find the real number solutions of the equation $x^\pi = \pi^x$.

Solution

The solutions that we seek are the x-coordinates of the points of intersection of the graphs of $y = x^\pi$ and $y = \pi^x$. As you can see from Figure 8, these graphs do not intersect at any point satisfying $0 \le x \le 2$, but they seem to be coincident for the x-values from slightly greater than 2 to at least 3. Are they really coincident or do the curves intersect at only some of the points indicated in Figure 8? How many points of intersection are there?

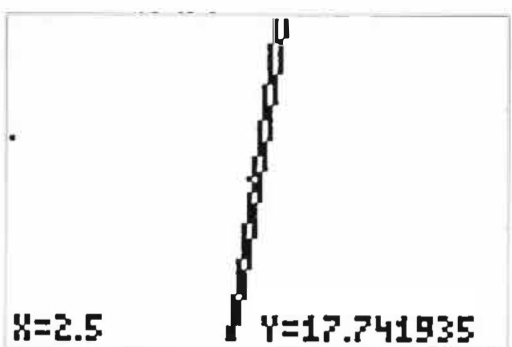
Figure 8



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = 0 Ymax = 20 Yscl = 2

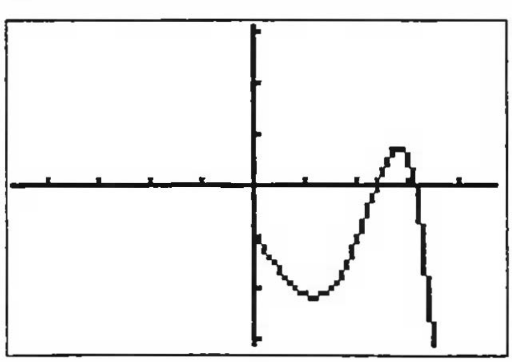
One way to proceed would be to "zoom in" on these graphs near $x = 2.5$, as shown in Figure 9. By continuing to zoom in, you find a point of intersection at $x = 2.3821791$. In addition, it is clear that there is another intersection point at (π, π^π). Additional zooming does not reveal any further solutions.

Figure 9



Xmin = 2.03 Xmax = 2.97 Xscl = 1
Ymin = 16.74193548 Ymax = 18.74193548 Yscl = 2

Figure 10



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = −3.1 Ymax = 3.1 Yscl = 1

There is another way to proceed which is more like the method used in Examples 1–4. The solutions of the equation $x^\pi = \pi^x$ are the same as the solutions of $x^\pi - \pi^x = 0$. The graph of $y = x^\pi - \pi^x$ in Figure 10 indicates two x-intercepts, corresponding to the two solutions which were found above.

It is natural to ask if the equation $x^\pi = \pi^x$ has more than the two solutions found above. The answer is "no." The reason depends on calculus and is part of some interesting history recounted in (Sved 1990).

Zooming Out

An ultimate type of intersection of two graphs occurs when they are coincident. This corresponds to equality of the functions being graphed. A currently popular method of verifying identities, especially trigonometric identities, is to check coincidence of the graphs of the left- and right-hand sides of an alleged identity. As you saw in Example 5, the apparent coincidence of portions of graphs in a figure generated by a graphing calculator may disappear when you take a closer look by zooming. Similarly, as Examples 6–8 show, zooming out can be used to distinguish between functions whose graphs may appear to be coincident when using a particular viewing window.

Example 6

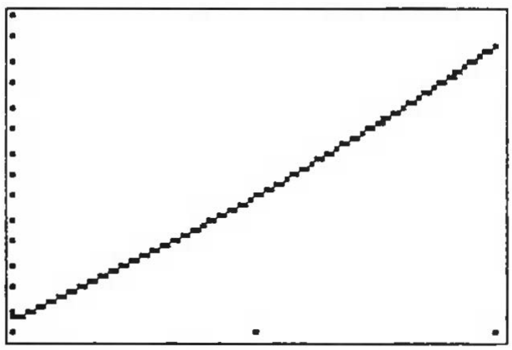
Suppose you view the graphs of $y = x^3$ *and* $y = 3x^2 - 2.99x + 0.99$ on a graphing calculator with window setting Xmin = 0.9, Xmax = 1.1, Xscl = 0.1, Ymin = 0.7, Ymax = 1.4 and Yscl = 0.05. Based on these graphs, would you conjecture that the functions f and g, given by $f(x) = x^3$ *and* $g(x) = 3x^2 - 2.99x + 0.99$, are equal? If so, zoom out to see if the new graphic evidence reinforces or disproves your conjecture. If possible, give a theoretical explanation for your new conclusion.

Solution

Figure 11 shows the graphs of f and g when viewed with the given window setting. To the naked eye, it appears that these graphs are coincident, and so one might conjecture on the basis of this evidence that $f = g$. However, if you zoom out, you obtain the graphs shown in Figure 12. Here, it is clear that the graphs of f and g are distinct, and so $f \neq g$.

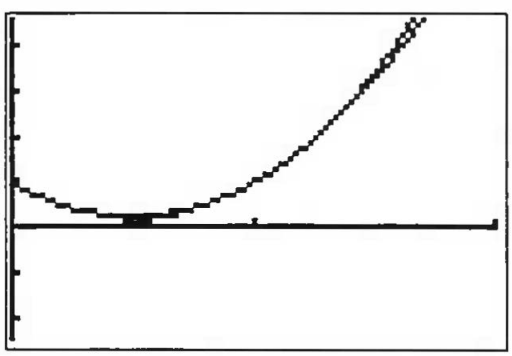
The same conclusion can be reached theoretically in a couple of ways. First, since the function $h = f - g$ is a third-degree polynomial, the Fundamental Theorem of Algebra tells us that h has at most three zeros. In particular, h is not identically zero, and so $f \neq g$. Second, f and g are unequal because they have different limits as $x \to -\infty$. Indeed, by the Leading Term Test (Dobbs and Peterson 1993, 152), $\lim_{x \to -\infty} f(x) = -\infty$ but $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} 3x^2 = \infty$.

Figure 11



Xmin = –0.9	Xmax =1.1	Xscl = 0.1
Ymin = 0.7	Ymax =1.4	Yscl = 0.05

Figure 12



Xmin = 0	Xmax = 2	Xscl = 0.1
Ymin = –2.45	Ymax = 4.55	Yscl = 0.05

Example 7

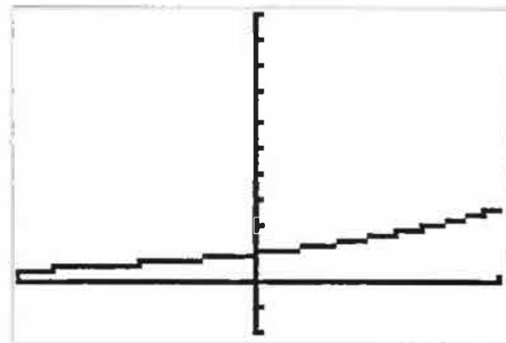
Follow the instructions of Example 6, for the functions f and g given by $f(x) = e^x$ and $g(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$ with the initial window setting Xmin = –1, Xmax = 1, Xscl = 1, Ymin = –2, Ymax = 10 and Yscl = 1.

Solution

Figure 13 shows the graph of f and g when viewed with the given window setting. To the naked eye, it appears that these graphs are coincident, and so one might conjecture on the basis of this evidence that $f = g$. However, if you zoom out, you obtain the graphs shown in Figure 14. Here, it is clear that the graphs of f and g are distinct, and so $f \neq g$.

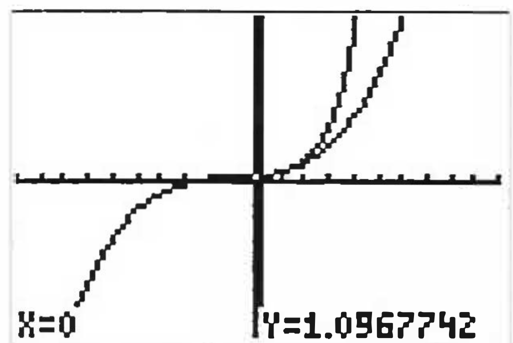
The same conclusion can be reached theoretically. Indeed, f and g are unequal because they have different limits as $x \to -\infty$. Of course, $\lim_{x \to -\infty} f(x) = 0$. However, by the Leading Term Test, $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{1}{6} x^3 = -\infty$.

Figure 13



Xmin = –1	Xmax =1	Xscl = 1
Ymin = –2	Ymax =10	Yscl = 1

Figure 14



Xmin = –10	Xmax = 10	Xscl = 1
Ymin = –56	Ymax = 64	Yscl =1

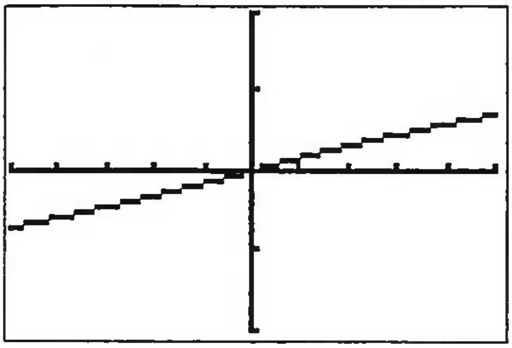
Example 8

Follow the instructions of Example 6, for the functions f and g given by $f(x) = \sin x$ and $g(x) = -\frac{1}{6}x^3 + x$ with the initial window setting Xmin = $-\pi/4$, Xmax = $\pi/4$, Xscl = $\pi/20$, Ymin = –2, Ymax = 2 and Yscl = 1.

Solution

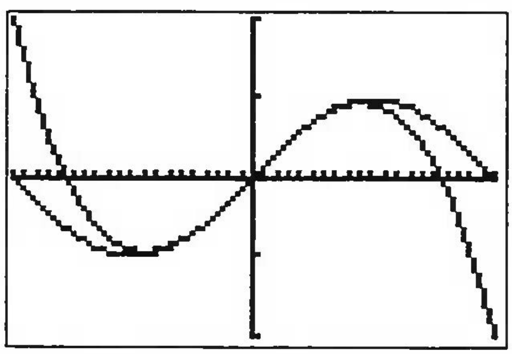
Figure 15 shows the graphs of f and g when viewed with the given window setting. To the naked eye, it appears that these graphs are coincident, and so one might conjecture on the basis of this evidence that $f = g$. However, if you zoom out by setting Xmin = $-\pi$, Xmax = π, and not changing the other settings, you obtain the graphs shown in Figure 16. Here, it is clear that the graphs of f and g are distinct, and so $f \neq g$.

Figure 15



Xmin = $-\pi/4$ Xmax = $\pi/4$ Xscl = $\pi/20$
Ymin = -2 Ymax = 2 Yscl = 1

Figure 16



Xmin = $-\pi$ Xmax = π Xscl = $\pi/20$
Ymin = -2 Ymax = 2 Yscl = 1

The same conclusion can be reached theoretically in a couple of ways. For instance, you can check that f and g have different limit behavior at $-\infty$ (or at ∞). Alternatively, f and g are unequal because they have different sets of zeros. Indeed, f has infinitely many zeros, while the Fundamental Theorem of Algebra tells us that g has at most three zeros.

In the final example, we see how entire intervals can be misinterpreted when using a graphing calculator to solve an inequality. The remedy, once again, involves zooming out.

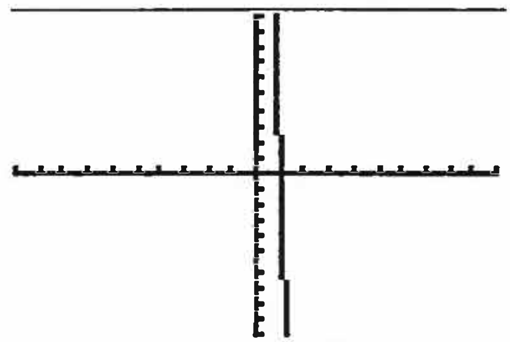
Example 9

Use a graphing calculator to solve the inequality $0.1x^3 - 3.4x^2 - 50.7x + 54 < 0$.

Solution

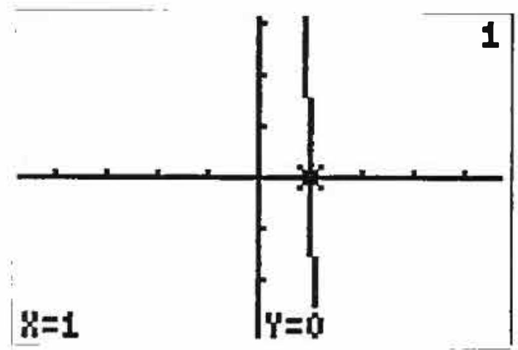
Using the default setting, we find the graph of the equation $y = 0.1x^3 - 3.4x^2 - 50.7x + 54$ shown in Figure 17. By changing the range settings to those indicated in Figure 18, and tracing, we see that this graph has an x-intercept at $x = 1$. Thus, since the solution of the inequality arises from the portion of the graph that lies below the x-axis, it appears from Figure 18 that the solution is the interval $(1,\infty)$.

Figure 17



Xmin = -10 Xmax = 10 Xscl = 1
Ymin = -10 Ymax = 10 Yscl = 1

Figure 18



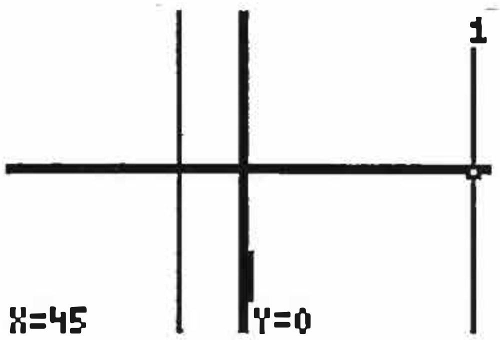
Xmin = -4.7 Xmax = 4.7 Xscl = 1
Ymin = -3.1 Ymax = 3.1 Yscl = 1

Is this the correct solution? Let's zoom out. By tracing on the resulting graph, as shown in Figure 19, you see that the graph intersects the x-axis a second time (at $x = -12$) and a third time at $x = 45$. So, now it seems as if the solution set is $(-\infty,-12)\cup(1,45)$. Is *this* the solution?

Will additional zooming out show that the graph turns around yet again? How much zooming out is enough? How many times can you expect the graph to turn? In general, first derivative information from calculus is needed to analyze turning points. In fact, in this example, you have zoomed out enough. The solution set is $(-\infty,-12)\cup(1,45)$.

In closing, it should be noted that both zooming in and zooming out procedures are often needed in analyzing one example. For instance, to solve the inequality $0.01x^3 - 1.0400x^2 + 2.0949999x - 1.019898 > 0$, one needs to zoom in near $x = 1$ to identify the zeros

Figure 18



Xmin = −46 Xmax = 48 Xscl = 1
Ymin = −31 Ymax = 31 Yscl =1

at $x = 0.99$ and $x = 1.01$, while one needs to zoom out to detect the zero at $x = 102$.

References

Dobbs, D. E., and J. C. Peterson. *Precalculus.* Dubuque, Iowa: Wm. C. Brown, 1993.

Sved, M. "On the Rational Solutions of $x^y = y^x$." *Mathematics Magazine* 63 (February 1990): 30–33.

Given the equations
$7x + 5y - z = 8$ and $y + z = 11$, find all the ordered natural number triplets which satisfy the two equations.