Analysis of Teaching Trigonometry in the Context of University Mathematics

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Trigonometry is a unit of study in the mathematics curriculum that is rich with content and concepts, contains clear interrelationships among its parts and provides us with important mathematical applications. Unfortunately, high school students conceive of trigonometry, not as a unified whole in which content and concepts are logically connected within the overall structure, but rather as a set of formulas that are difficult to memorize. They usually do not like trigonometry. Why? Because students do not understand it sufficiently. This lack of understanding is often evident among college or university students as they attempt to solve problems involving even the simplest applications of trigonometric concepts.

This article is an attempt to find some answers to these questions and apparent problems by analyzing the trigonometry units of the Alberta high school curriculum. Particular attention will be paid to the content of the high school curriculum that is essential in the study of calculus at postsecondary institutions.

Trigonometry is studied in Grades 10, 11 and 12. The basic textbooks for this unit of study, written by B. Kelly, B. Alexander and P. Atkinson, are [1] *Mathematics 10* (1987) (Chapter 14), [2] *Mathematics 11* (1990) (Chapter 10) and [3] *Mathematics 12* (1991) (Chapters 5 and 6), respectively. These textbooks are well written with some good examples and interesting applications of trigonometric functions. Despite that fact, students seem to experience difficulties with trigonometry.

The areas that seem to be most prevalent with respect to students experiencing difficulties are, in order of increasing difficulty, treated below.

Basic Formulas

It is interesting to note that students seem to know the reciprocal and quotient identities studied in the Section 10.2 of [2] and Sections 5.4 and 6.1 of [3]. However, if we want to further enhance the students' understanding of trigonometry, a discussion of the following questions would be of considerable benefit:

1. Can sine of positive x be negative?
2. Can sine squared of negative x be negative?
3. Can sine cube of negative x be negative?
4. Can sine of x squared be negative?
5. Can sine of x cube be negative?
6. Can sine of x be equal to negative cosine of the same x?
7. Can sine of x be equal to cosine of the same x?
8. Can sine squared of x cube be equal to the square of 2 (that is, 4)?
9. Can sine of x be equal to cosecants of the same x?
10. Can sine x be equal to 0.5?
11. Which is greater, sine squared of x or sine cube of x?
12. Which is greater, secant squared of x or secant cube of x?

These same questions should also be considered by substituting other trigonometric functions.

Students are also experiencing difficulties with other trigonometric identities. In particular, the Pythagorean identities (see Section 6.1 of [3]):

$$\sin^2 A + \cos^2 A = 1,\quad 1 + \tan^2 A = \sec^2 A,\quad 1 + \cot^2 A = \csc^2 A, \tag{1}$$

The double-angle identities (see Section 6.4 of [3]):

$$\sin 2A = 2 \sin A \cos A,\quad \cos 2A = \cos^2 A - \sin^2 A, \tag{2}$$

and, especially the following derivations from (1) and (2):

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A),\ \cos^2 A = \tfrac{1}{2}(1 + \cos 2A), \tag{3}$$

are difficult to do for students.

However, these formulas are basic in mathematics, and they are frequently used in different applications. Later on in their learning of mathematics and calculus in particular, students experience difficulties applying trigonometric formulas which they have studied in high school. Let us consider several examples:

- Find $\sin(\cos^{-1} 4/5)$ (No.13 in the Section 6.6, the Chapter "Inverse functions" of [4]).

The difficulty here consists in expressing cosine through sine using formula (1) above.

- Find $\lim_{x \to 0} \frac{1 - \cos x}{2x^2}$ (No. 44 in the Section 2.4, the Chapter "Derivatives" of [4]).

- Evaluate integrals $\int_0^{\pi/2} \sin^2 3x dx$ and $\int \cos^4 x dx$ (No.1 and 3 in the Section 7.2, the Chapter "Techniques of integration" of [4]).

Students need to know how to express $\sin^2 x$ through $\cos 2x$, using the formulas (1) and (2) above.

- Evaluate the integral $\int \sin^5 x \cos^5 x dx$ (No.11 in the Section 7.2 of [4])

Students need to know that formulas (2) are used several times.

To assist students with memorizing formulas (1)–(3) above and to enhance their understanding, it might be useful to ask students to derive the contents of Table 1 below. Doing a few exercises on a homework assignment is not sufficient to gain a good understanding of the interconnection of trigonometric functions.

Being able to choose the right sign for each function in each quadrant is the objective of Section 5.3 of [3] and Section 10.7 of [2]. With the exception of a small error in example 2 of Section 10.7 in [2], both sections are well done.

As students derive the function shown in Table 1, it is important to direct students' attention to choosing the sign near the root once again. Also a review of the questions 1–12 would provide students with additional help to choose the sign of the function properly.

By way of reviewing identities, the students' understanding could be further strengthened by developing another set of trigonometric formulas through $\tan(x/2)$ (so-called universal or Weierstrass substitution):

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}, \quad \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}, \quad \tan x = \frac{2\tan x/2}{1+\tan^2 x/2}. \tag{4}$$

These formulas are frequently used in sections of calculus [4], particularly in the chapter "Techniques of integration."

- Evaluate the integral $\int \frac{1}{3\sin x + 4\cos x} dx$ (No. 25 in Section 7.5 of [4]).

Table 1. **Interconnections Among Trigonometric Functions**

	sin	cos	tan	cot	sec	csc
sin		$\pm\sqrt{1-\cos^2 x}$	$\pm\frac{\tan x}{\sqrt{1+\tan^2 x}}$	$\pm\frac{1}{\sqrt{1+\cot^2 x}}$	$\pm\frac{\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
cos	$\pm\sqrt{1-\sin^2 x}$		$\pm\frac{1}{\sqrt{1+\tan^2 x}}$	$\pm\frac{\cot x}{\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\pm\frac{\sqrt{\csc^2 x-1}}{\csc x}$
tan	$\pm\frac{\sin x}{\sqrt{1-\sin^2 x}}$	$\pm\frac{\sqrt{1-\cos^2 x}}{\cos x}$		$\frac{1}{\cot x}$	$\pm\sqrt{\sec^2 x-1}$	$\pm\frac{1}{\sqrt{\csc^2 x-1}}$
cot	$\pm\frac{\sqrt{1-\sin^2 x}}{\sin x}$	$\pm\frac{\cos x}{\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$		$\pm\frac{1}{\sqrt{\sec^2 x-1}}$	$\pm\sqrt{\csc^2 x-1}$
sec	$\pm\frac{1}{\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\pm\sqrt{1+\tan^2 x}$	$\pm\frac{\sqrt{1+\cot^2 x}}{\cot x}$		$\pm\frac{\csc x}{\sqrt{\csc^2 x-1}}$
csc	$\frac{1}{\sin x}$	$\pm\frac{1}{\sqrt{1-\cos^2 x}}$	$\pm\frac{\sqrt{1+\tan^2 x}}{\tan x}$	$\pm\sqrt{1+\cot^2 x}$	$\pm\frac{\sec x}{\sqrt{\sec^2 x-1}}$	

- Prove the formula $\int \sec x dx = \ln \left|\frac{1 + \tan {}^{x}/_{2}}{1 - \tan {}^{x}/_{2}}\right| + C$ (No. 31.A in section 7.5 of [4]).

  At this point, it is also important that students practise by using the identities (1)–(4) identified above with angles different from A or $2A$, such as nA or $n^{A}/_{2}$. Not to engage in such practice would result in students not knowing that $\sin^2 5A + \cos^2 5A = 1$.

Related Angle and Other Identities

The next set of student problems is connected with the evaluation of concrete values of trigonometric functions. Calculation of trigonometric functions of concrete (special or general) angles is used in various applications. Students do not seem to remember the trigonometric ratios of special angles they had studied before in degrees in Section 10.5 of [2] and in radians in Section 5.5 of [3]. Using the related angle, cofunction and odd-even identities (Section 6.2 of [3]) provides only seven problems for students to practise. Moreover, the definition of odd-even function is not clear and should be written in general, not just using the properties of powers. It is also worthwhile for students to complete Table 2 below.

Table 2

Argument	Function			
x	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$-x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$
${}^{\pi}/_{2} + x$	$\cos x$	$-\sin x$	$-\cot x$	$-\tan x$
${}^{\pi}/_{2} - x$	$\cos x$	$\sin x$	$\cot x$	$\tan x$
$\pi + x$	$-\sin x$	$-\cos x$	$\tan x$	$\cot x$
$\pi - x$	$\sin x$	$-\cos x$	$-\tan x$	$-\cot x$
$3\,{}^{\pi}/_{2} + x$	$-\cos x$	$\sin x$	$-\cot x$	$-\tan x$
$3\,{}^{\pi}/_{2} - x$	$-\cos x$	$-\sin x$	$\cot x$	$\tan x$
$2p + x$	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$2p - x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$

Also answering questions 1–12 stated in the "Basic Formulas" section above and applying the rules which follow help students immensely in choosing the sign for each trigonometric function. The rule that joins together all identities mentioned in this section is very simple and useful to students. It is stated as follows:

1. Add or subtract 2π until the angle is between 0 and 2π, because the function and the sign stay the same if one adds to or subtracts from the angle 2π times any natural number.
2. Do not alter the function if the angle added to or subtracted from is 0 or π. Change the function into its cofunction if the angle is added to or subtracted from ${}^{\pi}/_{2}$ or ${}^{3\pi}/_{2}$.
3. The sign depends on the quadrant of the initial angle.

At the end of the chapter, students should engage in solving several exercises similar to the ones shown below. (We mark them as "✧" to distinguish them from university-level exercises)

✧ Verify the equality:

$2\sin^2 (3\pi - 2A) \cos^2 (5\pi + 2A) = {}^{1}/_{4} - 0.25 \sin({}^{5}/_{2}\pi - 8A)$.

✧ Check (fun exercises):

$\cos^2 x = 1 - \sin^2 x$,
$(\cos^2 x)^{3/2} = (1 - \sin^2 x)^{3/2}$,
$\cos^3 x + 3 = (1 - \sin^2 x)^{3/2} + 3$,
$(\cos^3 x + 3)^2 = ((1 - \sin^2 x)^{3/2} + 3)^2$ at $x = 180°$ and
$2^2 = 4^2$.

✧ Find (without a table) the values of the expressions when $x = {}^{\pi}/_{8}$:
  $\sin^4 x - \cos^4 x$,
  $\sin^6 x - \sin^6 ({}^{\pi}/_{2} - x)$,
  $\sin^6 x + \cos^6 (x + 4\pi)$.

These exercises have the following goal in mind: on the one hand, they represent a review of the "difference of squares" (Section 3.9 of [1]) or the "sum of difference of cubes" (Section 3.6 of [1]), and, on the other hand, they repeat and group together all basic trigonometric identities.

Advanced Formulas

Here we dwell on the sum and difference identities (Section 6.3 of [3]):

$$\begin{aligned} &\sin(A - B) = \sin A\cos B - \cos A\sin B,\\ &\sin(A + B) = \sin A\cos B + \cos A\sin B, \qquad (5)\\ &\cos(A - B) = \cos A\cos B + \sin A\sin B \text{ and}\\ &\cos(A + B) = \cos A\cos B - \sin A\sin B. \end{aligned}$$

It would be reasonable to derive their evident consequence

$$\begin{aligned} &\sin A\cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)],\\ &\sin A\sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)] \text{ and} \qquad (6)\\ &\cos A\cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]. \end{aligned}$$

As in all previous sections of this text, there is a lack of practice here. If students had more practice, they would not only come to memorize these formulas but to understand them as well. Also, taking the derivative of trigonometric functions or using them to solve problems on calculus would further enhance their understanding. Failure to provide students with extensive practice opportunities would result in students forgetting this content in a few days.

Trigonometric Equations

Only Section 6.5 of textbook [3] is devoted to general solutions of trigonometric equations. Inverse trigonometric functions and trigonometric inequalities are not covered. It is, therefore, natural that first-year university students are not able to construct the range of $\cos x > \frac{1}{2}$ or similar problems. The treatment of this important matter is totally inadequate. Knowing how to solve trigonometric equations and inequalities helps students to better understand trigonometric functions in general, and it refreshes their knowledge of graphs and graphing of trigonometric functions (Chapter 5 of [3]).

The four examples demonstrated in this section merely show the solution of the equation, but provide no general algorithm or formula. Only example 2 of Section 6.5 of [3] provides experience for students which involves a solution step where $\sin x = 2$, with no solution. It is important that problems of this nature be drawn to the students' attention. Perhaps by examining a graph similar to Figure 1, it will become evident that there are no real values of x satisfying the equation $\sin x = a$ (and $\cos x = a$) when $a > 1$. This can also be confirmed by sketching the graphs of functions $y = a$ and $y = \sin x$ (see Figure 1).

Given that students studied this graph in some detail in Chapter 5 of [3] and that the students' ability to find solutions to trigonometric equations using trigonometric tables or calculators is well developed, it would not be difficult to derive the general solution for trigonometric equations and corresponding trigonometric inequalities.

Being able to find the general solution of the simplest trigonometric equations, such as $\sin x = a$ and $\cos x = a$ for $a = 0$, $a = 1$ and $a = -1$ is extremely important. Incomplete or limited understanding will present difficulties to students in solving many problems in university calculus even simple connected trigonometric equations. The following examples illustrate this point:

- Sketching the graph of the function $y = x + \sin x$ (Exercise 34, Section 3.6 of [4])

To find intervals where the function is increasing or decreasing, everyone takes the derivative of the function, but then finds it difficult to solve the equation $\cos x + 1 = 0$. Finding intervals of concavity requires finding the general solution of the second trigonometric equation $-\sin x = 0$.

- Sketching the graph of the function $y = \dfrac{\cos x}{2 + \sin x}$ (Exercise 36, Section 3.6 of [4])

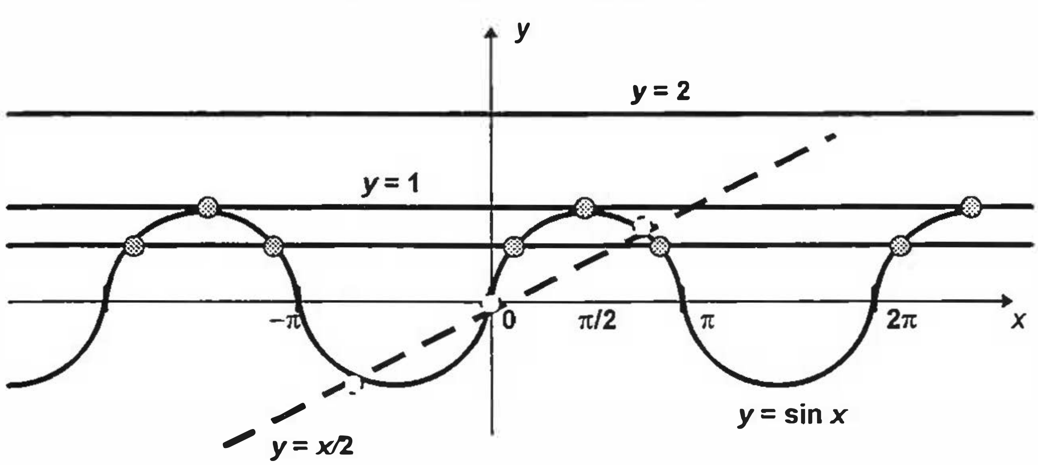
The problem is similar to the one above, but the equation is more complex. Before students solve this equation, they have to simplify it by using basic trigonometric formulas. Moreover, to define the domain of this function, some students try to solve the equation $2 + \sin x = 0$.

- Find the area between the curve $y = \sin x$, $y = \cos 2x$, $x = 0$, $x = \pi/4$ (Exercise 24, Section 5.1 of [4])

To find the intersection points, it is necessary to solve the equation $\sin x = \cos 2x$.

- Find the volume generated by rotation of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$ about x-axis (Exercise 28, Section 5.2 of [4])

Figure 1. **Graphic Method for Solving Trigonometric Equations**



This problem has a similar degree of difficulties.

A quick review of the following questions helps students in gaining a better understanding of equations:

13. Solve the equation $\sec x = 0.8$.
14. How many solutions are there for equations $\cos x = 0.5$ and $\tan x = 2$ on the interval $[9\pi, 11\pi]$?
15. How many times does the graph of $\tan x$ intersect the line $x = {}^{3\pi}/_{2}$?
16. Find a when $y = \cot(x + a)$ intersects the line $x = \pi$?
17. Sketch the graph $y = \sin 2$.
18. Does the equation $\sin x \cos x = \sin \alpha$ have a solution at $\alpha = 20°$ and $\alpha = 80°$?
19. Find solution of the equation $\cos^2 2x - \sin^2 2x = \cos\alpha$ where $\alpha = 20°$, $\alpha = 80°$.

Instead of stating "a trigonometric equation *usually has* infinitely many roots" (p. 289 of [3]), it is probably more appropriate to state, "equations *can have*," because the opposite statement—"no roots"—as in equation $\cos x = 2$ or a "finite number of roots"—as in $\sin x = {}^{x}/_{2}$, refer equally to the family of trigonometric equations. For more explanation, see Figure 1.

On page 290 of [3], it is stated that "no general methods exist for solving trigonometric equations." Many students probably find this statement intimidating. It would have been helpful to add that a graphic method as shown in Figure 1 (that is, drawing graphs of the left-hand and right-hand side of equations) can help students to analyze the existence and number of roots, as well as determine an approximate evaluation. In fact, this graphic method is being used following the above-noted statement, hence a reference to it would have reinforced the graphic method as a legitimate method.

Construction of graphs of trigonometric functions and their variations received much attention in seven sections of Chapter 5 of [3], whereas trigonometric identities and equations, which are very important, were only dealt with in five sections. In addition, sketching graphs and transformation of relations were already studied in some detail in Chapters 6 and 7 of textbook [2]. It would be reasonable to increase the study of trigonometric identities and equations at the expense of reducing graphing.

Although the treatment of trigonometry in textbooks [1] to [3] is comprehensive at the high school level, the main impediment to mastery is a lack of practice. How can students possibly memorize or understand these formulas when each section provides limited opportunities for practice and each section often contains more formulas than exercises? It is, therefore, not surprising when students get lost among all these trigonometric formulas and identities. Student practice, solving more complex problems and opportunities for reviewing the concepts are absolutely essential if students are to grasp trigonometry in general and identities and equations, specifically. In closing, the following examples contain all basic trigonometric formulas:

- ✧ Simplify: $\cos^2(\alpha + 2\beta) + \sin^2(\alpha - 2\beta) - 1$.
- ✧ Factor: $\sin 6x - 2\sqrt{3}\cos^2 3x + \sqrt{3}$.
- ✧ Verify the equality: $\dfrac{\cos 63^0 \cos 3^0 - \cos 87^0 \cos 27^0}{\cos 132^0 \cos 72^0 - \cos 42^0 \cos 18^0} = -\tan 24^0$.
- ✧ Simplify: $2 - \dfrac{\sin 8\beta}{\sin^4 2\beta - \cos^4 2\beta}$.
- ✧ Verify the equality: $\dfrac{\sin({}^{\pi}/_{2} + 30)}{1 - \sin(30 - \pi)} = \cot({}^{5}/_{4}\pi + {}^{3}/_{2}0)$.

From this brief review, it is evident that the high school mathematics program is filled with interesting and challenging problems that connect with the real world.

References

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