delta-k

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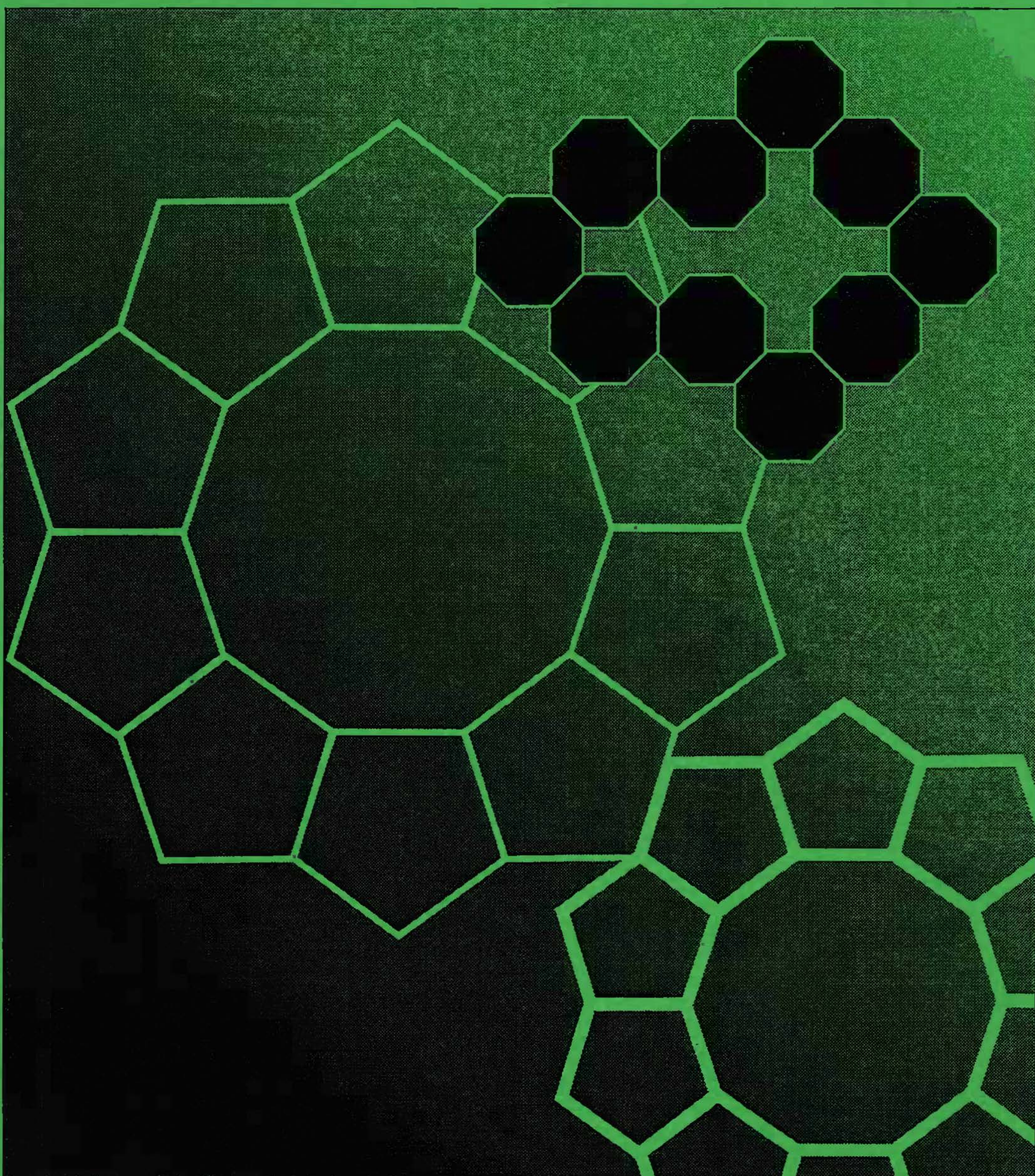

GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. Letters to the editor or reviews of curriculum materials are welcome.
7. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 34, Number 2 September 1997

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COMMENTS ON CONTRIBUTORS

Karen Bell is a graduate student in mathematics education at the University of Georgia, Athens.

David Burghes is a professor of education at the University of Exeter, England.

Thomas J. Cooney is a professor of mathematics education at the University of Georgia, Athens.

Lee Cross teaches assessment and methods classes for preservice teachers of students with disabilities at the University of Central Florida, Orlando.

David E. Dobbs is a professor of mathematics at the University of Tennessee, Knoxville.

David R. Duncan is a professor of mathematics at the University of Northern Iowa, Cedar Falls.

Diane Fisher-Cauble is a graduate student in mathematics education at the University of Georgia, Athens.

Helena Fung is a Grade 11 student at Archbishop MacDonald High School in Edmonton.

Hélène Gendron is a French Immersion teacher at Jubilee Junior High School in Edson.

Florence Glanfield is a doctoral student in the Department of Secondary Education at the University of Alberta, Edmonton.

Brendan Halloran is a Grade 11 student at Archbishop MacDonald High School in Edmonton.

Peter Höeg is the author of *Smilla's Sense of Snow* (1992).

Natali Hritonenko is a research associate for the Department of Physics and provides tutoring services in the Mathematics Department, both at the University of Alberta, Edmonton.

Michael C. Hynes is director of the Martin Marietta/University of Central Florida Academy for Mathematics and Science Education, Orlando.

Bonnie H. Litwiller is a professor of mathematics of the University of Northern Iowa, Cedar Falls.

Barbara Morrison is supervisor of mathematics with the Calgary Catholic Board of Education and is currently working with ITP Nelson.

Kelly Paul is an undergraduate student at the University of Alberta, currently completing Phase 3 student teaching.

John C. Peterson is a professor of mathematics at the Chattanooga State Technical Community College, Chattanooga, Tennessee.

Sandra M. Pulver is a professor of mathematics at Pace University in New York.

Nicholas Pyke is a journalist writing for the *Times Educational Supplement*, London, England.

Stephen Samogyi is a Grade 11 student at Archbishop MacDonald High School in Edmonton.

Wendy B. Sanchez is a graduate student in mathematics education at the University of Georgia, Athens.

Ajit Paul Singh is a Grade 11 student at Archbishop MacDonald High School in Edmonton.

Malcolm Swan is a lecturer in mathematics education at the University of Nottingham, Nottingham, England.

Harold Torrance is a mathematics specialist with the Kauai School District in Hawaii.

EDITORIAL

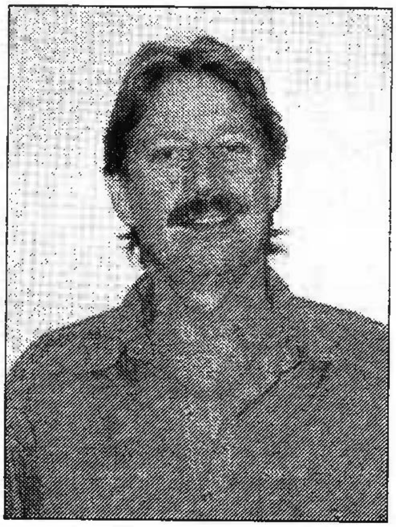
The 1996–97 school year will be over when you receive this issue of *delta-K*. I hope you are having an enjoyable and restful summer, preparing you for yet another school year in the fall. I hope you will find some time to reflect on your work and your teaching of mathematics and that, from this reflection, some contributions for the next issue of *delta-K* will flow.

As always, we are interested in receiving articles, teaching ideas, student submissions or anything else that might be of interest to our *delta-K* readers. While we appreciate submissions from all readers, we want to encourage Alberta educators, in particular, to share their work with us. I know that many good things are happening in the mathematics classrooms of our province. They must not go unnoticed because someone is waiting to use them in his or her teaching.

The introduction of the Western Canadian Protocol in mathematics and its implementation presents a significant challenge to us all. The successful implementation of this curriculum will, to a large measure, depend on our ability and willingness to share our good ideas and successes with one another. I hope that *delta-K* can serve as an effective vehicle to facilitate that sharing.

Klaus Puhlmann

From the President's Pen



At the MCATA annual general meeting in Red Deer in November 1996, proposed constitutional amendments were accepted by the general membership. These changes were subsequently ratified by ATA table officers. Changes will allow for shorter terms of office for MCATA table officers. This will relieve table officers of the lengthy commitment to specific duties on the Council and better support the opportunity for turnover of executive members. The revised constitution is accompanied by a slight change in structure and expectations of executive members to assist in the work of the Council.

The MCATA executive met January 17–18, 1997. We renewed our commitment to enhance our public relations initiatives, focus on providing leadership to our members, and support professional development opportunities and publication materials. A number of initiatives addressing professional development opportunities, publications and membership support are in place for the upcoming months.

The MCATA executive invites input from all members. If you have any concerns, suggestions, questions or comments, please feel free to contact any executive member.

George Ditto

The Right Angle

Alberta Continues Participation in SAIP

In 1991, the provincial and territorial ministers of education, as the Council of Ministers of Education, Canada (CMEC), agreed to undertake the School Achievement Indicators Program (SAIP). Alberta Education is continuing its participation in the program with the assessment of mathematics knowledge and skills of 13- and 16-year-old students. In all, 220 Alberta schools were selected to participate—110 for 13 year olds and 110 for 16 year olds.

This assessment of mathematics knowledge and skills will provide us with an excellent opportunity to show the education community, as well as the general public, the effectiveness of our system with regard to the learning of this discipline. The assessment, which occurred between April 21–May 9, 1997, has two separate components: a mathematics-content component and a problem-solving component. Selected students took part in only one of the two components.

Previous SAIP assessments were conducted in mathematics in 1993, reading and writing in 1994, and science in 1996. CMEC plans the release of the assessment results in December 1997. The results will be compiled at the national and provincial levels. No results will be published at the school board or school level.

Classroom Assessment Materials Project

The Classroom Assessment Materials Project (CAMP) is a collection of 45 English and 35 French assessment packages for Grades 1–11 (excluding Grades 3, 6 and 9). Through the collaborative efforts of Alberta Education, The Alberta Teachers' Association, an advisory committee representing key education organizations and hundreds of Alberta principals, teachers and students, high-quality assessment packages in the four core subject areas are now available.

Assessment packages are available in English for the following:

- Grades 1, 2, 4 and 5 Mathematics. These materials are blueprinted to the Western Canadian Protocol Curriculum.
- Grades 7 and 8 Mathematics are delayed by one year. They will be blueprinted to the new curriculum.
- Mathematics 10, 20, 13, 23, 14, 24 and 31.

Assessment packages in French will be available in September.

Kay Melville

In this section, we will share your reactions to articles and points of view on teaching and learning of mathematics. We appreciate your interest and value the views of those who write. The following articles have been submitted by readers for inclusion in this issue. Furthermore, two readers drew attention to the fact that in the May 1997 issue of delta-K *(Volume 34, Number 1) several problem statements and answers provided in the article "Calendar Math" (pp. 34–35) were incorrect. The problems for which the answers were incorrect and the comments submitted by the readers are as follows:*

Comments on "Calendar Math"

Problem 3: Susan drives from Calgary to Edmonton, a distance of 300 km, at a rate of 100 km/h. She drives back at a rate of 110 km/h. What was her average speed?

"The average speed is **not** determined by dividing the two speeds by the factor 2, as students will often do, but one has to take into account the fact that the car does not take the same amount of time to cover each part of the trip.

The average speed is the total distance divided by the total time.

Total distance is about 300 km each way.

Total time is 300 km/100 km/h + 300 km/110 km/h = 3.00 h + 2.73 h = 5.73 h. Average speed is 600 km/5.73 h = 104.71 km/h.

I suppose one could make the case that we might still round it off to 105 km/h, but it is instructive to teach students that constructing averages is not always simply arithmetically 'averaging' the numbers given."

Wytze Brouwer
Department of Secondary Education
University of Alberta

John Percevault, Professor of Education (Emeritus), University of Lethbridge submitted a similar comment on problem 3 as well as the following comments on problems 4, 15, 25 and 31.

Problem 4: Find the average of 151 whole numbers from 1–150, inclusive.

"Since when are there 151 whole numbers from 1–150, inclusive? This leads to the error in the answer."

Please note that it should read that there are 150 whole numbers from 1–150, inclusive. Hence, the answer should be 11,325 ÷ 150 = 75.5.

Problem 15: Find a pattern and determine the next number in this series: 77, 49, 36, 18, ___.

"The sequence 77, 49, 36, 18, ___ can be developed consistently across the sequence by multiplying the digits of the digit numerals. The correct answer is then 1× 8 = 8."

Problem 25: Jason has five friends at his birthday. To be polite, each person shakes hands with everyone else. How many handshakes will there be? (This is an excellent problem to solve by actual demonstration.)

"I can only assume that the instructions are ambiguous or the question is poorly worded. If the five invited guests are the only ones to shake hands, the answer of 10 is accepted. If each person shakes hands, the answer is 15. What is the intent?"

Problem 31: Put the appropriate signs between each of the following numbers so as to get an answer of 9.
8 4 2 5 = 9

"The answer which legitimately gives 9, does not correspond with the numerals given in the question."

One of the correct answers is (8 ÷ 4) + 2 + 5 = 9. Another correct answer is 8 + 4 + 2 – 5 = 9.

British Math Fails to Add Up

Nicholas Pyke

Progressive Methods Blamed for Gap Between England and the Continent

Modern teaching methods and sloppy thinking have undermined British math, leaving secondary pupils in England and Scotland trailing far behind their international peers, according to new research.

The most comprehensive study of its kind so far shows youngsters in a range of countries, including Germany, Hungary and Singapore, outstripping British teenagers. Schools in these countries insist on rigorous whole-class teaching and ban pupils from using calculators.

Academics are now making films of Hungarian math lessons to help British schools improve.

The findings, which have attracted interest from the Office for Standards in Education, come hard on the heels of a critical study by the National Institute for Economic and Social Research. This said that British 10 year olds are two years behind their continental counterparts.

Both pieces of research were funded by the Gatsby Charitable Foundation which staged a major math seminar in Birmingham. Conducted by Exeter University, it shows that British secondary pupils start from a lower level and then make less progress than students abroad—with alarming implications for the U.K.'s engineering and technological future.

Professor David Burghes from Exeter's school of education blamed inadequate teaching methods allowing pupils to "chop and change" math topics, without fully understanding the material.

"It is time to question our so-called 'progressive' methods," he said. "A much more sensible approach is needed—for example, we must not be afraid to say a pupil's work is wrong because it is so difficult to correct misconceptions introduced at any early age.

"We do seem to be underperforming in comparison with both European and Far Eastern countries. Since math plays such a central role in technological developments, it is a real concern for many that we are lagging so far behind."

The researchers from Exeter gave exactly the same math problems to 13 and 14 year olds in 17 countries over two to three years. They included England, Scotland, Germany, Hungary, Poland, Singapore, Japan, Thailand, Greece, Holland and Finland.

English 13 year olds scored, for example, only 11.3 out of 50 in algebra, and Scots scored 9.6. But Germans taking the same test managed 12.5, Polish children averaged 16.6, while those in Singapore got 23.9. By the age of 14, the differences had increased further. English pupils scored 14.4; Scots 13; Germans 17.6; Poles 24.9; while Singapore pupils reached 30.7. There were similar results in tests on shape and space, and number.

"Math in other continental countries is characterized by the teacher playing a central teaching role, not a management role as we see so often in the U.K.," said Professor Burghes.

"Whole-class interactive teaching is the norm with teachers adept at bringing everyone into a discussion—often choosing the stragglers to work through exercises. In short, they keep all the pupils on task.

"Math is always written and spoken clearly and precisely. Calculators are not used in primary schools and only allowed in secondaries when pupils have gained that all important feel for number."

Other countries, he said, back up the classroom work with homework and regular written tests.

"We no longer treat math as a precise and exact science. The discipline of actually writing equations correctly, for example, is not tested in the way it used to be, and in the way it still is on the continent.

"If a kid's got the right answer, we now tend not to worry about the working in between. German colleagues have been appalled by what they've seen going on in British classrooms."

Professor Burghes is also doubtful about the mathematical expertise among primary teachers. "The weaknesses do point back to the primary level, where many teachers have no more than a grade C at GCSE. I am a chief examiner in math and I know what a grade C means. Not very much."

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Why We Lag Behind in Math

David Burghes

There has been much speculation about the poor performance of U.K. pupils in school mathematics—complaints have come from engineers, scientists and others in higher education, as well as from employers of school-leavers. Much of the evidence presented has been, at best, anecdotal.

To seek the truth behind their criticism, we started the Kassel Project, funded mainly by the Gatsby Charitable Foundation, in which secondary pupils (aged 13-plus) in 17 countries have been taking tests in math over two to three years. The countries involved include England, Scotland, Germany, Hungary, Poland, Singapore, Japan, Thailand, Norway, Greece, Holland and Finland.

Not only do we have data on attainment in math at a particular age, but we can also see how well pupils of similar ability in each country progress year by year. Our aim has been to find the factors that give rise to enhanced progress, and in consequence, to make recommendations for math teaching in the U.K.

Our results, which were presented at the second Gatsby mathematics education seminar in Birmingham, have tended to confirm the anecdotal evidence, although there are some topics in which England is doing reasonably well. More important, however, we have been able to identify a number of key factors in which countries making good progress differ from England and Scotland, so that there is every chance that we might redeem the situation.

It should be added that we are not blaming teachers—in the main they have been trying to implement, in difficult circumstances, the advice being given by educationists, administrators and government.

In our project, all pupils take the same tests (translated where necessary) in number, algebra, and shape and space. Table 1 shows the results and total progress over the first year (age 13-plus to 14-plus) for pupils in England, Scotland, Germany, Poland and Singapore. For each of these countries, Table 1 shows the average score (out of 50 marks for each test) for representative samples of about 1,000 pupils in each country. The final column shows progress made over the year.

Not only were England and Scotland well behind in total attainment on these core topics on the first testing, but the progress made during the year was less than in other countries. It should also be noted that Singapore is doing so well that it begins to become more difficult for many of the pupils to show any real progress so their increase of 16.8 over the year is an excellent result.

The trends in progress can be seen in the examples in Table 2 of responses to individual questions on the tests.

It is also interesting to note the performance of different groups of pupils. In Germany, there are three types of schools: *Gymnasium* (academic); *Realschule* (technical); *Hauptschule* (vocational). The attainment and progress made by pupils in these schools from Year 1 to Year 2 of the project and the equivalent data for pupils in England are shown in Table 3.

Table 3 shows that the *Gymnasium* students are catching up on our able pupils; the middle-ability students in both countries show similar performance, both in attainment and progress; while the *Hauptschule* pupils are progressing much faster than similar-ability pupils in England, and from a slightly higher attainment level.

In summary, we really do seem to be underperforming in comparison with both European and far eastern countries. Since math plays such a central role in technological developments, it is a real

Table 1. **Average Score Out of 50**

	Number		Algebra		Shape and Space		Totals		Progress
Age	**13+**	**14+**	**13+**	**14+**	**13+**	**14+**	**13+**	**14+**	
England	17.6	20.2	11.3	14.4	15.4	19.9	44.3	54.5	10.2
Scotland	18.2	21.6	9.6	13.0	14.1	18.4	41.9	53.0	11.1
Germany	23.5	26.9	12.5	17.6	11.3	17.3	47.3	61.8	14.5
Poland	24.0	29.2	16.6	24.9	13.6	22.4	54.2	76.5	22.3

Table 2. **How They Scored—% of Correct Answers**

Questions	Scotland		England		Germany		Poland		Singapore	
	13+	14+	13+	14+	13+	14+	13+	14+	13+	14+
$70 \times 0.3 =$	34	42	23	34	65	71	77	80	83	85
$2.4 \times 1\frac{1}{4} =$	3	7	5	11	20	26	42	54	63	70
Simplify $\frac{\sqrt{147}}{\sqrt{3}}$	0	2	1	5	2	4	12	33	16	33
Solve for x $3x - 4 = 11$	48	63	50	65	60	76	61	72	84	88
Multiply out $(x + 1)(x - 2)$	0	9	1	11	5	31	25	39	21	57

Answers
21 / 3 / 7 / 5 / $x^2 - x - 2$

concern for many that we are lagging so far behind our economic competitors.

So why are we underachieving? There is probably no single answer to this, but our observation of mathematics teaching in good schools in this country and abroad, particularly in Germany, Hungary and Poland, does, at least, give us some clues. Math teaching in these countries, and in other continental countries, is characterized by the teacher playing a central teaching role, not a management role as we see so often in the U.K. Whole-class interactive teaching is the norm, with teachers adept at bringing everyone into a discussion—often choosing the stragglers to work through exercises, or the homework, on the blackboard. In short, they keep all the pupils on task.

Math is always written and spoken clearly and precisely—again in contrast to the rather sloppy trends now seen in the U.K. Calculators are not used in primary schools and only allowed in secondary schools when pupils have gained that all-important feel for numbers and have learned to use them correctly. Homework plays a key role in the learning process, and mental and written tests are given regularly. Both homework and tests are marked before the next math lesson so that any common mistakes can be used as teaching points.

Table 3. **Rate of Progress Over Two Years**

	Germany		England	
	Yr 1	Yr 2	Yr 1	Yr 2
Gymnasium (higher ability)	62	+21	72	+15
Realschule (middle ability)	43	+11	41	+11
Hauptschule (lower ability)	25	+10	22	+5

Another key factor, apparent in Singapore, is that it is made absolutely clear what should be taught and when. There is only one series of texts and practice books, and it is these which in essence provide the vastly enhanced expectations, compared with the U.K.

These factors provide us with the basis for recommendations for math teaching in the U.K. Some may sound rather old-fashioned, but it is time to question our so-called "progressive" methods. A much more sensible approach is needed to teaching mathematics. For example, we must not be afraid to say a pupil's work is wrong, because it is so difficult to correct misconceptions introduced at an early age (as tutors in higher education are finding now).

Despite the many recent negative reports about mathematics teaching, let me finish with the really good news. Since outlining our recommendations, we have offered to support secondary schools in putting them into place. We already have 100 schools keen to take part in the demonstration project, the Mathematics Enhancement Program, which is again backed by The Gatsby Charitable Foundation and also by some leading companies including Esso, the Post Office and British Steel.

It has been heartening to find schools receptive to our recommendations and keen to be involved. It is time to stop the criticism and instead invest in and support our math teachers. We need to help our pupils reach their mathematical potential, which means enhancing teaching and learning. Our future prosperity depends on investment in education, and math is a key subject which should not be neglected.

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From *Smilla's Sense of Snow*

Peter Höeg

The following is a passage comprising a number of excerpts taken from pages 121–22 of Smilla's Sense of Snow *by Peter Höeg.*

"Because the number system is like human life. First you have the natural numbers. The ones that are whole and positive. The numbers of a small child. But human consciousness expands. The child discovers a sense of longing, and do you know what the mathematical expression is for longing:"

"The negative numbers. The formalization of the feeling that you are missing something. And human consciousness expands and grows ever more, and the child discovers the in between spaces. Between stones, between pieces of moss on the stones, between people. And between numbers. And do you know what that leads to? It leads to fractions. Whole numbers plus fractions produce rational numbers. And human consciousness doesn't stop there. It wants to go beyond reason. It adds an operation as absurd as the extraction of roots. And produces irrational numbers."

"It's a form of madness. Because the irrational numbers are infinite. They can't be written down. They force human consciousness out beyond the limits. And by adding irrational numbers to rational numbers, you get real numbers."

"It doesn't stop. It never stops. Because now, on the spot, we expand the real numbers with imaginary square roots of negative numbers. These are numbers we can't picture, numbers that normal human consciousness cannot comprehend. And when we add the imaginary numbers to the real numbers, we have the complete number system. The first number system in which it's possible to explain satisfactorily the crystal formation of ice. It's like a vast open landscape. The horizons. You head toward them and they keep receding. . . ."

The sum of two numbers is 19 and the sum of their squares is 205. What are the numbers?

STUDENT CORNER

Mathematics as communication is an important curriculum standard, hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas, and to explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternate solution to a familiar problem, poetry about mathematics or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of The Alberta Teachers' Association to publish "insert title" in one of its publications. The student author must sign this statement, indicate the student's grade level, and provide an address and telephone number. Parental permission is required if the student is under age 18.

The following poems have been written by students from Archbishop MacDonald High School in Edmonton.

$y = x^2$

Since you are always manipulating
I am always responding
Although your values always change
I am always square of you
And we will form a parabola
Since you add a coefficient in front of you
I change more rapidly
However, when you link with other terms
I cannot depend on you
And we will degenerate
into . . . two parallel lines
. . . one line
. . . empty graph

Helena Fung, Grade 11

Poem of Mathematics

$Ax + By + C = 0$
In the beginning
a point
But point after point
these points form a line.

We see in this line
a relationship form,
Between the good force of *x*
and the evil of *y*.

These forces compete
with the mighty C,
yet altogether they form
an unlimited nothing.

Stephen Samogyi, Grade 11

"Jabberwocky" as an Equation

(Biting Jaws) + (Catching Claws) + 2(Fiery Eyes)
= Jabberwocky

Jabberwocky = Manxome Foe

[Beamish Boy + Vorpal Blade] • [Snicker - Snak]
= Jabberwocky - Head

Jabberwocky / Vorpal Blade = Frabjous Day

Frabjous Day
(Calloh)(Callay)

Brendan Halloran, Grade 11

The Conic Song— $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

"Let me be your abscissa,"
Said the *x* to the *y*.
"Let us jump co-ordinates
Just you and I."

"Shall we be circular,
hyperbolic or the two?"

"Can't we be a conic?
Oh please oh please, let's do,"

"Shall we frolic
On a double napped cone?"

"Yet we cannot leave our friend
The plane, dear plane, alone!"

So off they went, a single dot
Singing Ax^2 and Bxy
. . . and all the bunch.

While in a math class,
Far, far away
We're all out to lunch.

Ajit Paul Singh, Grade 11

This year, every issue of delta-K *will devote a section to the NCTM Standards. In this issue, the focus will be on the* Assessment Standards for School Mathematics *(1995).*

NCTM Standards in Action

Klaus Puhlmann

According to the NCTM *Assessment Standards for School Mathematics* (1995), assessment is the process of gathering evidence about a student's knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes. Evaluation, on the other hand, refers to the process of determining the worth of, or assigning a value to, something on the basis of careful examination and judgment.

The Assessment Standards have been designed to expand on and complement the Evaluation Standards. The Evaluation Standards propose that

- student assessment be aligned with, and integral to, instruction;
- multiple sources of assessment information be used;
- assessment methods be appropriate for this purpose;
- all aspects of mathematical knowledge and its connection be assessed; and
- instruction and curriculum be considered equally in judging the quality of a program.

Assessment involves several interrelated, but nonsequential, phases:

- Planning the assessment
  - ✧ What purpose does the assessment serve?
  - ✧ What framework is used to give focus and balance to the activities?
  - ✧ What methods are used for gathering and interpreting evidence?
  - ✧ What criteria are used for judging performances on activities?
  - ✧ What formats are used for summarizing judgments and reporting results?
- Gathering evidence
  - ✧ How are activities and tasks created or selected?
  - ✧ How are procedures selected for engaging students in the activities?
  - ✧ How are methods for creating and preserving evidence of the performances to be judged?
- Interpreting the evidence
  - ✧ How is the quality of the evidence determined?
  - ✧ How is an understanding of the performance to be inferred from the evidence?
  - ✧ What specific criteria are applied to judge the performances?
  - ✧ Have the criteria been applied appropriately?
  - ✧ How will the judgments be summarized as results?
- Using the results
  - ✧ How will the results be reported?
  - ✧ How should inferences from the results be made?
  - ✧ What action will be taken based on the inferences?
  - ✧ How can it be ensured that these results will be incorporated in subsequent instruction and assessment?

More specifically, there are six assessment standards which, when used, constitute a dynamic process that informs teachers, students, parents and others and supports each student's continuing growth in mathematical power. These are as follows:

1. Assessment should reflect the mathematics that all students need to know and be able to do.
2. Assessment should enhance mathematics learning.
3. Assessment should promote equality.
4. Assessment should be an open process.
5. Assessment should promote valid inferences about mathematics learning.
6. Assessment should be a coherent process.

As stated in *The Common Curriculum Framework for K–12 Mathematics* (Alberta Education 1995), our goals for students in school mathematics are to

- use mathematics confidently to solve problems,
- communicate and reason mathematically,
- appreciate and value mathematics,
- commit themselves to lifelong learning, and
- become mathematically literate adults, using mathematics to contribute to society. (p. 3)

Clearly connected to these goals and an important outcome is the development of positive attitudes toward mathematics among all students. Implicit in our vision are also clearly stated expectations of what our students must know and be able to use and how their progress is to be assessed. In order for school assessment practices to inform educators as they progress toward these goals, it is essential that we move away from the "rank order of achievement" approach in assessment toward an approach that is philosophically consistent with our overall vision of school mathematics and classroom practices.

From the three articles that follow, it is evident that a new approach to assessment is evolving in many schools and classrooms. The new assessment approaches endorse the setting of high expectations for all students, but a convergence of information from a variety of balanced and equitable sources. It is important that the new assessment practices allow much more information to be derived by teachers during the process of instruction. Teachers are the people who are in the best position to judge the development of students' progress, hence must be considered the primary assessors of students.

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Loose Cash

What is the largest sum of money—all in current coins and no silver dollars—that I could have in my pocket without being able to give change for a dollar, half dollar, quarter, dime or nickel?

Assessing Mathematics Learning for Students with Learning Differences

Lee Cross and Michael C. Hynes

The *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) advocates the alignment of the mathematics curriculum with instructional practices and assessment techniques. The authors clearly understood that this alignment would not occur without expanding the notion of assessment and making the process of assessment more meaningful for all students. Consequently, the guidelines for the evaluation of mathematics became a significant part of the curriculum standards and stated the following:

- Student assessment must be integral to instruction.
- Multiple means of assessment should be used.
- All aspects of mathematical knowledge and its connections should be assessed.

Teachers should be paying more attention to assessing what students know about mathematics and spending less time determining what they do not know. This attention is especially true for students who have experienced difficulty in learning mathematics. Constant reminders of failure only lead to low self-esteem, which can lead to lower achievement. Perhaps this downward spiral can be stopped by changing the emphasis of assessment from checking only for the correct answer to recording what students *know*, how they *think* about mathematics and how they *apply* mathematics to real-world problems.

Many students in our schools have learning problems in the area of mathematics. Many of these students are labeled handicapped and at-risk for school failure. These students may exhibit deficits in computational skills, spatial awareness, understanding of mathematical concepts, problem solving, and memory of procedures and strategies. A larger and larger percentage of these students are being taught in regular classrooms for all or most of the school day. Educators (Gartner and Lipsky 1987; Lilly 1988; Reynolds, Wang and Walberg 1987; Stainback and Stainback 1987; Wang, Reynolds and Walberg 1986) have advocated that adaptive instructional strategies be used to help these students succeed in regular classrooms. More recently, researchers have proposed that educators working with disabled students must adapt instruction in mathematics to that proposed by the curriculum standards (Cawley, Baker-Kroczynski and Urban 1992). However, adapting instruction alone is not sufficient. The methods of assessment must be adapted as well. These disabled students are penalized most by traditional paper-and-pencil tests, particularly when their performance is being compared with that of typical children at a particular grade or age. Clearly, if children learn mathematics with difficulty or use different methods of learning, standard paper-and-pencil assessments will not be sufficient to document learning and students' progress. Using alternative forms of assessment is essential to describe what a student has learned, how he or she learns best, under what conditions he or she learns, and his or her understanding of mathematical processes.

Some alternative-assessment forms appropriate for students with handicaps are observation, interviews, holistic scoring, checklists, portfolios and journals, as well as paper-and-pencil forms of assessment.

Assessing Through Observation

Many teachers have made observation an integral part of evaluation. They practise targeting one or two students at each lesson for observation. To record observations, sticky notes for computer labels are used. Names of students targeted for the lesson are written on one or two sticky notes. Blanks are available for writing spontaneous observations about other students. These observations are then pasted on a specific sheet for each student in a class notebook. Figure 1 illustrates how observations can be used to assess students' understanding of mathematics effectively.

For students with disabilities, using observation as an assessment technique gives the teacher a window to obtain student-performance information that cannot be gleaned from paper-and-pencil tests. Teachers can unobtrusively gain insight into the approach to the task as well as the persistence in completing the task. Additionally, information can be obtained about how students are constructing meaning from concrete manipulations, as in the illustration with James. All observation should be systematically recorded. The record can document achievement and communicate a student's success with mathematics to the student and others. These documented observations, when linked to paper-and-pencil test results, the results of interviews, the record of achievement on a checklist, and other assessment data, will give a more complete picture of a student's success in mathematics.

Interviewing

Students with disabilities often have difficulty in problem solving because their lack of fluency in reading can cause a misunderstanding of mathematical concepts, poor computational skills or poor dispositions toward learning mathematics. Alternative-assessment techniques should be used to determine the areas of problem solving in which the student has strengths as well as weaknesses. One alternative technique for assessing problem-solving abilities is interviewing students while they are in a problem-solving situation. Teachers can interview students informally in conversation while monitoring seatwork, or the teacher may plan a more structured, individual interview to survey the understanding of several students. Figure 2 illustrates assessing with an interview.

Holistic Scoring

Relying on observation or interview techniques to assess problem solving is not always effective. Interviews may be too time-consuming to use for an entire class or even with targeted students. An alternative to interviews is using a holistic-scoring technique. (See Figure 3 [Hynes 1990].) Three types of holistic scoring are generally recognized for evaluating mathematics learning: analytic scoring, focused holistic

Figure 1. **Using Observation to Assess James's Understanding of Area and Factors**

Mr. Mir has been working on geometry and measurement in his class and wants to assess students' understanding of calculating the area of rectangles. He is concerned about how to assess James, a student with learning disabilities who has difficulty making transitions from the concrete to the abstract. Mr. Mir was concerned that James might not be able to distinguish perimeter from area or apply a rudimentary formula to calculate the area. Previously, James had done poorly on written tests on area. Mr. Mir believes that the written tests do not reflect James's understanding of the mathematical concepts because James shows more insight during class presentations and discussions. Mr. Mir decides to assess James's understanding of area by observing him working with manipulatives while the rest of the class works at the abstract level.

Mr. Mir requests, "Class, make a rectangle that is 4 units by 6 units. Record the area of this rectangle, and the dimensions of all rectangles that have the same area as the first rectangle, using only whole-number dimensions."

Mr. Mir observes James using square tiles to make his 4-by-6 rectangle. James makes the rectangle and records that the area of the rectangle is 20. Mr. Mir notes that James added 4 + 6 + 4 + 6 to get 20. As James tries to respond to Mr. Mir's direction to make more rectangles, he seems confused. Mr. Mir asks the class to take time out. "Each member of the class can ask someone sitting nearby two questions about the problem." James interacts with the student in front of him in an acceptable manner and asks how the other student got 24. After the class returns to work, Mr. Mir notes that James has erased his first answer and written the correct answer.

As James makes other rectangles with an area of 24, Mr. Mir writes that he seems to have grasped a concrete understanding of area but failed to show all the possible rectangles with an area of 24.

Teacher Assessment. James demonstrated some understanding of area and the ability to find the area of the rectangle using concrete materials. He is also able to represent the area of a rectangle symbolically when allowed to use concrete materials.

Recommendations. James seemed to benefit from using manipulatives. He could make the described rectangles but he seems to confuse area and perimeter. Continued use of the manipulatives will be necessary to help him make this distinction. Since James was not able to make all the rectangles for an area of 24, he may need more work on the factors of 24. He may know the factors and not be able to connect this problem and the factors of a number. More observation is needed. The written work given James was assigned to help determine if he is progressing in relating abstract number sentences to models and pictures. If time allows, interviewing James about his understanding might be helpful.

scoring and general-impression scoring. (More information on holistic scoring can be found in *How to Evaluate Progress in Problem Solving* [Charles, Lester and O'Daffer 1987].) The focus of holistic scoring is on the process rather than on the correct answer. Students are given some credit for employing all or part of the correct steps in the problem, even if they get the wrong answer. Of course, if the student executes an appropriate process and also gets the correct answer, more points are awarded. In short, students are given credit for what they know. In implementing holistic-scoring techniques, it becomes imperative that students show all written work and record the thinking processes they used to solve the problem. Since writing problem situations is often difficult for students with disabilities, these students may need to work in pairs or cooperative groups, have an adult assist in recording their work or act out the solutions while showing the abstract solutions. Figure 4 presents a solution to a problem by two students, and the teacher's assessment and recommendation are shown in Figure 5.

Checklist

When assessing progress in learning, teachers should be primarily concerned with the content of mathematics. Another dimension of learning mathematics, however, should also be assessed. Students' dispositions toward learning mathematics are important, too. Students' confidence during mathematics learning, their willingness to persevere in mathematical tasks and their inclination to monitor their own thinking and performance all are important in the evaluation process (NCTM 1989). These dispositions are usually assessed as students engage in instructional activities in the classroom. For example, as they work in cooperative groups, teachers should note the ability of students to function properly during the instructional activity.

Mrs. Locklear has been using cooperative groups in her mathematics class two to three times a week for several weeks. Groups in her class solve problems, build models, practise with activities and study for weekly tests. She has seen increases in

Figure 2. **Using Interviewing to Assess José's Ability to Solve Two-Step Problems**

Since José has reading difficulties, Ms. Ryerson presents the following problem to him orally while pointing to the important facts.

"You and your dad go fishing. Your dad catches 4 fish and throws back 2 because the fish were too small. You catch 6 good-sized eating fish. Late in the afternoon, you and your dad go home and give your mom the fish to cook for supper. How many fish did your mom have to cook for supper?"

The first step for Ms. Ryerson was to ask José to tell her about the problem. By explaining the problem, he has completed the first step in problem solving—understanding the problem. José's response indicates that he is supposed to tell how many fish were cooked for supper. The teacher then continues the interview process by asking José how he would find out how many fish should be cooked. José's response will indicate whether he can select the correct operations and plan the solution. This question will probably lead to calculating the answer or solving the problem. José explains that he will add 4 and 2 and 6. At this point, the teacher suggested that José could use some fish counters to retell the story of the fishing trip. The teacher gives José a red paper labeled "dad" and a green paper labeled "José." José is asked to retell the story, placing the fish counters on the appropriate paper. As José retells the story with the manipulatives, he takes the correct action to indicate he understood the meaning of throwing back fish. However, in continuing the abstract solution of the problem, he adds 4, 2 and 6 correctly.

Teacher Assessment. When given two-step problems, José is able to identify the question the problem is asking. Initially, he does recognize one of the correct operations; however, José does not appear to comprehend the problem. In the retelling procedure, the teacher observes that José appears to comprehend the problem, but he is unable to transfer this comprehension and the physical action to the mathematical operations.

Recommendations. José needs more experience with two-step problem-solving exercises. He appears to profit from acting out the problem. His experiences should include working with another student to share their understanding of problems and creating his own two-step problems. He also needs to improve his conceptual understanding of operations. This student needs to experience real-world situations that indicate the operations needed. He needs to experience many models of subtraction: how many more, subset and comparison, as well as the take-away model.

Figure 3. **A Sample Focused Holistic-Scoring Scale for a Problem-Solving Assignment**

0 points

1. Problem is not attempted or the answer sheet is blank.
2. The data copied are erroneous and no attempt has been made to use that data.
3. An incorrect answer is written and no work is shown.

1 point

1. The data in the problem are recopied but nothing is done.
2. A correct strategy is indicated but not applied to the problem.
3. The student tries to reach a subgoal but never does.

2 points

1. An inappropriate method is indicated and some work is done, but the correct answer is not reached.
2. A correct strategy is followed but the student does not pursue the work sufficiently to get the solution.
3. The correct answer is written but the work either is not intelligible or is not shown.

3 points

1. The student follows a correct strategy but commits a computational error in the middle, which leads to an incorrect solution.
2. The student uses a correct strategy but ignores or misunderstands some conditions and never reaches a solution.
3. The correct answer is given and the work gives some evidence that an appropriate method was used. However, the implementation of the strategy is not clear.

4 points

1. The student uses an appropriate method and implements it correctly but commits a computational error toward the end and obtains an incorrect answer.
2. The student follows a correct method and performs the necessary work but toward the end loses sight of the answer or does not label the answer appropriately.
3. The student makes an error in copying. Except for this error, the work shows complete understanding of the method and implementation, even though an incorrect answer is reached.

5 points

1. The student has followed a correct method, performed appropriate computations and labeled answers correctly.

achievement in some of the slower students in her class. Mrs. Locklear has worked with her students on such social skills necessary for cooperative-group work as negotiating, complimenting one another and accepting criticism. Although she thinks that most groups are working well together, she is concerned that a few individuals and their groups might not be responding well to this strategy. She has noticed that some of the students with learning problems may not be actively involved in their groups. A month ago, Mrs. Locklear developed a checklist to assess students while they worked in cooperative groups (Figure 6). Figure 7 highlights an assessment and recommendation of the checklist.

A checklist supplies a strategy to record data systematically. Whereas the checklist in this example is used to assess students who are working in cooperative groups, checklists have many potential uses in assessment. These devices are excellent assessment tools to support teacher observation (Charles, Lester and O'Daffer 1987). In cooperative learning, group members can use a checklist to rate their peers in the group. Checklists can also be used for self-appraisal.

Filling out a checklist every day on every student is not necessary; however, a teacher who chooses to use checklists to assess students should use them periodically to assess students' progress on the attainment of concepts and skills as well as dispositions. Repeated observation using checklists will make patterns of behavior more apparent.

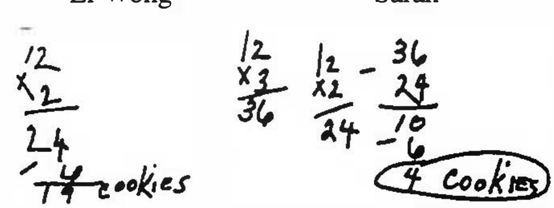
Figure 4. **Using Focused Holistic Scoring to Assess Li Wong and Sarah's Ability to Solve Multistep Problems**

Li Wong and Sarah were given the following problem in written form to solve independently.

Mother made a batch of cookies.
She sent 2 dozen cookies to school with her daughter Ginger. Mother also gave 6 cookies to the children after school.
If mother had made 3 dozen cookies, how many cookies will she have left for dessert at dinner?

Li Wong and Sarah's papers contained the following answers and solutions.

Li Wong | Sarah



Journals

Mathematics instruction should furnish opportunities for students to communicate their understanding of mathematics. Journal writing is a communication format that allows students to reach agreement among themselves about the use of mathematical terms and to recognize the importance of shared understanding of mathematical ideas. Writing about mathematics helps students clarify their understanding and gives teachers valuable information from which instructional decisions can be made (NCTM 1989). Figure 8 describes one use of journal writing to improve understanding of mathematical ideas.

Journals are an excellent means of self-assessment. Students with disabilities often have difficulty expressing themselves in writing. With structure and guidance they can, and should, develop reflective skills. Often students with mild handicaps are so accustomed to receiving feedback from others, particularly negative feedback, that they fail to take responsibility for their own understanding of content. Journals help develop this important skill.

Journals should be kept on a consistent basis. Ideally, daily entries should be made in the journal. Teachers of self-contained classrooms can have students keep one journal for all subjects.

Students can write in a special mathematics section or communicate in a special color. If journals are used as part of the assessment program, they should be continually monitored by the teacher to identify any instructional needs of the student. Systematically gathering and analyzing information from the students' journals is necessary. Some teachers collect three or four different journals each day to read and then respond to the students' entries. In addition to the three or four targeted students, teachers may allow one or two students voluntarily to submit their journals if they have something they want the teacher to read and respond to quickly.

Teachers who function in a departmental situation can collect two or three journals from each class. This tactic requires that the teacher do no more "grading" than if one whole class turned in an assignment for grading. When students with learning differences are in the mathematics class, monitoring the journals of these children more often than those of other students may be necessary.

Adapting Assessment Techniques

Table 1 lists characteristics that impede progress in mathematics for students with mild handicaps, along with suggested alternative-assessment procedures for these students. Even these procedures may need some adaptation to be appropriate for students with disabilities. Some suggestions for adaptation appear in the third column. As students with learning problems become more proficient in mathematics, many of the adaptations may be phased out, or less obtrusive methods may be selected.

Figure 5. **Teacher's Assessment of Student Work**

Teacher Assessment. Although Li Wong did not get the correct answer to the problem, he did convert 2 dozen to the exact number of cookies. He then subtracted 6, the number of cookies given to the children after school, to determine the number of cookies left for dinner, but he made a computational error. However, Li Wong forgot to consider the total number of cookies. Mother made 36 cookies. The teacher made a decision to give Li Wong 2 points because he used an inappropriate method, did complete the problem and reached an incorrect answer.

Using the focused holistic-scoring scale, the teacher gave Sarah 4 points. She used the correct strategy but near the end committed a computational error, which caused an incorrect answer.

Li Wong and Sarah received some points for their solutions even though they failed to obtain the correct answer. Both students attempted to apply the correct strategies in solving the problem. Thus, their ability to use the correct problem-solving processes were recognized. Analyzing the processes both students used was fairly easy, since they showed a great deal of their work.

Recommendations. Both Li Wong and Sarah may benefit from problem-solving experiences in pairs or cooperative groups. Computational errors are a barrier to completing the problem correctly for both students. By working together or with other students, Li Wong and Sarah may detect and correct these errors. Likewise, the discussion of a group may have allowed Li Wong to see another step in the problem. Additionally, both students need some practice with the algorithms for operations. If these students work on problem solving independently, they should be encouraged to use a calculator so that their computational skills do not interfere with the problem-solving process.

Figure 6. **Cooperative-Learning Checklist**

Skills	Names					
Followed directions	☐	☐	☐	☐	☐	☐
Stayed on task	☐	☐	☐	☐	☐	☐
Explained ideas clearly to others	☐	☐	☐	☐	☐	☐
Supported ideas of others	☐	☐	☐	☐	☐	☐
Developed a plan	☐	☐	☐	☐	☐	☐
Engaged in constructive criticism	☐	☐	☐	☐	☐	☐
Persisted in completing the assignment	☐	☐	☐	☐	☐	☐
Performed the following roles:						
Checker	☐	☐	☐	☐	☐	☐
Recorder	☐	☐	☐	☐	☐	☐
Leader	☐	☐	☐	☐	☐	☐
Summarizer	☐	☐	☐	☐	☐	☐

Figure 7. **Using a Checklist to Assess a Student's Disposition Toward Learning Mathematics**

While students are working in groups of four, Mrs. Locklear walks from group to group and uses the checklist to assess the individual students in the group. She has had an opportunity to assess each group two or three times.

Teacher Assessment. Mrs. Locklear notes that Tyrone, a low achiever, is not really participating in his group. He is slow to get on task and seems to be easily frustrated. He rarely supports others and does not share his ideas in the group. Although he was assigned the role of checker, he rarely assumes the responsibility. In contrast. Latesha, a student with mild mental retardation, seems to function well in her cooperative group. Latesha gets on task quickly and supports other group members. She has also assumed the role of checker, but her resource teacher had practised that role with her.

Recommendations. Mrs. Locklear feels that Tyrone would benefit from the group experience but is not taking advantage of the opportunity. She decides that tomorrow she will take aside all the students who are checkers and do some role-playing with them to give some strategies for the checker role. Asking one of the members of Tyrone's group to encourage him to participate might be helpful. Perhaps Tyrone would do better in another group. Latesha needs no adjustment at this time, only continued support and encouragement.

Figure 8. **Using Journals to Assess Students' Ability to Communicate Mathematical Ideas**

Miss Brant, a Grade 5 teacher, has been having her students write in their mathematics journals daily as a form of self-assessment. In the initial journal-writing experiences, Miss Brant instructed her students to write about what they had learned and what was difficult for them. While reading the students' journals, she realized that many were having difficulty expressing themselves in writing. For those students who needed more structure, she decided to begin the daily journal assignment using the questions that follow. Miss Brant suggested to Donny and Enrico, two students who attend the special-education resource class, that they may want to copy the problem down to answer questions 1 and 2. Miss Brant told them not to worry about spelling every word correctly. She stressed how important it was to get their ideas in writing.

1. What did you find easy in this lesson?
2. What was the most difficult for you today?
3. What new thing did you learn today?

Teacher Assessment. In reviewing Donny and Enrico's journals, Miss Brant discovered that both boys were having difficulty with equivalent fractions. On Tuesday and Wednesday, both boys said that "3/4 = ?/8" was the most difficult equation for them to solve. Enrico's response to the third question was, "I hate fractions!" Donny's response to it was "Nothin'." However, when Miss Brant asked Donny to write a sentence about what he learned that day, he responded, "The bigger the bottom number the smaller the piece."

Recommendations. Miss Brant realizes that she can gain some insight into both boys' progress by continuing to monitor their journals carefully. At the same time, she wants the boys to use the journals as a way to describe their own progress. She decided that she will spend time with some of the students in extending their journal-writing skills. Miss Brant decided that she will also begin charting the areas that students frequently identify as difficult. Miss Brant views the students' journals as valuable information about students' progress and the need for further instruction.

These assessment procedures should not be viewed as alternatives but rather as examples of appropriate assessment procedures for the student who learns mathematics differently. These assessment techniques should be used systematically if sound decisions are to be made about instructing children with disabilities.

Summary

To determine the progress of, and make appropriate educational decisions for, students with disabilities or at-risk students, teachers should use assessment techniques that accurately determine the students' progress *in spite of their learning differences.*

Table 1. **Adapting Assessment Techniques for Students with Learning Difficulties**

Examples of Learning Difficulties	Alternative-Assessment Procedures	Examples of Assessment Adaptations
Reading Difficulty	Interview Observation & questioning Journals	Students use pictures or manipulatives. Teacher presents problems orally. Students dictate journal entries to an aide who records them. Students respond by acting out interpretations of problems or solutions.
Computational difficulty	Interview Observation Journals	Students use manipulatives in skill-development activities. Students use calculators in problem solving. Teacher provides positive feedback about progress.
Difficulty translating concrete understanding to abstract level	Interview Checklists Journals	Students use manipulatives to create an abstract solution Teacher records repeated successes at the abstract level before verifying mastery. Students' entries use pictures. Teacher has students explain pictures orally.
Learns more slowly	Holistic scoring	Teacher to— provide mnemonics for steps during assessment, give fewer questions or problems on tests, test in pairs or in cooperative groups, and allow students to complete graded assignments at home.
Difficulty remembering procedural steps	Interview Observation Holistic scoring	Teacher to— color code steps on tests. remind students of self-monitoring strategies. provide mnemonics for steps during assessment.
Fear of failure	Observation Self-appraisal checklist Interview	Teacher to— remind students to use stress-reduction techniques. remind students to use "self-talk" techniques. ask easy questions initially to build up students' confidence.
Lack of such number-sense concepts as "more than," "less than," or the value of multidigit numbers	Interview Holistic scoring Observation Journals	Teacher to— encourage the use of manipulatives give partial credit in estimation exercises. color code to focus attention on place value. encourage expressions about numbers.

Alternative-assessment procedures are not only for students with disabilities but are appropriate for *all students*. However, given the learning characteristics of many students who are challenged in one way or another, it is imperative that alternative-assessment procedures be used.

The authors are not suggesting that pencil-and-paper assessments be abandoned. Information gained from pencil-and-paper assessments, however, is not sufficient to assess the mathematics learning for these students.

An assortment of appropriately applied assessment strategies should be used to gain a comprehensive view of the abilities of students with disabilities.

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A mathematical truth is in and by itself neither simple nor complicated.

—Émile Lemoine

Assessing Mathematical Processes: The English Experience

Malcolm Swan

Historically, assessment has been concerned with measuring how completely students have mastered knowledge and skills. More recently, however, increasing attention is being placed on how effectively students can tackle unstructured problems and investigate novel, open-ended situations. Here, the focus of assessment is on how well students have acquired those processes or strategies that guide the choice of appropriate skills and enable students to explore unfamiliar situations. In short, we are becoming more aware of the need to assess how well students perform as mathematicians in addition to how well they have learned mathematics.

Whereas the knowledge of facts and skills can be assessed through short, closed questions, the existence of strategic skills can be assessed only through more open tasks that require students to make choices, to reason and to explain. Traditional forms of timed, written examinations do not allow sufficient time for students to pursue their own lines of inquiry, meet dead ends, plan approaches and so on. A system is needed that allows students adequate time to tackle extended tasks, collaborate with peers, reflect and redraft ideas and polish products. The system also needs to be manageable and rigorous so that the assessments are made efficiently and reliably. For the system to have status, we must also ensure that different assessors use common standards that are externally validated. This article describes such a system, warts and all, that is currently being used throughout England.

Specifying an Assessment System

Traditionally in England, assessment has been norm-referenced. At the end of a period of assessment, all we could say about students was how well they performed in comparison with their peers. We were unable to say anything objective about what the student was able to do. It was impossible to know whether, in the long term, standards were rising or falling. For these reasons, the government decided to introduce a set of national criteria against which assessments would be made (DFE/WO 1991). Mathematics currently has five attainment targets: Using and Applying Mathematics, Number, Algebra, Shape and Space, and Data Handling. The last four targets list the facts, skills and concepts to be assessed in traditional content areas (for example, "can find a fraction of a quantity"); the first lists the mathematical processes to be assessed (for example, "can make generalizations"). Having a separate list for processes has the advantage that strategic skills are given more status than if they are merely absorbed into content lists, but the danger also exists that they are taught and assessed separately from "real mathematics."

Each attainment target has a 10-level hierarchical description. Teachers are required to use these levels to describe how students' concepts and skills develop through their school careers. The levels are not age related. In any mathematics class can be found a range of levels of performance in any individual attainment target. Similarly, an individual student will perform differently across different attainment targets. Teachers are required to keep careful records documenting each student's profile and progress. For students at the age of 7, 11, 14 and 16, the results of these assessments, together with the results of externally supplied national tests, are made public.

The laudable desire to pinpoint precisely what students know and can do has unfortunately led to a proliferation of criteria; the system has become so unwieldy that it is currently undergoing simplification (SCAA 1994). A second problem is that the original intention of the levels—monitoring performance outcomes—has often been misinterpreted, by teachers and textbook writers, as prescribing the order in which mathematics must be taught. This misconception has led to whole classes being taught level 5 before level 6, making nonsense of the differentiation in ability that exists within a class.

Because this article is concerned with assessing mathematical processes, attention is focused on just the first attainment target, Using and Applying Mathematics, which is subdivided into three strands for the purpose of assessment. These strands are (1) applications, (2) communication and (3) reasoning, logic and proof. The strands are described more fully in Table 1. In the original documents, examples are attached to each statement to aid their interpretation.

Table 1. **Attainment Target 1: Using and Applying Mathematics**

Level	Strand 1: Applications	Strand 2: Communication	Strand 3: Reasoning, Logic and Proof
10	Explore independently and constructively a new area of mathematics.	Apply mathematical language and symbolism confidently when handling abstract concepts. Present logical and concise accounts of work resulting from an independent exploration of a new area of mathematics, commenting on alternative solutions.	Handle abstract concepts of proof and definition when exploring independently a familiar or new area of mathematics.
9	Coordinate a number of features or variables of solving problems.	Use mathematical language and symbolism effectively when presenting logical accounts of work. Produce concise justifications of their solutions to complex problems.	Justify their solutions to problems involving a number of features or variables.
8	Make reasoned choices when exploring a mathematical task.	Use mathematical language and symbolism effectively when presenting logical accounts of work, stating reasons for choices made.	Understand the role of counterexamples in disproving generalizations or hypotheses.
7	Follow new lines of inquiry when investigating within mathematics itself or when using mathematics to solve a real-life problem.	Use appropriate mathematical language and notation when solving real-life problems or commenting on generalizations or solutions.	Examine and comment constructively on generalizations or solutions.
6	Pose their own questions or design a task in a given context.	Examine critically the mathematical presentation of information.	Make a generalization giving some degrees of justification.
5	Carry through a task by breaking it down into smaller, more manageable tasks.	Interpret information presented in a variety of mathematical forms.	Make a generalization and test it.
4	Identify and obtain information necessary to solve problems.	Interpret situations mathematically, using appropriate symbols of diagrams.	Give some justification for their solutions to problems. Make generalizations.
3	Find ways of overcoming difficulties when solving problems.	Use or interpret appropriate mathematical aspects of everyday language in a precise way. Present results in a clear and organized way.	Investigate general statements by trying out some examples.
2	Select materials and the mathematics to use for a practical task.	Talk about work or ask questions using appropriate mathematical language.	Respond appropriately to the question "What would happen if . . .?"
1	Use mathematics as an integral part of practical classroom tasks.	Talk about their own work and respond to questions.	Make predictions on the basis of experience.

Assessing Mathematical Strategies

The way in which students are taught from age 11 is deeply affected by the style of the examination that they will eventually face at age 16. In England, probably even more so than in the United States, the examination system drives the curriculum through its backwash effect. How the assessment of the Using and Applying Mathematics strand is carried out by different regional examination boards is presented next.

Teachers currently have a choice, depending on the examination board for which they opt. They may assess students through

- extended tasks that are prescribed by the examination board;
- extended tasks of their own choice but that are bounded within prescribed themes, such as one statistical study, one pure investigation or one with real-life practical applications;
- their own selection of evidence from students' portfolios of work; and
- an externally set examination paper consisting of a number of shorter questions.

This final method is currently proving the most controversial. It is hard to see how such processes as breaking tasks down into manageable steps, posing questions or exploring independently can be assessed in traditional examination formats, and every attempt made to date has proved unsatisfactory.

This article focuses on the use of more extended tasks that are completed in a normal classroom-working atmosphere. Suitable tasks are open-ended, are accessible to the least able students and also afford opportunities to stretch the more able students.

In most situations, teachers are expected to perform the assessments. The job of ensuring the consistency of standards falls on a system called "moderation." Within individual schools, teachers hold standardization meetings at which they award scores to the students' portfolios and rank them in order of merit. Samples of these portfolios are then required by the examination board for inspection by area moderators. Their job is to ensure that different schools are applying similar standards. They will not attempt to alter the ranking of students within a school, but they may adjust all the scores to bring them into line with those from other schools.

A Sample Task

The following activity was sent to all schools to exemplify the type of task that supports the assessment of strategic skills. See Figure 1. Students are given octagonal tiles made from cardboard and work on the problem in groups. They may decide to investigate the number of tiles in each loop or the number of free edges inside, outside or in total. This initial investigation can then be extended according to the capabilities of students. For example, pupils might focus on the following concepts:

- The different loops that can be made from a specific number of tiles
- The shapes inside loops and the shapes made by joining the centre of each tile (see Figure 2)
- The ways of recording shapes (Figure 3)
- The symmetrical properties of the tiles and loops
- Some ways to change the rules for making the loops
- The angle sum of the shapes inside the loops
- Whether all regular polygons can make rings (Figure 4)
- How to prove that an odd number of octagon tiles will not make a loop
- Using Logo to draw the shapes obtained

Assessing the Work

Figure 5 is part of a response from one student, Alison. She starts by sketching the inside shapes made by loops of four, six, and eight octagons and finds that no loops exist for three, five or seven octagons. She then looks for patterns in the number of inside and outside free edges. The reader might try to assess this response with the "Using and Applying Mathematics" attainment target (Table 1) before reading on.

To assess this work, teachers match aspects of this performance to the general criteria in each strand. This task is not always straightforward.

Consider strand 1, Applications. Alison has thought about the shapes that she can make with three, four, five, six, seven and eight octagons in turn. She has thus broken down the investigation into stages. This process could be said to show evidence that she can "carry through a task by breaking it down into smaller, more manageable tasks" (level 5). She has clearly not posed her own questions or followed new lines of inquiry, so no evidence demonstrates that she has reached levels 6 or 7.

Look at strand 2, Communication. Her work uses diagrams, tables and text. Perhaps this presentation suggests that she can "interpret situations mathematically, using appropriate symbols or diagrams" (level 4) or "interpret information in a variety of mathematical forms" (level 5). Although hard to judge, level 5 perhaps requires a wider variety of forms than that shown here, such as graphical and symbolic forms. Alison has not yet shown that she can examine critically the mathematical presentation of information (level 6). This ability could perhaps be shown if she is asked to criticize the presentation of her work and invited to redraft it.

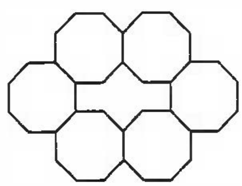
Next examine strand 3, Reasoning. Her conclusion that "you multiply the number of octagons by 6" to get the total number of free edges shows that Alison can "make a generalization and test it" (level 5). She has not explained why this pattern occurs—for each tile in a loop, two edges are touching and six are free—so she has not yet shown that she can "give some degree of justification" (level 6).

Figure 1. **A Task Supporting the Assessment of Strategic Skills. Reproduced with permission of SCAA (1992).**

Octagon loops

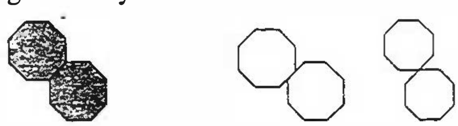
Making the loops

This is an octagon loop.
It is made by joining octagon tiles together.
There is only one space in the middle of the loop.



Rules for joining the tiles

Line up the edges exactly. Not like this . . . or like this . . .

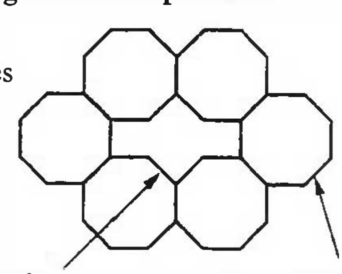


Make sure that each tile only touches two others. Not like this

Make some loops. Use any number of tiles up to 10. Try 5 different loops.

Looking at number patterns

6 tiles



10 inside free edges 26 outside free edges

Total number of free edges 36

Can you tell how many free edges there will be in your next loop before you make it?
Investigate

This analysis leads to several important points:

- Matching specific performances to general criteria is a subjective business. A wide variety of interpretations may be given to each level's descriptor. In practice, however, teachers are given examples of assessed students' work to guide their judgments, and the process of consultation helps teachers reach a consensus "feeling" for what a level's descriptor means. For situations in which externally supplied tasks are used, task-specific descriptors are often given.
- The demand, or level, of a process criterion is not meaningful unless it is related to a particular context. Finding a generalization of the number of inside free edges, $3n - 8$, is much harder than finding a generalization for the total number of free edges, $6n$, where n is the number of tiles used. Thus a level cannot really be attached to "Make a generalization and test it" unless the context is specified more closely. Again, assessment falls back on the judgments of teachers.
- Even if levels were well defined, assigning a level to a student on the basis of a single fragment of evidence is clearly not possible. In addition, it is unclear whether the "best" or a "typical" performance of the student should be assessed. One examination board, for example, specified that the teacher select "two pieces of evidence which represent the best sustainable work of the student in each strand."
- If students are unaware of which aspects of performance are being assessed, they are unlikely to display these aspects. In the foregoing example, we cannot say whether Alison was able to extend the problem or use a graph. She was not asked to do these things. One alternative is to introduce new scaffolded prompts, but this addition destroys the openness of the task. A second alternative is to make students more aware of the criteria on which they will be judged. Some teachers have developed student-friendly versions of the criteria or even samples of assessed work for students to discuss.
- Most tasks do not permit students to display performance at the full range of levels. In the foregoing task, students had no opportunity to reach the highest levels in "communication," for example. This limitation means, therefore, that a carefully balanced range of tasks must be offered.

Concluding Comments

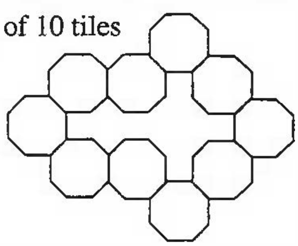
So What Is All This Assessment For?

Ongoing assessment is intended to be formative, whenever individual profiles are kept of each student's progress. In many situations, these profiles have ensured more continuity when students change schools or move between classes. Many teachers, however, see all this record keeping as an unnecessary form of bureaucracy that adds nothing of significance to what they already know about their students. Many have simply refused to participate in the assessment.

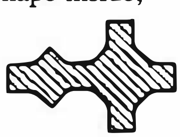
In some schools, especially those in which students are involved in their own assessment, these profiles can serve to inform and motivate students. In one school, for example, each student is given a list of specific content-and-process learning objectives at the start of each extended mathematics activity. These objectives are taken directly from the National Curriculum Framework and are translated into simple, task-specific English. At the end of the activity, students assess their own performance against these targets. The teacher then assesses the work and discusses any discrepancies with each student privately. Assessed portfolios of the students' best work are built. Over a period of time, students thus become much

Figure 2. **Loops, Octagons and Resulting Shapes**

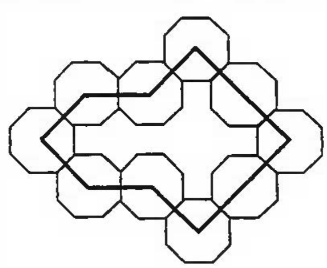
This loop of 10 tiles



produces this shape inside;



when the centres of the octagons are joined,



this shape results.

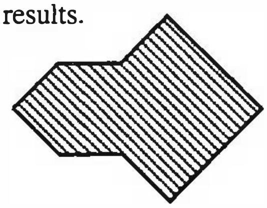


Figure 3. **Encoding the Perimeters**

The perimeter of the shape inside this loop may be described by this code: 1, 2, 4, 1, 3, 1, 3, 1, 4, 2. It could also be described by its area.

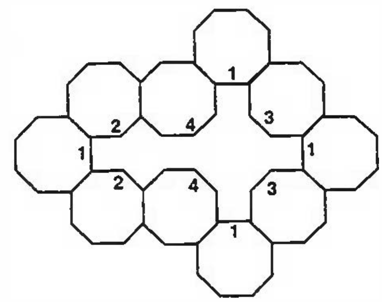


Figure 4. **Students Are Asked Whether Regular Polygons Can Make Rings**

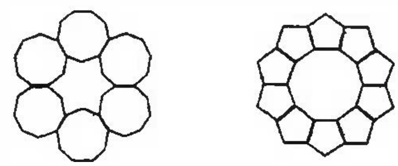
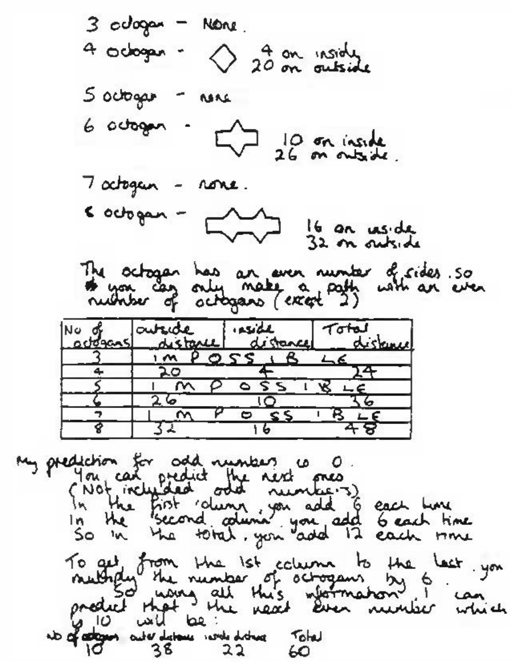


Figure 5. **Alison's Response to the Task**



more aware of their own progress. One noticeable outcome is that students begin to realize that strategic skills are legitimate goals for learning.

Unfortunately, when assessments are made for summative purposes, levels for process get added, weighted and combined with content scores to give relatively meaningless lettered grades. We still have "grade A" students and "grade G" students. Norm-referencing still plays an important part in checking that grade boundaries are "accurately" placed, since populations are not expected to change significantly from year to year. This tendency to reduce people to numbers or letters for selection purposes is unjustifiable. Any system of assessment can be turned from one that celebrates positive achievements into one that just placed crude, meaningless labels on people.

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What is the solution of the following cryptarithm puzzle?

```
  MIX
  FUN
 +AND
 MATH
```

The Demands of Alternative Assessment: What Teachers Say

Thomas J. Cooney, Karen Bell,
Diane Fisher-Cauble and Wendy B. Sanchez

Margaret is an experienced high school mathematics teacher who is always searching for ways to "get the kids turned on to math." As a result, she is very concerned about understanding better the diverse student populations she teaches, and she recognizes that her traditional teacher-centred methods of teaching and assessing often did not allow her to reach all her students. For the past three years, she has been involved in an inservice program on alternative assessment, and she has used many new techniques, such as portfolios and open-ended questions, to enrich the mathematical environment in her classroom. She recalls her earlier days of teaching, which she describes as lecture- and teacher-oriented. Then, her students seldom worked together. She assigned them mostly algorithmic exercises, which also dominated her tests and quizzes. Although her students were generally successful, something was missing for Margaret. She wanted students to share her excitement about mathematics, but this enthusiasm was rarely apparent. She "wasn't brave enough to go outside the traditional setting" until she became involved with other like-minded teachers who shared her desire to teach and assess mathematics differently.

The following open-ended question contrasts with the narrow questions that Margaret typically asked before becoming involved with alternative assessment. Her students' responses enriched Margaret's understanding of what mathematics they were really learning.

Problem: Write the equation of a parabola that opens downward and whose vertex is in the second quadrant. Support your selection.

Karen responded in the following manner.

$$y = 2x^2$$

x	y
–1	2
–2	9
–.5	.5

David's answer follows:

$$-3x^2 = -4(5)y$$

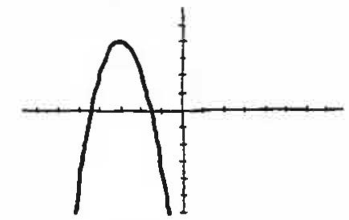
I got this answer because I know a parabola has an x-squared term. I put a –3 in front of the x-squared term and a 5 in front of the y term to make it in the second quadrant. Then I put a –4 in front of the $5y$ to make it open down.

Alisha's solution included a graph (see Figure 1):

$y = 2 - (x + 3)^2$

Its [sic] a parabola because the x term is squared and the y term isn't. The 2 modifies the y-coordinate and the 3 modifies the x-coordinate. This make [sic] the vertex be in the second quadrant. The negative sign in front of the parentheses makes it open downward.

Figure 1. **Alisha's graph**



Although their grades had not differed dramatically when these students were assessed on algorithmic tasks, Margaret realized that they had quite different understandings about graphing parabolas. Karen was vaguely aware that a parabola involves squaring one of the variables, but she was unable to produce an equation that would meet the constraints of the problem. Margaret did notice that Karen attempted to place her parabola in the second quadrant through her choice of data points. David was aware that certain numbers in the equation would determine the location of the parabola's vertex, although the vertex of his parabola remained at the origin because he worked only with the coefficients of x^2 and y. Like Karen, he failed to check his reasoning by drawing the parabola. Margaret was impressed with the thoroughness of Alisha's response. Alisha realized how to create a parabola that opened downward and how the vertex could be translated into the second quadrant. Further, she supported the equation that she created by actually drawing the parabola. Margaret decided to share this response with the other students to convey her expectations about what constitutes an excellent response.

These and other students' responses were eye-openers for Margaret because she could see the range of students' understanding of the relationship between the equations and graphs of parabolas. This realization encouraged her to change both her teaching and her assessment in many ways. She now routinely uses open-ended questions in class as well as on tests and quizzes as she encourages students to explain their reasoning. She admits that her classroom is sometimes chaotic, but she wants her students to explore and discuss mathematics. She has found herself trying "to think about all the things I may be not asking the students to tell me." Interestingly, she has found that "my students seem to enjoy mathematics more, which has enriched my teaching." Somehow, the extra time and effort for teaching and assessing mathematics in this way have been a good investment for both Margaret and her students.

The Demands of Alternative Assessment

The recent reform associated with alternative assessment has many facets, including teachers' use of portfolios, open-ended questions, journal writing and projects designed to elicit students' ability to communicate mathematically. The intent of alternative assessment is at least twofold. First, it encourages students to think more deeply about the mathematics they are learning. Second, it is a valuable tool for teachers to use in revising and redirecting their teaching when necessary. Teachers who have implemented the assessment standards have found that they have a better idea of their students' understanding of mathematical concepts (Seeley 1994, 5). Hancock (1995) suggests that integrating instruction and assessment in a continuous cycle blurs the boundaries between assessment and instruction, thus resulting in a more coherent instructional program.

But the advocated shift away from "treating assessment as independent of curriculum or instruction" and toward "aligning assessment with curriculum and instruction" (NCTM 1995, 83) comes with a price. Without question it places additional burdens on teachers in terms of planning for instruction and assessing students' performances. The following section focuses on issues that teachers have raised when implementing alternative-assessment methods.

Teachers' Perspectives of Alternative Assessment

Over the past several years we have interviewed teachers, observed their teaching and assessment methods and worked with them as they implemented various aspects of alternative assessment. Every teacher implemented some aspect of the assessment standards, outlined in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), but no teacher implemented all the recommendations. Teachers had to find what was comfortable for them as they reformed their teaching and assessment methods. Here, we focus on five issues that were central to most teachers: (1) loss of predictability in the classroom, (2) content coverage, (3) familiarizing students with the teacher's expectations, (4) dealing with increased demands on time and (5) communicating with parents.

Loss of Predictability in the Classroom

It seems fair to say that most classrooms are grounded in predictability. Students expect it; most teachers demand it. Yet alternative assessment creates a climate of unpredictability because we can never be sure what students are going to say when we ask about their mathematical thinking. This unpredictability can be a challenge for teachers, especially those who feel more comfortable in a well-organized and predictable classroom. Alternative assessment does not doom one to a chaotic state, but a teacher-centred classroom with limited mathematical objectives is contrary to what alternative assessment is all about.

Loss of predictability makes the use of alternative assessment risky for some teachers. For Margaret, it was not a problem. She admitted that sometimes her classroom was chaotic, but "planned" chaos was within her zone of acceptability. In contrast, another teacher, Ellen, thought that the risk was too great and relegated alternative assessment to open-ended questions used as warm-up activities or as bonus questions on tests. She recognized her reluctance but was concerned about engaging in open-ended questions in class discussions because of their inherent unpredictability. Observations of Ellen's classroom indicated a tightly controlled, teacher-centred classroom with tests that focused on predictable algorithms or questions that had been well rehearsed during class. Another teacher expressed her concern this way: "If an administrator comes in to observe you and if he doesn't realize what you are doing, then he is going to mark you down because your class is kind of loud."

Still, some teachers find it a relief to loosen the reins. One teacher stated that becoming involved in alternative assessment has allowed her to let go of always requiring a particular answer and of making the students "practise a million problems" because she knows that as students work with a particular topic, they will get practice "where it's needed."

Content Coverage

Another issue is related to covering the mathematics content. Common wisdom suggests that trade-offs are necessary between emphasizing higher-level-thinking skills and focusing on drill and practice. Yet project teachers maintain that they have no evidence that an emphasis on alternative assessment inhibits test performance. Content coverage and performance on high-stake tests are exceedingly important issues for teachers. This concern is particularly acute for teachers of upper-level mathematics courses, which heavily emphasize preparing students to "take the next course" and to score well on standardized tests. Our observations suggest that as teachers teach higher-level content, they develop a certain conservatism with respect to implementing alternative-assessment methods. Middle school teachers, for example, are much more likely to use journal writing and portfolios than are secondary teachers, perhaps because their grading schemes generally place less emphasis on tests.

Familiarizing Students with the Teacher's Expectations

Many teachers saw a need to include more open-ended questions in their discussions with students. In part, they believed that it was "better mathematics," but some also thought that it was necessary for the students to "practise" answering open-ended questions before they encountered them on tests. The teachers wanted to be fair. Margaret, for example, wanted to share Alisha's response on the parabola problem with other students so that they would have a clear idea about what she regarded as a high-quality response. The issue of grading and subjectivity and the corollary of being consistent are major concerns to teachers and students alike.

To some extent the issue of consistency can be defused through developing and using scoring rubrics. A few teachers involved students in creating scoring rubrics with some success. Shepard (1995) states that such an approach affords a context for helping students learn to assess their own learning and to be reflective. Clarke (1995) advocates negotiating rubrics with students by giving them examples of high-, average-, and low-quality performance and then having them assess other responses. Too, this approach helps students identify qualities that characterize outstanding performance, increasing the likelihood that they will demonstrate such performance. Nevertheless, some students—particularly those who have always done well with traditional, algorithmic tasks that lead to a single, numerical right answer—are anxious and concerned about being assessed by the use of open-ended questions.

Dealing with Increased Demand on Time

Teachers who are considering alternative assessment frequently ask, "How much time does it take?" The use of alternative assessment can appear overwhelming. Indeed, we encountered no teachers who used portfolios, journal writing, projects and tests that were dominated by open-ended questions. Different teachers have different comfort levels with each approach. Nearly 40 percent of one teacher's test items were open-ended, but he used little else that could be characterized as alternative assessment. In contrast, another teacher made extensive use of journal writing and projects, but her tests emphasized problems with single, numerical answers. These contrasts have to do with an individual teacher's comfort level when using various facets of alternative assessment. An experienced project teacher emphasized to teachers new to the project to "start small and go from there. Select just one class to do portfolios with . . . because until you've been through the process of evaluating completed portfolios, you don't really know what you're getting into. So start small, feel comfortable and then expand."

We found that presenting a context for teachers to take a simple, initial step was a significant help in addressing the time issue. For example, questions of the form "What's wrong with this?" are easy to generate yet elicit responses that reveal much about students' understanding of mathematics (see Cooney, Badger and Wilson 1993). Most project teachers felt comfortable asking such questions and found that the time it took to grade them was not prohibitive. Still, the problem of creating good, open-ended items is significant for many teachers. Our project teachers profited from sharing items used by others: "I've tried this and here's what happened and here's why this is a good item." A good bank of items was very helpful to the teachers.

Communicating with Parents

Parents are understandably concerned about their children's grades. They complain when they feel unable to help their children, especially on more open-ended tasks with which they are not familiar.

Some parents have difficulty interpreting their children's grades when alternative assessment is used, expressing the view that the grades are too subjective. One teacher explained that her students "are given a grade on their math content, on their written explanation, and [on] their visual product." But some parents consider that "appearance or neatness and things like eye contact are just a lot of fluff." Parents also question whether projects involved significant mathematics. However, teachers have found that parents, once educated, are "surprised at the [amount of] mathematics involved." One teacher, Sally, found considerable support from parents once they had a sense of what she was trying to accomplish. When she explained the survey and data analysis that she had expected her students to do and the rubric that she had used to score the students' work, the parents were quite impressed. Sally summarized their reactions in the following way: "They realized what a marketable skill it was for a student to be able to present raw data mathematically and attractively using computer software. They felt I was fair."

Several teachers commented that alternative assessment has facilitated their communication with parents. One teacher was confronted by a set of parents who were concerned that their child was placed with the "slower students." She showed them their child's portfolio to support her position that their child did not have the requisite skills for a more advanced class. Another teacher recalled showing parents their child's responses to open-ended items that suggested that "the child can solve the problems but can't show her reasoning. When I ask these questions, I'm determining whether or not your child understands."

Several teachers have found that parents like alternative-assessment tasks that engage students in writing activities. Portfolios, journals, nonroutine problems and open-ended questions have brought "lots of positive parent reaction. Parents like journal entries, the writing. They like the fact [that] students are learning more than multiple-choice answers. . . . They see the purpose of essay questions, that the students must understand."

Professional Development and Alternative Assessment

"The best thing that's happened to me in my career," offered Marsha, "was working with other teachers on using alternative assessment." Despite all the reservations and the demands placed on their time, teachers see a value in alternative assessment that goes beyond the benefit to their students. They see themselves in a different professional light—as being empowered. Another teacher summed up their feelings in this way: "I'm glad to see we're doing it, but it's hard. This is one of the hardest things that I've ever tried to do. It's more difficult than changing textbooks and learning a new curriculum because basically you're changing your whole teaching style." Even those teachers who consider themselves "movers and pushers" of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) find that the time required to "be creative" is demanding but "we're going to have more students turned on to mathematics, and that's [our] goal." The teachers realize that they can change their ways of teaching and assessing mathematics. Another teacher, like Margaret, reflected on how she formerly taught mathematics. "You wouldn't see my students working cooperatively as much. . . I used to teach in lecture form. . . it was a right or wrong answer." Lastly, many teachers expressed appreciation for the increased communication with their students that alternative assessment had afforded them.

A Closing Comment

If the project teachers have such high regard for alternative assessment, why are many other teachers reluctant to try this approach? One teacher's answer to this question was that teachers like to stick to the "old ways because it's easier to go page by page." Another teacher believed that teachers may not be aware of what students can do. "I now know what they're capable of. When I first did this, I didn't know what the students could do." Perhaps the strongest endorsement for alternative assessment was offered by a teacher who stated, "It's assessment. I hope that other teachers will learn that—it's assessment, not alternative assessment." Although the demands are many, we hope that what is uncommon today will become common tomorrow in the way that we teach and assess our students—as it was for many of the project teachers.

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Municipal Park

A municipal park has recently been opened in the ancient city of Danechester. The park is rectangular in shape. When I asked about its dimensions, I was given two somewhat odd items of information. The first: The diagonals of the park, plus its longer sides, were together equal to seven times one of the shorter sides. The second: the length of one diagonal exceeded that of one of the shorter sides by just 250 metres. What is the area of the park?

Une sortie mathématique!

Hélène Gendron

Objectif: Mesurer la hauteur approximative des arbres, poteaux, édifices, etc. qui sont près de l'école en utilisant leurs ombres.

Durée: Une classe

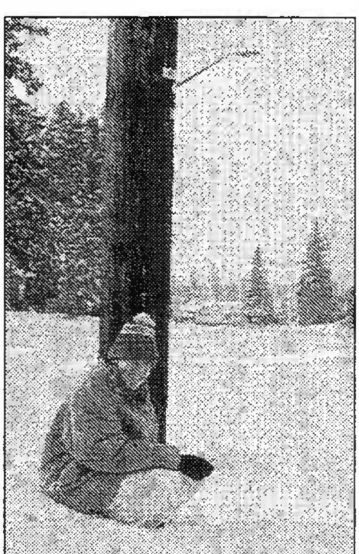
Cette sortie peut être présentée comme amorce à une unité de trigonométrie en Math 9 ou comme application du concept des rapports en Math 8.

Matériel et préparatifs

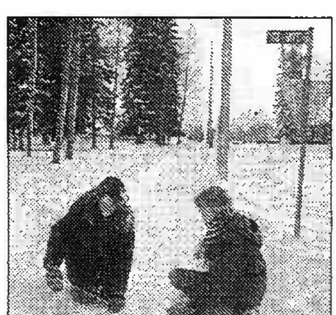
- Remplir tous les papiers nécessaires à la sortie. Étant donnée la nécessité d'avoir une journée ensoleillée, expliquer à votre directeur que la sortie se fera entre tel et tel jour et que la date officielle sera décidée le matin même.
- Avertir les élèves de se tenir prêts pour la sortie: Avoir les bottes, tuques et mitaines dans les casiers tous les jours, même si ce n'est pas la mode.
- Emprunter les rubans à mesurer au professeur d'éducation physique qui en a pour mesurer la longueur des pistes (en athlétisme). Si l'activité se fait dans la neige, ce serait une bonne idée de laisser sécher le ruban à mesurer avant de le rembobiner. Sinon il serait peut-être difficile de le ravoir l'année suivante.
- Choisir à l'avance les arbres, poteaux, édifices que vous allez faire mesurer. S'assurer que les ombres sont toutes à l'horizontal (qu'elles ne se projettent pas sur un mur par exemple) et ne traversent pas de rues! Attention aux groupes d'arbres parce qu'il peut être difficile de déterminer quelle ombre appartient à quel arbre.

Méthode

- Diviser la classe en deux ou trois équipes.
- Assigner une ombre à mesurer par équipe.
- Spécifier que les élèves ne doivent pas quitter le terrain de l'école. Indiquer aussi que les élèves sans vêtements appropriés contre le froid ne peuvent pas sortir!
- Les élèves présentent leurs résultats à l'écrit dans un genre de laboratoire de science:
  - Leur hypothèse (estimation)
  - Leur méthodologie (communication mathématique) incluant leur tâche dans cette méthodologie
  - Leurs résultats (calculs)
  - Leur analyse (liens):
    - Pourquoi ça marche?
    - Pourquoi l'heure de la journée change les données? Est-ce que cela change le résultat?
    - Quel est le degré de précision? Est-ce que cela peut-être amélioré? Comment?



L'élève tient une extrémité du ruban à mesurer pendant qu'une partenaire trouve l'autre extrémité de l'ombre.



Les élèves mesurent le poteau de la 12e avenue.

Les élèves sont rentrés pour calculer les résultats au chaud.

Is This a Math Class or an Art Class?

Harold Torrance

Over the years, students have hurled that question at me as I try to teach the value of setting up certain types of problems with a drawing or simple sketch. I often answer it with the old saying "a picture is worth a thousand words." Can this be true for mathematics? Is a picture really worth a thousand words? The answer may be a resounding "yes."

Humans are visual beings. The graphic artists employed by large marketing firms have known this for many years. When they really want to get our attention as consumers, what do they do? The artists create a picture to accompany the product they are selling. Ideally, it's a drawing that catches our attention at first glance and holds it until their message has been conveyed. So how does this age-old marketing concept help us in the classroom? The eye sends the brain thousands of pieces of information with each glance we take. So why not harness some of this potential by using drawings as a tool for solving mathematics problems?

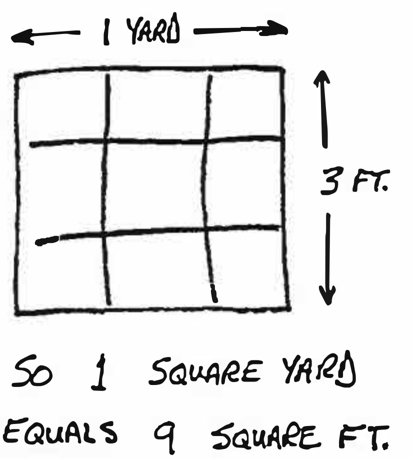
I teach students to consider employing a drawing or sketch as a routine part of problem solving. It has proven to be an effective technique for many students, even if only used to supplement other methods of approaching problems. Drawings can be particularly effective for students when confronted with problem types that fall into these two categories: 1) a problem type they don't normally encounter and therefore have little or no experience base to draw upon and 2) problems having lots of information which is difficult to organize. Sometimes using a drawing will enable students to solve a problem that they did not even know how to begin. The drawing provides a framework on which to attach the known information. With that in place, a process for solving the problem is more likely to emerge.

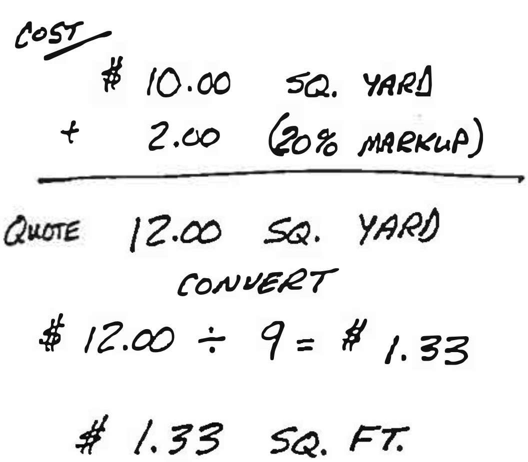
The problems that follow are much simpler to understand and solve once a visual reference has been produced. The sketches are left "rough" as they might look on a chalkboard or student's paper.

Problem 1

A contractor buys floor covering wholesale at cost of $10 per square yard. The contractor then marks the floor covering price up by 20 percent before quoting the price to customers. The customers are always given the price per square foot, as the contractor believes this is easier for the customer to understand. What is the price the contractor quotes per square foot of floor covering?

In this problem, the common mistake comes not from computing the percentage markup, but in treating an area problem as if it were a linear measurement problem. Students will add $2 to the price per square yard, then proceed incorrectly by dividing this figure by 3, since there are 3 feet in a yard. That step yields the incorrect answer since there are 9 square feet contained in 1 square yard. When shown the drawing below, most students immediately see their error.

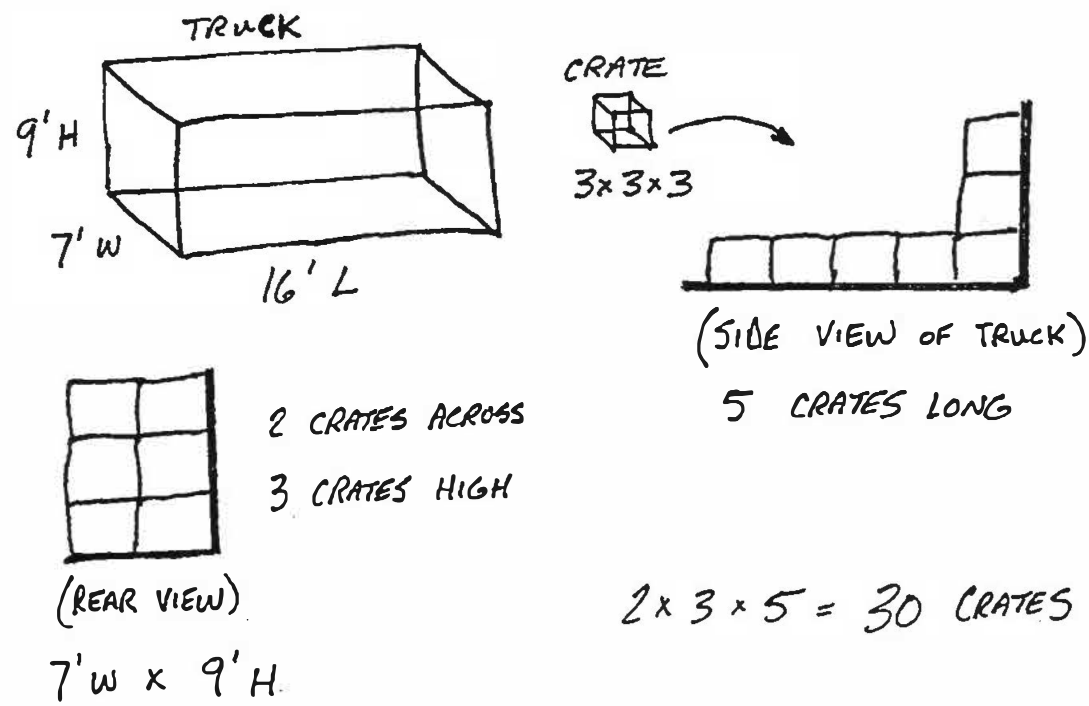




Problem 2

A trucking company uses trucks with cargo dimensions of 7 feet wide, 16 feet long and 9 feet high. What is the maximum number of cube-shaped crates measuring 3 feet by 3 feet by 3 feet that will fit in the truck's cargo space?

Students often approach this problem confidently, computing the volume of the cargo space, then dividing that figure by each crate's volume of 27 cubic feet. But the crates are not a liquid item which can simply be poured into the truck until the volume is fully maximized. Their dimensions are both fixed and rigid, so the incorrect answer turns out to be a bit on the high side. A drawing clearly indicates that the crates may be stacked inside the truck 3 high, 2 across and 5 running the length.



Problem 3

A seafood wholesaler pays the following amounts for fish: $2.29/pound for tuna, $1.79/pound for mackerel, $2.07/pound for premium cod, $1.59/pound for small cod and $3.50/pound for swordfish. What is the total amount spent for the fish if the following quantities of each were purchased: 339 pounds swordfish, 1,202 pounds mackerel, 154 pounds premium cod, 874 pounds tuna and 561 pounds small cod?

In this problem a drawing would be of little use, but some mechanism for organizing the varying quantities and prices would clearly give a foothold for getting started. Here, a simple table seems to be the ideal approach. Begin by listing the fish in the left-hand column. Add columns for the prices paid, quantities purchased and tallied amounts.

	PRICE PD.		LBS. BOUGHT		
TUNA	2.29	x	874	=	2001.46
MACKEREL	1.79	x	1202	=	2151.58
PREM. COD	2.07	x	154	=	318.78
SM. COD	1.59	x	561	=	891.99
SWORDFISH	3.50	x	339	=	1186.50
			TOTAL SPENT		$ 6550.31

Word Problems a Problem? WHAC 'Em

Harold Torrance

No, not the students! WHAC is the name of a strategy that students can use for solving word problems. It provides a systematic method for approaching and solving even the most difficult problems. WHAC is also easy for students to learn initially and put into use.

Many students encounter difficulty correctly solving word problems. This is a situation often encountered at all levels from elementary through high school. Numerous books and research studies have outlined causes for this inability to solve word problems, but few strategies have been offered to actually address the problem. WHAC is a simple strategy that I've used with quite good success in both elementary and middle grades classrooms. Students are given a set procedure to use for tackling any word problem. The approach is designed to analytically guide them through a given problem in organized steps.

The only prerequisite for using this strategy is a simple matching of objective to entry behavior. For example, if the problem will eventually call for a student to add two numbers, then the student must already be capable of computing the sum of two numbers. For the WHAC strategy to work successfully, the student's entry behavior must correspond with the computational operation which is being called for in the problem. Computation is usually the easiest part of mathematics for students anyway. WHAC helps students with the most difficult, higher-order reasoning by giving them an initial foothold to the problem.

The WHAC strategy is composed of a four-part process that is used virtually the same way each time a word problem is attempted. This prevents students from rambling without a place to start, or trying to answer a problem without first knowing what is being asked. I've found that students learn the process quickly and are able to use it successfully for solving problems.

What follows is an explanation of WHAC and an example of how it is actually used. The strategy can be taught as a group activity or to individual students as needed. I've used it for both enrichment and remediation purposes, though I think it is best used as a whole-class activity.

WHAC stands for What, How, Action, Complete. These four words outline the four-step process:

1. What is the question asking?
2. How do you go about solving it?
3. Action is taken to do the actual computation.
4. Complete the problem by answering it.

Example Problem

A bicycle shop has 23 red bikes, 14 blue bikes, 11 black bikes and 9 green bikes in stock. They can special order other colors, but none are currently in the store. What is the total number of bikes in the shop which are either red or green?

Step 1. What Is the Question Asking?

After the problem has been read through, show students how to focus on this one part of the problem. They should be able to see that the question simply asks how many bikes are either red or green. I typically ask students to jot down a shortened version of the question so that it is not forgotten while they are doing the actual computation.

How many bikes are red or green only?

Step 2. How Do You Go About Solving It?

This particular problem asks only for the total number of bikes that are either red or green. Students should be able to see that the information given about blue and black bikes will not be used. The information about ordering other colors is also superfluous. At this point the plan for solving the problem is written out by the student.

Combine the number of red bikes with the number of green bikes. Add them.

Step 3. Action Is Taken to Do the Actual Computation

Here the problem is set up and computation done.

23 red bikes
\+ 9 green bikes
32 bikes which are either red or green

Step 4. Complete the Problem by Answering It

In this step it is useful to have students refer back to Step 1. They can then fashion an answer based on what they previously determined was being asked in the problem. For word problems, I have asked students to answer with a complete sentence.

The total number of bikes in the shop which are either red or green is 32.

As students become familiar with thinking through a process for solving word problems, they will become more comfortable in working even multistep problems. WHAC gives students a tangible place to start a problem, replacing frustrated guessing with a deliberate course of action. Through greater confidence, students are empowered to do their best.

Wreck of the Hesperus

When an emigrant ship, bearing the ill-fated name Hesperus, was wrecked off Turtle Island, the authorities voted to donate $1,000 to the survivors. This sum was divided among the men and women who were eligible, and their 20 children, in accordance with an agreed scale. Each child received the same number of dollars; each woman received six times as much as a child; each man, $5 more than each woman. There were exactly twice as many men as women. The Hesperus carried 118 passengers. How many survivors were there?

The X and Y Files

Kelly Paul

The use of themes in teaching mathematics has the potential to increase students' interest in mathematical content. It is always a challenge for teachers to find connections that students find motivating. I believe that linking mathematical content to a well-known person, a popular TV show, a celebrated movie or a familiar phrase can have a powerful effect.

I am a student from the University of Alberta completing my preservice education. In my mathematics methods course we have been discussing the use of themes to engage students' interest. I used a theme for the Systems of Equations unit in Math 10 that is based on the popular TV show *The X Files*. Because this unit involved solving systems of linear equations in two unknowns, I modified the title to "The X and Y Files." Here are some of the examples I created which incorporate the theme into a lesson dealing with the substitution and elimination method for solving linear systems.

Example 1. Supernatural Beings have landed on Earth and have entered a virus into the U.S. government computer system, but in doing so they left a number trail behind them. Mulder and Scully are trying to use the number trail to find the latitude and longitude of the Beings' position. They know that the sum of two of the numbers is 176, and the difference of the two numbers is 48. By solving by substitution, can you help Mulder and Scully find the two numbers? (Hint: let x be the latitude and y be the longitude.)

Example 2. Mulder and Scully leave their office in separate cars on their way to the scene of an unusual event. Mulder travels on a path given by $3(x + 2y) = 48$. Scully's path is $(1/4)(x - 3y) = (1/2)x - 5$. Will they meet somewhere along the way? At what point?

Example 3. Mulder and Scully have information contacts in Halifax. They both call their own contact. Scully talks for 12 minutes and it costs her \$12.24. Mulder talks for 5 minutes and it costs him \$5.52. To find the cost k, for the first 3 minutes and the charge n for each minute after that, use these equations:

$k + 9n = 12.24$
$k + 2n = 5.52$

Find the basic charge and the charge per minute of Mulder and Scully's long-distance company.

Example 4. Scully has \$3.70 in nickels and quarters. She has 4 more nickels than quarters. Mulder wants to borrow \$2.50 in quarters. Does Scully have enough quarters for him? How many quarters and nickels does she have?

Example 5. Mulder is 4 times as old as his secret daughter. Five years ago he was 7 times as old as his secret daughter. Let x be Mulder's age and y be his secret daughter's age. What are their ages right now?

There are endless possibilities in creating themes such as this one. As new movies, TV shows and personalities become popular, new themes can be added to an already vast creativity bank. Another theme that I have used stems from the recently re-released *Star Wars* movie. For the Math 13 unit of Coordinate Geometry, I used this theme and titled it "May the slope be with you." A simple stroke of the pen can transform a common example into one which fits with the theme.

> If Han Solo is traveling on a path represented by the equation $y = 3/4x + 2$ and Darth Vader is traveling on a path that passes through the points (5,1) and (9,4), are their paths going to cross?

Star Wars could also serve as the basis for a theme for the Math 20 Power unit that involves exponents. Here the slogan could be, "May the power be with you."

The key to using a theme is to find a topic that interests students at whatever level you are teaching. Using the popular movie *Star Wars* for a theme caused interesting discussions in my Math 13 class. Students became involved in writing the examples with me by volunteering characters and then the mathematics. I have students who are coming up to me in the hallways asking me if I've seen the latest re-released movie of the trilogy and asking me to use a certain character in my examples the next day. When I first used some of these examples in my Math 13 class, there were cheers or groans, the students were making connections, adding meaning and facilitating recall, but most importantly, they were having fun.

Two Facets of the Linear Regression Process

David R. Duncan and Bonnie H. Litwiller

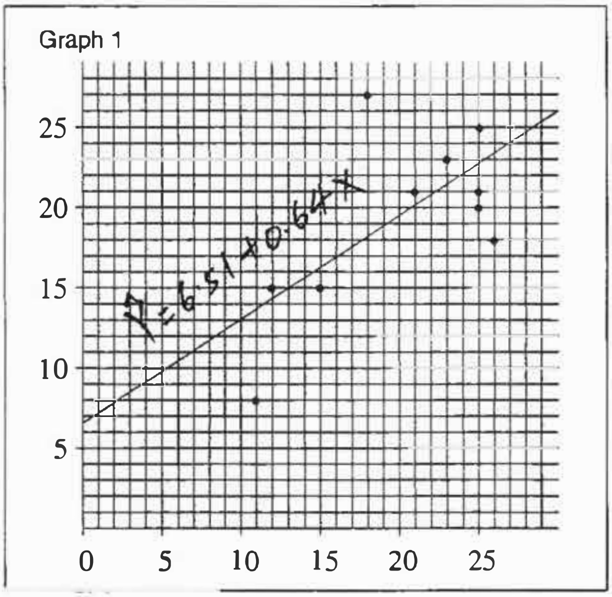
Mathematics teachers are always looking for ways to incorporate statistical concepts into their classrooms. Since calculators can readily produce the equation of the "least-squares" regression line for a data sample, this topic is sometimes introduced in secondary school mathematics.

We will show two examples that illustrate facets of the regression process that are appropriate for the secondary school level. Since regression involves graphing, these examples also connect to algebra concepts.

Example 1

Consider the following set of 10 pairs of scores. Each pair represents a different student in a class. For a given student, the x-value is the score on test 1 and the y-value is the score on test 2. Set of scores = {(11,8), (12,15), (15,15), (18,27), (21,21), (23,23), (25,20), (25,21), (25,25), (26,18)}. The built-in statistical capability of the TI-85 (or some other calculator) reports that the regression line has the equation: $\hat{y} = 6.51 + 0.64x$. This line provides the "best" linear fit to the observed data.

Graph 1 displays the original 10 points and the regression line.



Have your students produce this graph with the given points and the equation cited above.

In what sense does this regression line provide the best line of fit for the data? The criterion used in statistics is that the best line of fit should minimize the sum of the squares of the deviations of each true y value from the $\hat{y}$ value predicted by the regression line.

Table 1 reports the 10 original (x,y) pairs, the predicted y-value ($\hat{y}$) for each point, the difference between the true y and the predicted y-value ($\hat{y}$) for each point, the difference between the true y and the predicted $\hat{y}$ $(y - \hat{y})$, and the square of these differences $(y - \hat{y})^2$.

The sum of the entries of the $(y - \hat{y})^2$ column is approximately 159. The theory of linear regression asserts that the sum of $(y - \hat{y})^2$ column is minimized when the regression line $\hat{y} = 6.51 + 0.64x$ is used. Any other regression line, even though it might appear to the eye to better approximate the data, will yield a larger $(y - \hat{y})^2$ sum and hence be less effective overall.

Table 1

x	y	$\hat{y} = 6.51 + 0.64x$	$(Y - \hat{y})$	$(Y - \hat{y})^2$
11	8	13.55	−5.55	30.8025
12	15	14.19	0.81	0.6561
15	15	16.11	−1.11	1.2321
18	27	18.03	8.97	80.4609
21	21	19.95	1.05	1.1025
23	23	21.23	1.77	3.1329
25	20	22.51	−2.51	6.3001
25	21	22.51	−1.51	2.2801
25	25	22.51	2.49	6.2001
26	18	23.15	−5.15	26.5225

Table 2
Using the Regression Line $\hat{Y} = X$:

X	Y	$\hat{Y} = X$	$(Y - \hat{Y})$	$(Y - \hat{Y})^2$
11	8	11	−3	9
12	15	12	3	9
15	15	15	0	0
18	27	18	9	81
21	21	21	0	0
23	23	23	0	0
25	20	25	−5	25
25	21	25	−4	16
25	25	25	0	0
26	18	26	−8	64
				204

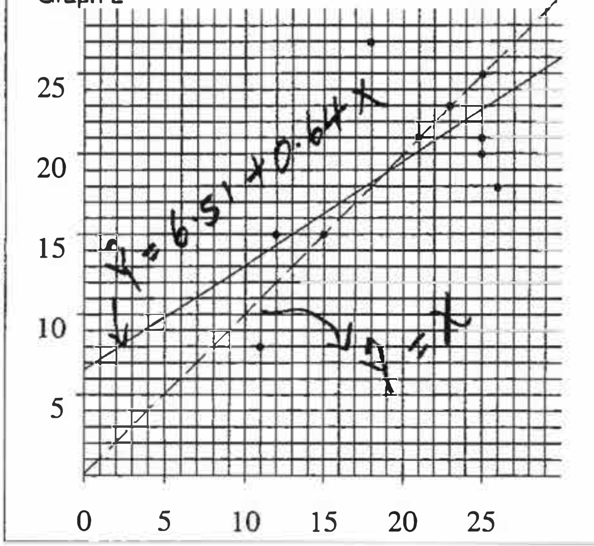
To test this assertion we note that 4 of the 10 points satisfy $y = x$. Consequently, one might suppose that the regression line $\hat{y} = x$ might better approximate the data than the line yielded by the calculator. Is this true?

Table 2 reports calculations similar to those of Table 1 for the alternative regression line.

Note that the $(y - \hat{y})^2$ column sums to 204; this is greater than the total of 159 of Table 1. Even though the alternative regression line has the advantage of predicting four y values exactly, in total, it does a worse job of fitting the data linearly than does the original regression line. Graph 2 displays both regression lines and the original data on the same axes.

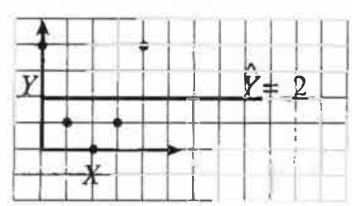
Is it obvious to your students that the solid graph fits the data better than the broken one? Opinions based on the appearance of Graph 2 must be tested by calculations as we have done in Tables 1 and 2.

Graph 2



Example 2

Consider the following small sample of paired scores: {(0,4), (1,1), (2,0), (3,1), (4,4)}. The TI-85 reports the linear regression equation to be $\hat{y} = 2 + 0x$ or $\hat{y} = 2$. These data and the regression line are depicted in Graph 3. The theoretical meaning of this horizontal regression line is that no linear pattern could be detected; the regression equation then predicts the mean value of y for each of the x values. Although one might say that this regression line does not predict the y values well, it does a better job by minimizing the sum of $(y - \hat{y})^2$ entries than would any other straight line. Graph 3 displays the data points and the regression line.



The striking facet of these data points is that they lie exactly on the parabola $y - (x - 2)^2$. The use of a parabola to predict the ys would have predicted perfect results. However, the linear regression process built into the TI-85 looks only for a straight line to fit the data; it is oblivious to any other possibility. This demonstrates the inappropriateness of rigidly using a procedure that may not fit a given situation.

Find other regression models in textbooks or in your calculator. On what assumptions are they built?

How Much Zooming Is Enough?

David E. Dobbs and John C. Peterson

The ability to use the "zoom in" and "zoom out" features of a graphing calculator enables us to focus on the important parts of a graph. As the examples given below show, the zooming features can be particularly useful when we are trying to solve equations or inequalities, as well as when trying to determine if two algebraic expressions define the same function. In viewing the graphs which arise from algebraic expressions, we need to know not only when to begin zooming but also when to stop. We can best make such practical decisions by using theory appropriately. To illustrate the procedure, we begin with five "zooming in" examples, and then give four "zooming out" examples.

Zooming In

Example 1 shows how zooming in on a "flat" part of a graph may reveal numerous x-intercepts and, thus, numerous zeros of the corresponding function.

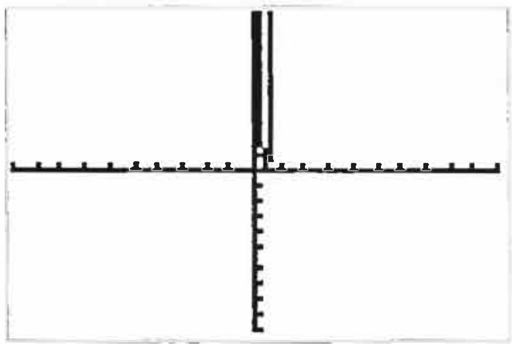
Example 1

Use a graphing calculator to solve the equation $1000x^4 - 1780x^3 + 1187.9x^2 - 352.262x + 39.1644 = 0$.

Solution

The real number solutions of the equation are the x-coordinates of the x-intercepts of the graph of $y = 1000x^4 - 1780x^3 + 1187.9x^2 - 352.262x + 39.1644$. Using the default setting of a TI-82 graphing calculator, we obtain the graph in Figure 1. How many times does this graph intersect the x-axis? The answer is not clear by inspecting Figure 1.

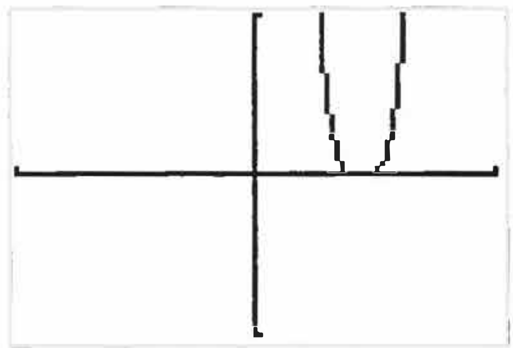
Figure 1



Xmin = −10	Xmax = 10	Xscl = 1
Ymin = −10	Ymax = 10	Yscl = 1

If you zoom in once (with both zoom factors set at 10), you find a graph like the one shown in Figure 2. Here, it looks as if there is a flat part of the graph which runs along an interval of the x-axis, but it is not clear whether the graph ever falls below the x-axis.

Figure 2

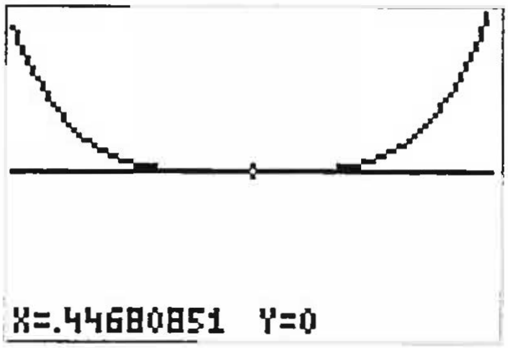


Xmin = −1	Xmax = 1	Xscl = 1
Ymin = −1	Ymax = 1	Yscl = 1

Tracing to a point near the middle of the flat part and then zooming in again, you get the graph in Figure 3. Using the trace feature again, you find a point on the graph with coordinates given by $x = 0.45106383$ and $y = -2.215 \times 10^{-6}$. Since this y-coordinate is negative, it seems as if the graph must cross the x-axis at least twice. (In fact it must, by applying the Intermediate Value Theorem for continuous functions.) Additional use of the trace feature on Figure 3 reveals enough sign changes of the y-values to indicate that the graph crosses the x-axis at least four times.

What would happen if you zoomed in and traced again? Would you find that the graph crosses the x-axis six (or possibly eight) times? How many x-intercepts are concealed within the "flat" parts of the graphs in Figures 1–3? How much zooming in is enough? The answer, in this case, depends on a theoretical result. According to one statement of the Fundamental Theorem of Algebra (Dobbs and Peterson 1993, 164–65), an nth degree polynomial has, counting multiplicities, exactly n complex roots. Thus, at least for this example, you have zoomed in enough. You have concluded that the given equation has four solutions and each of these is a real number. This can also be seen by inspecting Figure 4. (By the way, the

Figure 3



Xmin = −0.346808511 Xmax = 0.546808511 Xscl = 1
Ymin = −0.1 Ymax = 0.1 Yscl = 1

Figure 4

Xmin = −0.35 Xmax = 0.5 Xscl = 1
Ymin = −0.0001 Ymax = 0.0001 Yscl = 1

solutions are really rational numbers, namely $x = 0.43$, 0.44, 0.45 and 0.46.)

The method in Example 1 needs to be fine-tuned in case the underlying polynomial has a multiple root. Examples 2 and 3 show how the Factor Theorem, together with zooming in, deals with such situations.

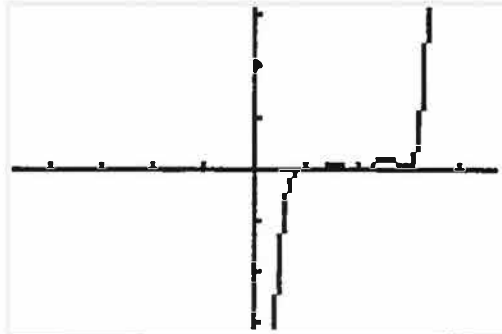
Example 2

Using a graphing calculator to solve the equation $x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996 = 0$.

Solution

As in Example 1, we first try to find all the real number solutions, by investigating the x-intercepts of the graph of $y = x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996$. An initial view of this graph is shown in Figure 5. As you can see, the graph seems to have flat parts on the x-axis near $x = 1$ and $x = 2$; also, it appears that the graph narrowly misses hitting the x-axis near $x = 3$.

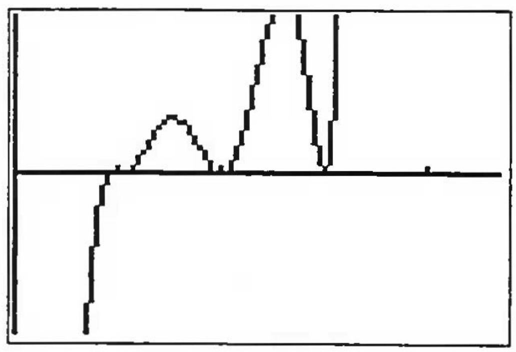
Figure 5



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = −3.1 Ymax = 3.1 Yscl = 1

By changing the window settings, you can obtain the graph in Figure 6. Although this view of the graph does not resolve the question, it does confirm our impression that we need to zoom in near $x = 1$, $x = 2$ and $x = 3$. By zooming repeatedly, you can check that there are solutions at $x = 1$, $x = 2$, $x = 2.99$ and $x = 3.01$. No matter how much more you zoom in (or out), you will not find any evidence of additional real number solutions.

Figure 6



Xmin = 0 Xmax = 4.7 Xscl = 1
Ymin = −0.2 Ymax = 0.2 Yscl = 1

According to the Fundamental Theorem of Algebra, the given seventh-degree polynomial has seven roots. If we have found all the real solutions (1, 2, 2.99, 3.01), you might suppose that three nonreal complex solutions remain to be found. However, according to the Conjugate Root Theorem (Dobbs and Peterson 1993, 168), nonreal complex zeros of a polynomial with real coefficients come in complete conjugate pairs, although 3 is not an even number! What's wrong?

In fact, there are no nonreal complex solutions in this example. We actually already have all seven solutions—all that is needed is a more careful application of the Fundamental Theorem of Algebra and the Linear Factor Theorem (Dobbs and Peterson 1993, 165). The point is that nth degree polynomials have exactly n roots if the roots are counted according to their multiplicities. In this example, 1 is a root with multiplicity three, 2 has multiplicity two, and 2.99 and 3.01 each have multiplicity one. Thus, we have found the even solutions, since $3 + 2 + 1 + 1 = 7$.

The above conclusions about multiplicity could possibly be conjectured by studying Figure 6, although the differences in behavior near $x = 2$ and near $x = 3$ may not be geometrically evident from that figure. But these differences can be determined algebraically by using the Factor Theorem, as follows. Let's consider $x = 1$. By substitution, you can check that $x = 1$ satisfied the given equation. So, by the Factor Theorem (Dobbs and Peterson 1993, 141), $x - 1$ is a factor of $x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996$. By division, you find the quotient, $x^6 - 12x^5 + 57.9999x^4 - 143.9994x^3 + 192.9987x^2 - 131.9988x + 35.9996$. Next, by substitution, you can check that $x = 1$ is a root of this sixth-degree polynomial. So, by the Factor Theorem, $x = 1$ is a root of the given seventh-degree polynomial of multiplicity at least two. Continuing in this way, you find that $x^7 - 13x^6 + 69.9999x^5 - 201.9993x^4 + 336.9981x^3 - 324.9975x^2 + 167.9984x - 35.9996 = (x - 1)^3(x^4 - 10x^3 + 36.9999x^2 - 59.9996x + 35.9996)$. By calculation, you can check that $x = 1$ is not a root of $x^4 - 10x^3 + 36.9999x^2 - 59.9996x + 35.9996$. Thus, the process of successive divisions stops, and 1 is indeed a root with multiplicity three. The other assertions are verified similarly.

Example 3

Use a graphing calculator to solve the equation $x^6 + 2.989x^5 + 2.96699x^4 + 0.96697x^3 - 0.01103x^2 - 0.00001x = 0$.

Solution

This example is somewhat similar to Example 2. It turns out that -1 is a root of multiplicity three, while -0.001, 0 and 0.01 are each roots with multiplicity one. Thus, the only solutions (real or complex) are $x = -1, -0.001, 0$ and 0.01.

In the next example, matters become somewhat complex.

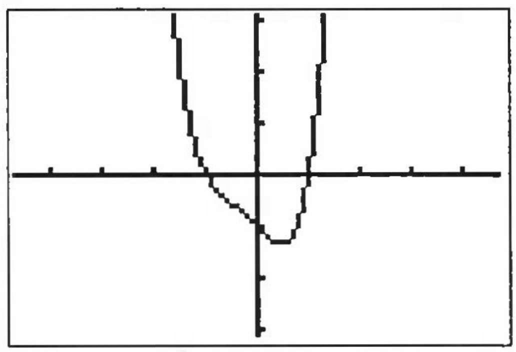
Example 4

Use a graphing calculator to solve the equation $x^4 + x^3 - x - 1 = 0$.

Solution

The graph of $y = x^4 + x^3 - x - 1$ in Figure 7 suggests that there are solutions near $x = -1$ and $x = 1$. By substitution, it is easy to verify that $x = -1$ and $x = 1$ are indeed solutions. Using division, as in Examples 2 and 3, you can check that $r_1 = -1$ and $r_2 = 1$ are each roots of multiplicity one. So, by the Fundamental Theorem of Algebra, two roots are still missing. The two missing roots—let's call them r_3 and r_4—satisfy $x^4 + x^3 - x - 1 = (x + 1)(x - 1)(x - r_3)(x - r_4)$, according to the Linear Factor Theorem. By division, r_3 and r_4 are the roots of $\frac{x^4 + x^3 - x - 1}{(x + 1)(x - 1)} = x^2 + x + 1$.
Hence, by the Quadratic Formula, r_3 and r_4 are given by $\frac{-1 \pm \sqrt{3}i}{2}$. No amount of inspection of the graph in Figure 7 would reveal these nonreal complex solutions. In summary, the solution set of the given equation is $\left\{-1, 1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}\right\}$.

Figure 7



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = −3.1 Ymax = 3.1 Yscl = 1

The analysis in Examples 1–4 was possible in part because each of the functions being graphed was a polynomial. As Example 5 shows, "zooming in" can be applied to solve equations involving nonpolynomial functions, even though the algebraic theory of such functions is more complicated than that of polynomials.

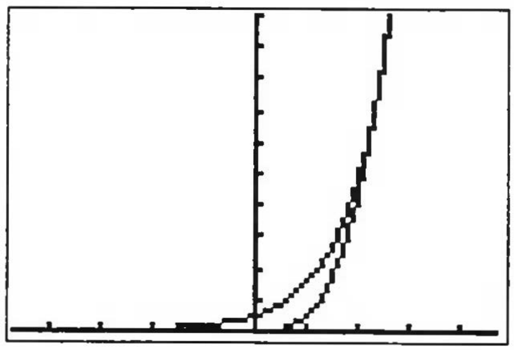
Example 5

Using a graphing calculator to find the real number solutions of the equation $x^\pi = \pi^x$.

Solution

The solutions that we seek are the x-coordinates of the points of intersection of the graphs of $y = x^\pi$ and $y = \pi^x$. As you can see from Figure 8, these graphs do not intersect at any point satisfying $0 \le x \le 2$, but they seem to be coincident for the x-values from slightly greater than 2 to at least 3. Are they really coincident or do the curves intersect at only some of the points indicated in Figure 8? How many points of intersection are there?

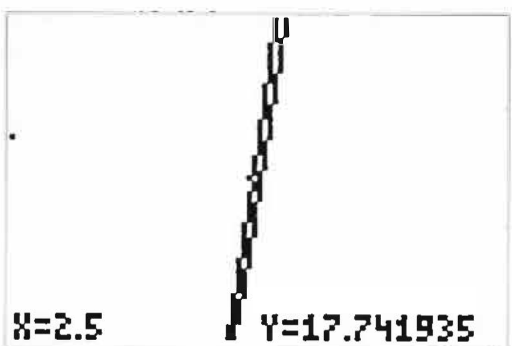
Figure 8



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = 0 Ymax =20 Yscl = 2

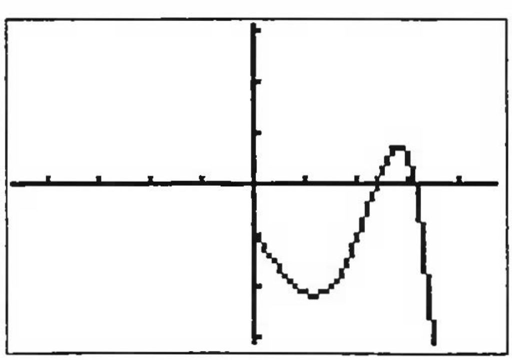
One way to proceed would be to "zoom in" on these graphs near $x = 2.5$, as shown in Figure 9. By continuing to zoom in, you find a point of intersection at $x \approx 2.3821791$. In addition, it is clear that there is another intersection point at (π, π^π). Additional zooming does not reveal any further solutions.

Figure 9



Xmin = 2.03 Xmax = 2.97 Xscl = 1
Ymin = 16.74193548 Ymax = 18.74193548 Yscl = 2

Figure 10



Xmin = −4.7 Xmax = 4.7 Xscl = 1
Ymin = −3.1 Ymax = 3.1 Yscl = 1

There is another way to proceed which is more like the method used in Examples 1–4. The solutions of the equation $x^\pi = \pi^x$ are the same as the solutions of $x^\pi - \pi^x = 0$. The graph of $y = x^\pi - \pi^x$ in Figure 10 indicates two x-intercepts, corresponding to the two solutions which were found above.

It is natural to ask if the equation $x^\pi = \pi^x$ has more than the two solutions found above. The answer is "no." The reason depends on calculus and is part of some interesting history recounted in (Sved 1990).

Zooming Out

An ultimate type of intersection of two graphs occurs when they are coincident. This corresponds to equality of the functions being graphed. A currently popular method of verifying identities, especially trigonometric identities, is to check coincidence of the graphs of the left- and right-hand sides of an alleged identity. As you saw in Example 5, the apparent coincidence of portions of graphs in a figure generated by a graphing calculator may disappear when you take a closer look by zooming. Similarly, as Examples 6–8 show, zooming out can be used to distinguish between functions whose graphs may appear to be coincident when using a particular viewing window.

Example 6

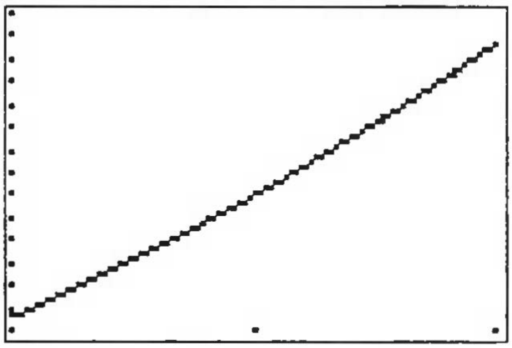
Suppose you view the graphs of $y = x^3$ and $y = 3x^2 - 2.99x + 0.99$ on a graphing calculator with window setting Xmin = 0.9, Xmax = 1.1, Xscl = 0.1, Ymin = 0.7, Ymax = 1.4 and Yscl = 0.05. Based on these graphs, would you conjecture that the functions f and g, given by $f(x) = x^3$ and $g(x) = 3x^2 - 2.99x + 0.99$, are equal? If so, zoom out to see if the new graphic evidence reinforces or disproves your conjecture. If possible, give a theoretical explanation for your new conclusion.

Solution

Figure 11 shows the graphs of f and g when viewed with the given window setting. To the naked eye, it appears that these graphs are coincident, and so one might conjecture on the basis of this evidence that $f = g$. However, if you zoom out, you obtain the graphs shown in Figure 12. Here, it is clear that the graphs of f and g are distinct, and so $f \neq g$.

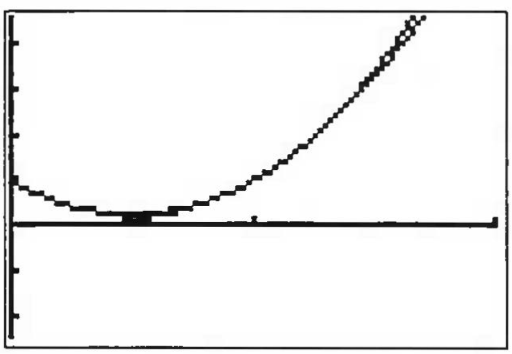
The same conclusion can be reached theoretically in a couple of ways. First, since the function $h = f - g$ is a third-degree polynomial, the Fundamental Theorem of Algebra tells us that h has at most three zeros. In particular, h is not identically zero, and so $f \neq g$. Second, f and g are unequal because they have different limits as $x \to -\infty$. Indeed, by the Leading Term Test (Dobbs and Peterson 1993, 152), $\lim_{x \to -\infty} f(x) = -\infty$ but $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} 3x^2 = \infty$.

Figure 11



Xmin = –0.9 Xmax = 1.1 Xscl = 0.1
Ymin = 0.7 Ymax = 1.4 Yscl = 0.05

Figure 12



Xmin = 0 Xmax = 2 Xscl = 0.1
Ymin = –2.45 Ymax = 4.55 Yscl = 0.05

Example 7

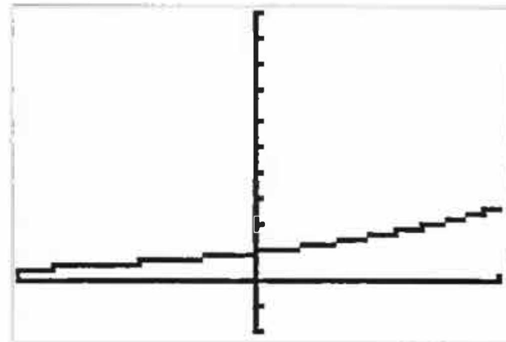
Follow the instructions of Example 6, for the functions f and g given by $f(x) = e^x$ and $g(x) = {}^1/_6 x^3 + {}^1/_2 x^2 + x + 1$ with the initial window setting Xmin = –1, Xmax = 1, Xscl = 1, Ymin = –2, Ymax = 10 and Yscl = 1.

Solution

Figure 13 shows the graph of f and g when viewed with the given window setting. To the naked eye, it appears that these graphs are coincident, and so one might conjecture on the basis of this evidence that $f = g$. However, if you zoom out, you obtain the graphs shown in Figure 14. Here, it is clear that the graphs of f and g are distinct, and so $f \neq g$.

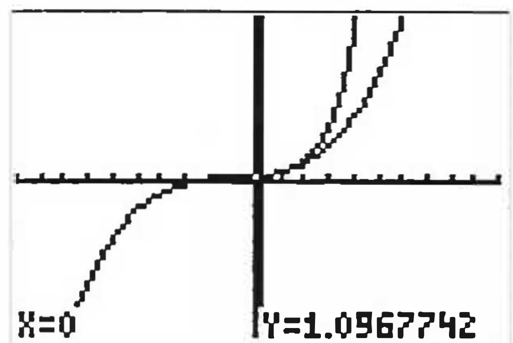
The same conclusion can be reached theoretically. Indeed, f and g are unequal because they have different limits as $x \to -\infty$. Of course, $\lim_{x \to -\infty} f(x) = 0$. However, by the Leading Term Test, $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} {}^1/_6 x^3 = -\infty$.

Figure 13



Xmin = –1 Xmax = 1 Xscl = 1
Ymin = –2 Ymax = 10 Yscl = 1

Figure 14



Xmin = –10 Xmax = 10 Xscl = 1
Ymin = –56 Ymax = 64 Yscl = 1

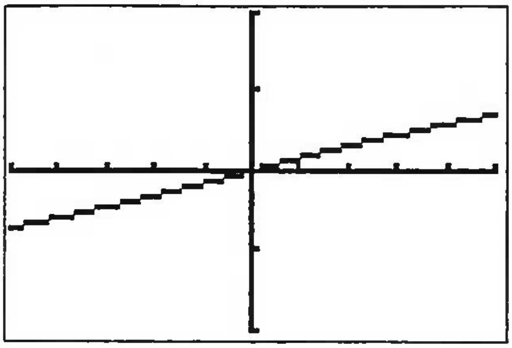
Example 8

Follow the instructions of Example 6, for the functions f and g given by $f(x) = \sin x$ and $g(x) = -{}^1/_6 x^3 + x$ with the initial window setting Xmin = $-\pi/4$, Xmax = $\pi/4$, Xscl = $\pi/20$, Ymin = –2, Ymax = 2 and Yscl = 1.

Solution

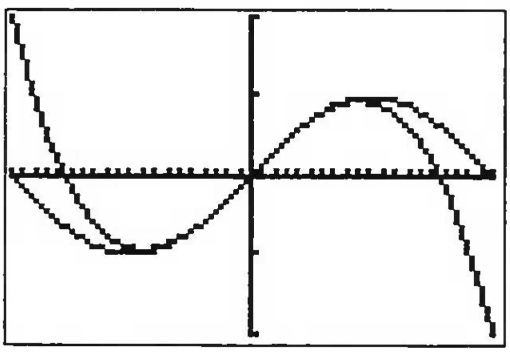
Figure 15 shows the graphs of f and g when viewed with the given window setting. To the naked eye, it appears that these graphs are coincident, and so one might conjecture on the basis of this evidence that $f = g$. However, if you zoom out by setting Xmin = $-\pi$, Xmax = π, and not changing the other settings, you obtain the graphs shown in Figure 16. Here, it is clear that the graphs of f and g are distinct, and so $f \neq g$.

Figure 15



Xmin = $-\pi/4$	Xmax = $\pi/4$	Xscl = $\pi/20$
Ymin = −2	Ymax = 2	Yscl =1

Figure 16



Xmin = $-\pi$	Xmax = π	Xscl = $\pi/20$
Ymin = −2	Ymax = 2	Yscl =1

The same conclusion can be reached theoretically in a couple of ways. For instance, you can check that *f* and *g* have different limit behavior at $-\infty$ (or at ∞). Alternatively, *f* and *g* are unequal because they have different sets of zeros. Indeed, *f* has infinitely many zeros, while the Fundamental Theorem of Algebra tells us that *g* has at most three zeros.

In the final example, we see how entire intervals can be misinterpreted when using a graphing calculator to solve an inequality. The remedy, once again, involves zooming out.

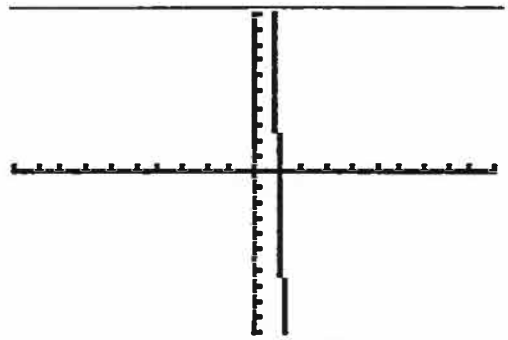
Example 9

Use a graphing calculator to solve the inequality $0.1x^3 - 3.4x^2 - 50.7x + 54 < 0$.

Solution

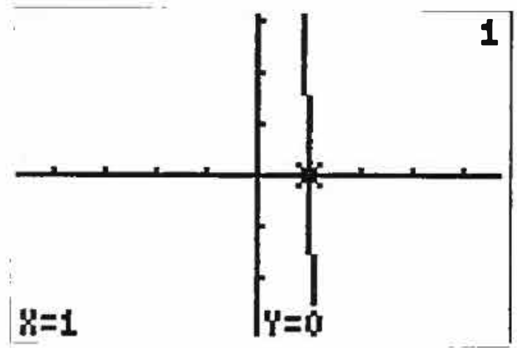
Using the default setting, we find the graph of the equation $y = 0.1x^3 - 3.4x^2 - 50.7x + 54$ shown in Figure 17. By changing the range settings to those indicated in Figure 18, and tracing, we see that this graph has an x-intercept at $x = 1$. Thus, since the solution of the inequality arises from the portion of the graph that lies below the x-axis, it appears from Figure 18 that the solution is the interval $(1,\infty)$.

Figure 17



Xmin = −10	Xmax =10	Xscl = 1
Ymin = −10	Ymax =10	Yscl = 1

Figure 18



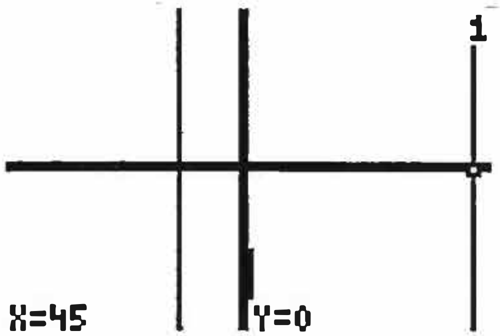
Xmin = −4.7	Xmax = 4.7	Xscl = 1
Ymin = −3.1	Ymax = 3.1	Yscl =1

Is this the correct solution? Let's zoom out. By tracing on the resulting graph, as shown in Figure 19, you see that the graph intersects the x-axis a second time (at $x = -12$) and a third time at $x = 45$. So, now it seems as if the solution set is $(-\infty,-12)\cup(1,45)$. Is *this* the solution?

Will additional zooming out show that the graph turns around yet again? How much zooming out is enough? How many times can you expect the graph to turn? In general, first derivative information from calculus is needed to analyze turning points. In fact, in this example, you have zoomed out enough. The solution set is $(-\infty,-12)\cup(1,45)$.

In closing, it should be noted that both zooming in and zooming out procedures are often needed in analyzing one example. For instance, to solve the inequality $0.01x^3 - 1.0400x^2 + 2.0949999x - 1.019898 > 0$, one needs to zoom in near $x = 1$ to identify the zeros

Figure 18



Xmin = −46 Xmax = 48 Xscl = 1
Ymin = −31 Ymax = 31 Yscl =1

at $x = 0.99$ and $x = 1.01$, while one needs to zoom out to detect the zero at $x = 102$.

References

Dobbs, D. E., and J. C. Peterson. *Precalculus.* Dubuque, Iowa: Wm. C. Brown, 1993.

Sved, M. "On the Rational Solutions of $x^y = y^x$." *Mathematics Magazine* 63 (February 1990): 30–33.

Given the equations
$7x + 5y - z = 8$ and $y + z = 11$, find all the ordered natural number triplets which satisfy the two equations.

Analysis of Teaching Trigonometry in the Context of University Mathematics

Natali Hritonenko

Trigonometry is a unit of study in the mathematics curriculum that is rich with content and concepts, contains clear interrelationships among its parts and provides us with important mathematical applications. Unfortunately, high school students conceive of trigonometry, not as a unified whole in which content and concepts are logically connected within the overall structure, but rather as a set of formulas that are difficult to memorize. They usually do not like trigonometry. Why? Because students do not understand it sufficiently. This lack of understanding is often evident among college or university students as they attempt to solve problems involving even the simplest applications of trigonometric concepts.

This article is an attempt to find some answers to these questions and apparent problems by analyzing the trigonometry units of the Alberta high school curriculum. Particular attention will be paid to the content of the high school curriculum that is essential in the study of calculus at postsecondary institutions.

Trigonometry is studied in Grades 10, 11 and 12. The basic textbooks for this unit of study, written by B. Kelly, B. Alexander and P. Atkinson, are [1] *Mathematics 10* (1987) (Chapter 14), [2] *Mathematics 11* (1990) (Chapter 10) and [3] *Mathematics 12* (1991) (Chapters 5 and 6), respectively. These textbooks are well written with some good examples and interesting applications of trigonometric functions. Despite that fact, students seem to experience difficulties with trigonometry.

The areas that seem to be most prevalent with respect to students experiencing difficulties are, in order of increasing difficulty, treated below.

Basic Formulas

It is interesting to note that students seem to know the reciprocal and quotient identities studied in the Section 10.2 of [2] and Sections 5.4 and 6.1 of [3]. However, if we want to further enhance the students' understanding of trigonometry, a discussion of the following questions would be of considerable benefit:

1. Can sine of positive x be negative?
2. Can sine squared of negative x be negative?
3. Can sine cube of negative x be negative?
4. Can sine of x squared be negative?
5. Can sine of x cube be negative?
6. Can sine of x be equal to negative cosine of the same x?
7. Can sine of x be equal to cosine of the same x?
8. Can sine squared of x cube be equal to the square of 2 (that is, 4)?
9. Can sine of x be equal to cosecants of the same x?
10. Can sine x be equal to 0.5?
11. Which is greater, sine squared of x or sine cube of x?
12. Which is greater, secant squared of x or secant cube of x?

These same questions should also be considered by substituting other trigonometric functions.

Students are also experiencing difficulties with other trigonometric identities. In particular, the Pythagorean identities (see Section 6.1 of [3]):

$$\sin^2 A + \cos^2 A = 1,\quad 1 + \tan^2 A = \sec^2 A,\quad 1 + \cot^2 A = \csc^2 A, \tag{1}$$

The double-angle identities (see Section 6.4 of [3]):

$$\sin 2A = 2 \sin A \cos A,\quad \cos 2A = \cos^2 A - \sin^2 A, \tag{2}$$

and, especially the following derivations from (1) and (2):

$$\sin^2 A = ½ (1 - \cos 2A),\ \cos^2 A = ½ (1 + \cos 2A), \tag{3}$$

are difficult to do for students.

However, these formulas are basic in mathematics, and they are frequently used in different applications. Later on in their learning of mathematics and calculus in particular, students experience difficulties applying trigonometric formulas which they have studied in high school. Let us consider several examples:

- Find $\sin(\cos^{-1} 4/5)$ (No.13 in the Section 6.6, the Chapter "Inverse functions" of [4]).

The difficulty here consists in expressing cosine through sine using formula (1) above.

- Find $\lim_{x \to 0} \frac{1 - \cos x}{2x^2}$ (No. 44 in the Section 2.4, the Chapter "Derivatives" of [4]).

- Evaluate integrals $\int_0^{\pi/2} \sin^2 3x dx$ and $\int \cos^4 x dx$ (No.1 and 3 in the Section 7.2, the Chapter "Techniques of integration" of [4]).

Students need to know how to express $\sin^2 x$ through $\cos 2x$, using the formulas (1) and (2) above.

- Evaluate the integral $\int \sin^5 x \cos^5 x dx$ (No.11 in the Section 7.2 of [4])

Students need to know that formulas (2) are used several times.

To assist students with memorizing formulas (1)–(3) above and to enhance their understanding, it might be useful to ask students to derive the contents of Table 1 below. Doing a few exercises on a homework assignment is not sufficient to gain a good understanding of the interconnection of trigonometric functions.

Being able to choose the right sign for each function in each quadrant is the objective of Section 5.3 of [3] and Section 10.7 of [2]. With the exception of a small error in example 2 of Section 10.7 in [2], both sections are well done.

As students derive the function shown in Table 1, it is important to direct students' attention to choosing the sign near the root once again. Also a review of the questions 1–12 would provide students with additional help to choose the sign of the function properly.

By way of reviewing identities, the students' understanding could be further strengthened by developing another set of trigonometric formulas through $\tan(x/2)$ (so-called universal or Weierstrass substitution):

$$\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}, \quad \cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}, \quad \tan x = \frac{2 \tan x/2}{1 + \tan^2 x/2}. \tag{4}$$

These formulas are frequently used in sections of calculus [4], particularly in the chapter "Techniques of integration."

- Evaluate the integral $\int \frac{1}{3 \sin x + 4 \cos x} dx$ (No. 25 in Section 7.5 of [4]).

Table 1. **Interconnections Among Trigonometric Functions**

	sin	cos	tan	cot	sec	csc
sin		$\pm\sqrt{1-\cos^2 x}$	$\pm\frac{\tan x}{\sqrt{1+\tan^2 x}}$	$\pm\frac{1}{\sqrt{1+\cot^2 x}}$	$\pm\frac{\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
cos	$\pm\sqrt{1-\sin^2 x}$		$\pm\frac{1}{\sqrt{1+\tan^2 x}}$	$\pm\frac{\cot x}{\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\pm\frac{\sqrt{\csc^2 x-1}}{\csc x}$
tan	$\pm\frac{\sin x}{\sqrt{1-\sin^2 x}}$	$\pm\frac{\sqrt{1-\cos^2 x}}{\cos x}$		$\frac{1}{\cot x}$	$\pm\sqrt{\sec^2 x-1}$	$\pm\frac{1}{\sqrt{\csc^2 x-1}}$
cot	$\pm\frac{\sqrt{1-\sin^2 x}}{\sin x}$	$\pm\frac{\cos x}{\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$		$\pm\frac{1}{\sqrt{\sec^2 x-1}}$	$\pm\sqrt{\csc^2 x-1}$
sec	$\pm\frac{1}{\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\pm\sqrt{1+\tan^2 x}$	$\pm\frac{\sqrt{1+\cot^2 x}}{\cot x}$		$\pm\frac{\csc x}{\sqrt{\csc^2 x-1}}$
csc	$\frac{1}{\sin x}$	$\pm\frac{1}{\sqrt{1-\cos^2 x}}$	$\pm\frac{\sqrt{1+\tan^2 x}}{\tan x}$	$\pm\sqrt{1+\cot^2 x}$	$\pm\frac{\sec x}{\sqrt{\sec^2 x-1}}$	

- Prove the formula $\int \sec x dx = \ln \left|\frac{1 + \tan {}^x/_2}{1 - \tan {}^x/_2}\right| + C$ (No. 31.A in section 7.5 of [4]).

  At this point, it is also important that students practise by using the identities (1)–(4) identified above with angles different from A or $2A$, such as nA or $n^A/_2$. Not to engage in such practice would result in students not knowing that $\sin^2 5A + \cos^2 5A = 1$.

Related Angle and Other Identities

The next set of student problems is connected with the evaluation of concrete values of trigonometric functions. Calculation of trigonometric functions of concrete (special or general) angles is used in various applications. Students do not seem to remember the trigonometric ratios of special angles they had studied before in degrees in Section 10.5 of [2] and in radians in Section 5.5 of [3]. Using the related angle, cofunction and odd-even identities (Section 6.2 of [3]) provides only seven problems for students to practise. Moreover, the definition of odd-even function is not clear and should be written in general, not just using the properties of powers. It is also worthwhile for students to complete Table 2 below.

Table 2

Argument	Function			
x	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$-x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$
${}^\pi/_2 + x$	$\cos x$	$-\sin x$	$-\cot x$	$-\tan x$
${}^\pi/_2 - x$	$\cos x$	$\sin x$	$\cot x$	$\tan x$
$\pi + x$	$-\sin x$	$-\cos x$	$\tan x$	$\cot x$
$\pi - x$	$\sin x$	$-\cos x$	$-\tan x$	$-\cot x$
$3\,{}^\pi/_2 + x$	$-\cos x$	$\sin x$	$-\cot x$	$-\tan x$
$3\,{}^\pi/_2 - x$	$-\cos x$	$-\sin x$	$\cot x$	$\tan x$
$2p + x$	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$2p - x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$

Also answering questions 1–12 stated in the "Basic Formulas" section above and applying the rules which follow help students immensely in choosing the sign for each trigonometric function. The rule that joins together all identities mentioned in this section is very simple and useful to students. It is stated as follows:

1. Add or subtract 2π until the angle is between 0 and 2π, because the function and the sign stay the same if one adds to or subtracts from the angle 2π times any natural number.
2. Do not alter the function if the angle added to or subtracted from is 0 or π. Change the function into its cofunction if the angle is added to or subtracted from ${}^\pi/_2$ or ${}^{3\pi}/_2$.
3. The sign depends on the quadrant of the initial angle.

At the end of the chapter, students should engage in solving several exercises similar to the ones shown below. (We mark them as "✧" to distinguish them from university-level exercises)

✧ Verify the equality:

$2\sin^2(3\pi-2A)\cos^2(5\pi+2A) = {}^1/_4 - 0.25\sin({}^5/_2\pi - 8A)$.

✧ Check (fun exercises):

$\cos^2 x = 1 - \sin^2 x$,
$(\cos^2 x)^{3/2} = (1 - \sin^2 x)^{3/2}$,
$\cos^3 x + 3 = (1 - \sin^2 x)^{3/2} + 3$,
$(\cos^3 x + 3)^2 = ((1 - \sin^2 x)^{3/2} + 3)^2$ at $x = 180°$ and
$2^2 = 4^2$.

✧ Find (without a table) the values of the expressions when $x = {}^\pi/_8$:
$\sin^4 x - \cos^4 x$,
$\sin^6 x - \sin^6({}^\pi/_2 - x)$,
$\sin^6 x + \cos^6(x + 4\pi)$.

These exercises have the following goal in mind: on the one hand, they represent a review of the "difference of squares" (Section 3.9 of [1]) or the "sum of difference of cubes" (Section 3.6 of [1]), and, on the other hand, they repeat and group together all basic trigonometric identities.

Advanced Formulas

Here we dwell on the sum and difference identities (Section 6.3 of [3]):

$$\begin{aligned} &\sin(A - B) = \sin A\cos B - \cos A\sin B,\\ &\sin(A + B) = \sin A\cos B + \cos A\sin B, \qquad (5)\\ &\cos(A - B) = \cos A\cos B + \sin A\sin B \text{ and}\\ &\cos(A + B) = \cos A\cos B - \sin A\sin B. \end{aligned}$$

It would be reasonable to derive their evident consequence

$$\begin{aligned} &\sin A\cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)],\\ &\sin A\sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)] \text{ and} \qquad (6)\\ &\cos A\cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]. \end{aligned}$$

As in all previous sections of this text, there is a lack of practice here. If students had more practice, they would not only come to memorize these formulas but to understand them as well. Also, taking the derivative of trigonometric functions or using them to solve problems on calculus would further enhance their understanding. Failure to provide students with extensive practice opportunities would result in students forgetting this content in a few days.

Trigonometric Equations

Only Section 6.5 of textbook [3] is devoted to general solutions of trigonometric equations. Inverse trigonometric functions and trigonometric inequalities are not covered. It is, therefore, natural that first-year university students are not able to construct the range of cos $x > ½$ or similar problems. The treatment of this important matter is totally inadequate. Knowing how to solve trigonometric equations and inequalities helps students to better understand trigonometric functions in general, and it refreshes their knowledge of graphs and graphing of trigonometric functions (Chapter 5 of [3]).

The four examples demonstrated in this section merely show the solution of the equation, but provide no general algorithm or formula. Only example 2 of Section 6.5 of [3] provides experience for students which involves a solution step where $\sin x = 2$, with no solution. It is important that problems of this nature be drawn to the students' attention. Perhaps by examining a graph similar to Figure 1, it will become evident that there are no real values of x satisfying the equation $\sin x = a$ (and $\cos x = a$) when $a > 1$. This can also be confirmed by sketching the graphs of functions $y = a$ and $y = \sin x$ (see Figure 1).

Given that students studied this graph in some detail in Chapter 5 of [3] and that the students' ability to find solutions to trigonometric equations using trigonometric tables or calculators is well developed, it would not be difficult to derive the general solution for trigonometric equations and corresponding trigonometric inequalities.

Being able to find the general solution of the simplest trigonometric equations, such as $\sin x = a$ and $\cos x = a$ for $a = 0$, $a = 1$ and $a = -1$ is extremely important. Incomplete or limited understanding will present difficulties to students in solving many problems in university calculus even simple connected trigonometric equations. The following examples illustrate this point:

- Sketching the graph of the function $y = x + \sin x$ (Exercise 34, Section 3.6 of [4])

To find intervals where the function is increasing or decreasing, everyone takes the derivative of the function, but then finds it difficult to solve the equation $\cos x + 1 = 0$. Finding intervals of concavity requires finding the general solution of the second trigonometric equation $-\sin x = 0$.

- Sketching the graph of the function $y = \dfrac{\cos x}{2 + \sin x}$ (Exercise 36, Section 3.6 of [4])

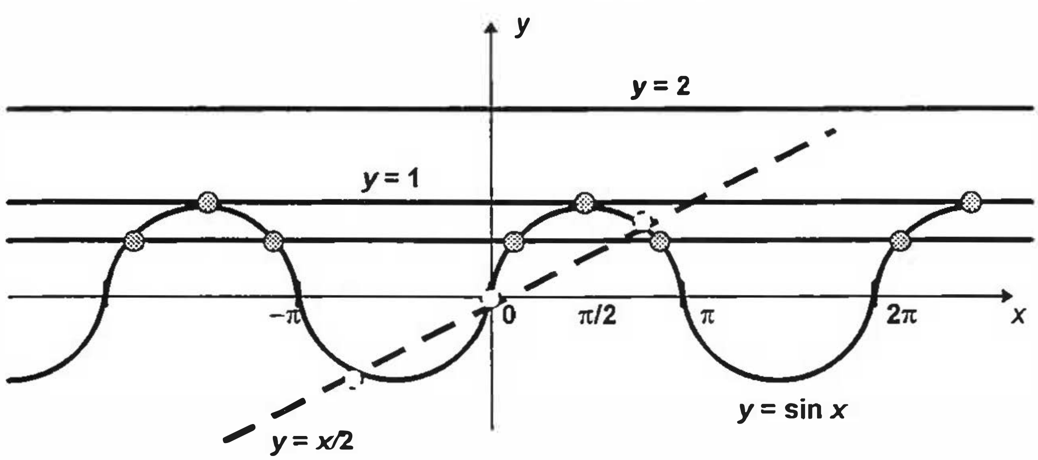
The problem is similar to the one above, but the equation is more complex. Before students solve this equation, they have to simplify it by using basic trigonometric formulas. Moreover, to define the domain of this function, some students try to solve the equation $2 + \sin x = 0$.

- Find the area between the curve $y = \sin x$, $y = \cos 2x$, $x = 0$, $x = \pi/4$ (Exercise 24, Section 5.1 of [4])

To find the intersection points, it is necessary to solve the equation $\sin x = \cos 2x$.

- Find the volume generated by rotation of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$ about x-axis (Exercise 28, Section 5.2 of [4])

Figure 1. **Graphic Method for Solving Trigonometric Equations**



This problem has a similar degree of difficulties.

A quick review of the following questions helps students in gaining a better understanding of equations:

13. Solve the equation $\sec x = 0.8$.
14. How many solutions are there for equations $\cos x = 0.5$ and $\tan x = 2$ on the interval $[9\pi, 11\pi]$?
15. How many times does the graph of $\tan x$ intersect the line $x = {}^{3\pi}/_{2}$?
16. Find a when $y = \cot(x + a)$ intersects the line $x = \pi$?
17. Sketch the graph $y = \sin 2$.
18. Does the equation $\sin x \cos x = \sin\alpha$ have a solution at $\alpha = 20°$ and $\alpha = 80°$?
19. Find solution of the equation $\cos^2 2x - \sin^2 2x = \cos\alpha$ where $\alpha = 20°$, $\alpha = 80°$.

Instead of stating "a trigonometric equation *usually has* infinitely many roots" (p. 289 of [3]), it is probably more appropriate to state, "equations *can have*," because the opposite statement—"no roots"—as in equation $\cos x = 2$ or a "finite number of roots"—as in $\sin x = {}^{x}/_{2}$, refer equally to the family of trigonometric equations. For more explanation, see Figure 1.

On page 290 of [3], it is stated that "no general methods exist for solving trigonometric equations." Many students probably find this statement intimidating. It would have been helpful to add that a graphic method as shown in Figure 1 (that is, drawing graphs of the left-hand and right-hand side of equations) can help students to analyze the existence and number of roots, as well as determine an approximate evaluation. In fact, this graphic method is being used following the above-noted statement, hence a reference to it would have reinforced the graphic method as a legitimate method.

Construction of graphs of trigonometric functions and their variations received much attention in seven sections of Chapter 5 of [3], whereas trigonometric identities and equations, which are very important, were only dealt with in five sections. In addition, sketching graphs and transformation of relations were already studied in some detail in Chapters 6 and 7 of textbook [2]. It would be reasonable to increase the study of trigonometric identities and equations at the expense of reducing graphing.

Although the treatment of trigonometry in textbooks [1] to [3] is comprehensive at the high school level, the main impediment to mastery is a lack of practice. How can students possibly memorize or understand these formulas when each section provides limited opportunities for practice and each section often contains more formulas than exercises? It is, therefore, not surprising when students get lost among all these trigonometric formulas and identities. Student practice, solving more complex problems and opportunities for reviewing the concepts are absolutely essential if students are to grasp trigonometry in general and identities and equations, specifically. In closing, the following examples contain all basic trigonometric formulas:

- ✧ Simplify: $\cos^2(\alpha + 2\beta) + \sin^2(\alpha - 2\beta) - 1$.
- ✧ Factor: $\sin 6x - 2\sqrt{3}\cos^2 3x + \sqrt{3}$.
- ✧ Verify the equality: $\dfrac{\cos 63^0 \cos 3^0 - \cos 87^0 \cos 27^0}{\cos 132^0 \cos 72^0 - \cos 42^0 \cos 18^0} = -\tan 24^0$.
- ✧ Simplify: $2 - \dfrac{\sin 8\beta}{\sin^4 2\beta - \cos^4 2\beta}$.
- ✧ Verify the equality: $\dfrac{\sin({}^{\pi}/_{2} + 30)}{1 - \sin(30 - \pi)} = \cot({}^{5}/_{4}\pi + {}^{3}/_{2}0)$.

From this brief review, it is evident that the high school mathematics program is filled with interesting and challenging problems that connect with the real world.

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Diophantine Analysis and Linear Indeterminate Problems

Sandra M. Pulver

Greek mathematics is regarded worldwide for its geometric character and has gained fame in this field. However, during the late Alexandrian period, about A.D. 250, when Greek science and philosophy were on the decline as a whole, and with them mathematics, algebra began to emerge as the main topic of interest.

Not much is known about the life of Diophantus except that he died at the age of 84 and had a son who died during his middle years. This too is not certain for it was provided rather cleverly by a rhymed problem that appeared in a later collection of Greek puzzles. The known titles of the works of Diophantus are the *Arithmetics* in 13 books, the *Porisms* and a study on polygonal numbers. The *Porisms* have been lost and only part of the *Polygonal Numbers* exists. However, six or seven books of the *Arithmetics* have been preserved, and it is through them that Diophantus makes his contribution to and mark on the world of mathematics.

In the theory of Diophantine analysis, two closely related problems are treated. In the first $f(x, y, z, \ldots)$ is a given polynomial in the variables $x, y, z, \ldots$ with rational (usually integral) coefficients. The equation $f(x, y, z, \ldots) = 0$ is called a Diophantine equation when it has to be determined which rational numbers $x, y, z, \ldots$ satisfy it. Usually further restrictions are made by requiring that $x, y, z, \ldots$ be integers, and sometimes it is required that they consist of positive integers. If we have several functions $f_i(x, y, z \ldots)$, in number less than the number of variables, then the set of equations $f_i(x, y, z \ldots) = 0$ is called a Diophantine system of equations. The term Diophantine became the name for such analysis because many of the problems in the *Arithmetics* call for a solution in rational numbers. Diophantus looked for rational solutions; that is, he did not insist on having a solution in integers as is customary in most of the recent work in Diophantine analysis.

Diophantus usually dealt with problems in which one had to find a set of 2, 3 or 4 numbers such that different equations involving them in the first, second and third degrees are squares, cubes and so on. The simplest nonlinear Diophantine equation may have no solution, any finite number of an infinity of solutions. For example, $x^2 + y^2 + 1 = 0$ has no rational solution and $x^2 + y^2 - 1 = 0$ has infinite number of rational solutions but a finite number of integral ones which are trivial.

Linear indeterminate problems are ones that occur commonly in puzzles. They lead to one or more linear equations where the number of unknowns is greater than the number of equations. If there were no restraints on the kind of values the solutions could take, one could give arbitrary values to some of the variables and find the others in terms of them. Because of the nature of these problems, the solutions are limited to integers and usually positive ones so they are called linear Diophantine equations. But even with these limitations, there may be none, several or even an infinite number of solutions. Solving these equations involves a number of repeated reductions.

The first type of equation is a single linear one—$ax + by = c$—in two unknowns. The following is a trivial example: $x + 5y = 14$, which may be written $x = 14 - 5y$.

This shows that any integral value of y substituted above will give an integral value for x. If it is required to have positive solutions, then $y > 0$ and $x = 14 - 5y > 0$ and $y < {}^{14}/_{5}$. Thus $y = 1, 2$ and $x = 9, 4$, respectively.

So when one of the coefficients of x and y is one, the solution is immediate. Thus, the method for solving linear indeterminate equations is to reduce them to this simple form.

It is not certain if indeterminate problems originated within a single culture, but if they did, it seems likely that India should be considered as a source. The following appears in Mahaviracarya's *Ganita-Sara-Sangraha* which was probably composed around A.D. 850.

> In the forest 37 heaps of apples were seen by the travelers. After 17 fruits were removed the remainder was divided evenly among 79 persons. What is the share obtained by each?

If x is the number of fruits in each heap, and y the share obtained by each person, then $37x - 17 = 79y$.

Since x has the smaller coefficient, we solve for x, and by taking out the integral parts of the fractional coefficients, we obtain $x = \frac{17 + 79y}{37} = 2y + 1 + \frac{5y - 20}{37}$.

Because x and y are integers, the quotient

$$t = \frac{5y - 20}{37}$$

is integral. Now we have to find integers y and t such that $37t = 5y - 20$. This equation is of the same type as $37x - 17 = 79y$ but with smaller numbers. This equation can be further simplified because both $5y$ and 20 are divisible by 5 and $37t$ must also have this factor. Since 37 is prime to 5, t must be divisible by 5 and we write $t = 5z$ and when this is substituted in $37t = 5y - 20$ we can cancel by 5 and have the simpler equation $37z = y - 4$. This gives $y = 37z + 4$, $x = 79z + 9$ as the general solution. The problem however will only allow positive integers. So,

$$37z + 4 > 0,\ 79z + 9 > 0$$
$$z > -{}^4/_{37},\ z > -{}^9/_{79}.$$

This shows that all values $z = 0, 1, 2, \ldots$ will give positive solutions in $y = 37z + 4$ and $x = 79z + 9$. This problem illustrates the fact that even when the solutions are required to be positive, there may be an infinite number of solutions. It also shows how simplifications are available in the solution of indeterminate problems.

Often the number of equations is one less than the number of unknowns. The procedure is to eliminate some of the unknowns until one winds up with a single equation with two unknowns which is the case above.

In medieval times, problems of this kind were called "problems coeci" probably referring to the fact that they related to scenarios in which people paid bills, as in the following problem from Christoff Rudolff in 1526.

At an inn, a party of 20 persons pay a bill for 20 groschen. The party consists of men (x), and women (y) and maidens (z), each man paying 3, each woman 2 and each maiden ½ groschen. How was the party composed? The equations are $x + y + z = 20$, $3x + 2y + {}^z/_2 = 20$. We double the second equation and subtract the first from it to obtain $5x + 3y = 20$ or $3y = 20 - 5x$. Once again we simplify (by substituting $y = 5u$) to obtain:

$$3u = 4 - x$$
$$x = 4 - 3u,\ y = 5u,\ z = 16 - 2u.$$

For a positive solution

$$x = 4 - 3u > 0,\ u < {}^4/_3$$
$$y = 5u > 0,\ u > 0$$
$$z = 16 - 2u > 0,\ u < 8.$$

This provides a unique solution in which $u = 1$ and $x = 1$, $y = 5$ and $z = 14$.

Then there are those problems in which the number of unknowns is at least two greater than the number of equations. In this case also, one can eliminate some of the unknowns and end up with a single equation with several unknowns. For example, there may be two equations and four unknowns and one of them may be eliminated to obtain a single equation with three unknowns. However, in the case of one equation with three unknowns, the general solution will contain two parameters instead of one as in the previous problems.

Diophantus worked extensively with the Pythagorean theorem trying to find right triangles with integral sides. However, one doesn't have to be restricted to integers because if any rational solution had been found, the three numbers could be written with a common denominator

$$a = \frac{a_i}{m},\ b = \frac{b_i}{m},\ c = \frac{c_i}{m}$$

and $a_i^2 + b_i^2 = c_i^2$ would be an integral solution.

It is enough to find primitive integral solutions of the Pythagorean equation. (Primitive solutions are those in which there is no factor, d, common to a, b and c because if there were, then the equation could be canceled by d^2.) In order for a primitive solution to exist, any two of the numbers a, b and c must be relatively prime. If a and b had a common factor x, both sides of the Pythagorean equation would be divisible by x^2. But then c is divisible by x which contradicts the assumption that the solution was primitive.

It will be determined that in a primitive solution a, b and c, the numbers a and b can't both be odd. This is so because of the following theorem:

> The square of a number is either divisible by 4 or leaves a remainder of 1 when divided by 4.

This is proven by the fact that every number is of the form $2n$ or $2n + 1$ and when they are squared, the results are $4n^2$ and $4n^2 + 4n + 1$. If a and b were both odd, both sides of the Pythagorean equation, $a^2 + b^2 = c^2$ would leave the remainder 2 when divided by 4 which contradicts the theorem.

Now, a will be even and b and c are odd since there are no common factors. Then the equation is $a^2 = c^2 - b^2 = (c + b)(c - b)$. Both sides are divisible by 4 since a is of the form $2n$ and when one divides by this factor, one gets

$$\frac{a^2}{2} = \left(\frac{c + b}{2}\right) \times \left(\frac{c - b}{2}\right)$$

The two factors on the right are relatively prime because any common factor, d, would divide their sum and difference. But since

$$\frac{c + b}{2} + \frac{c - b}{2} = c \qquad \frac{c + b}{2} - \frac{c - b}{2} = b$$

and b and c are relatively prime, d must equal 1.

When the two numbers on the right in

$$\frac{a^2}{2} = \left(\frac{c+b}{2}\right) \times \left(\frac{c-b}{2}\right)$$

are relatively prime, their prime factorizations are different and their products can't be a square unless each of them is a square. So,

$$\frac{c+b}{2} = u^2 \quad \frac{c+b}{2} = v^2$$

and by substituting above in we get

$$\frac{a^2}{2} = \left(\frac{c+b}{2}\right) \times \left(\frac{c-b}{2}\right)$$

$a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$.

To verify that this is a primitive solution, we see that any common factor of b and c has to divide their sum and difference. However, $c + b = 2u^2$, $c - b = 2v^2$ and since u and v are relatively prime, 2 is the only common factor which is eliminated when one of the numbers is odd and the other even.

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In the equations $a + b + c = d + e + f = g + h + i$, is it possible to substitute the natural numbers 1, 2, 3, . . . , 8, 9 in the place of the variables?

Students Creating Stories in Math Classes

Florence Glanfield

To: jste@somserver.ab.ca
From: glanfiel@gpu.srv.ualberta.ca
Subject: School Mathematics Program Review
Dear Jean:

As I sit down to write this e-mail message, I can't help but tell you how the story that I am going to share with you keeps being replayed in my mind. It has become a part of my lived experience and history—it has become a part of my life.

Last November I had the chance to spend five days in a K–9 school as part of a school mathematics program review. During these five days, I talked with many wonderful teachers and students, and this story emerged from a conversation I had with the Grade 7 math teacher.

During that particular week, the Grade 7 math classes were studying decimals and fractions, and the teacher and I were discussing ways that we could have students think about how decimals and fractions are used in "real life." We decided to ask the students to write a story about what it would be like to spend a day without decimals and fractions in their world.

The next day this idea was shared with students. I was lucky enough to be in one of the Grade 7 classes as the idea was broached—and I was blown away at how taken the students were. Hands shot up and students moved toward the board as the teacher recorded the types of things that would be affected if decimals and fractions did not exist. They enthusiastically offered their ideas—picture a group of about 20 Grade 7 students sitting on the floor around the teacher, eagerly wanting their ideas to be put on the board. "Eagerly" is probably not even a good word to use to describe the action that was happening in that room—students kept talking about their ideas, and they physically kept moving with excitement. When I saw this, I was reminded of being in a crowd at the end of a bicycle race. One of the sponsors was giving away free products by standing on the stage and tossing the products into the crowd—the crowd would shift and disperse in the excitement of possibly getting one.

In this Grade 7 classroom, no idea was rejected; they were all written on the board. The following list was generated from the class discussion:

- Grocery store—groceries, money
- Time
- Transportation—gas, speed, bus fare, mileage, odometer, roads, lanes, no cars
- Tests—percentages, no marks, anecdotal, no math
- Business—graphs
- Food—sharing
- Athletics—no time keeping, no distance running, penalties
- Weather
- Cooking
- Entertainment—movies, radio, TV stations
- Age
- Construction
- Calendar
- Mass
- Criminal penalties
- No science
- No numbers at all because all numbers can be written as decimals

At one point in the activity a student turned to me and said, "Our whole world would be affected—my father is a doctor, he wouldn't have a job." As I think about that statement, I realize that telling this story on paper misses the multidimensional thinking and interacting that was happening in that classroom when the list was being generated. The list only became a short script of the conversations that students were having with one another and with themselves as they were thinking about the implications of not having decimals or fractions. I was amazed at how quickly the 40-minute period passed. The students were asked to write their story for homework to be read in class the next day.

So what happened the next day? All students completed their homework and were ready to share their story—even the student whose first language is not English. I have only selected two stories to share with you, I hope you enjoy reading them.

Story 1

Today I arrived on Galorp. This planet does not have a decimal and fractions system! It is most peculiar, and confusing. To explain how life goes on without this, I give you a day in the life of the Galorplings. Without further ado, here it is.

The alien wakes up, and chooses whether the nine on its clock means it is late or early for work; they don't understand 9:30! The alien then gets into its car thing, (very primitive) and goes. When I—in my role as the child—asked how fast we were traveling they were unable to figure out the answer. They bought gas and were ripped off because they cannot compute without the decimal system. I then assumed my role as the young school child. In terms of marks all they could tell me was how many were right and wrong. When I asked the teacher what my percentage was, she asked if I was feeling okay, and sent me to the nurse. These Galorplings sure are strange! It must be uncouth to ask to share food, for when I did I received this evil stare. After I arrived back at our family shelter (another grueling trip in the car), I found their entertainment unbearably limiting. They cannot pick up even half the 620 channels we have, because theirs are all in whole numbers. These Galorpians do not even know how old they are, all they know is they turn from one whole to another when the weather warms up. If one asks what is in between, one is told not to be so impudent. The science here is only to the level of grade schools, and the university students have problems with that. I conclude that this underdeveloped civilization poses no threat to Saroah.

Reporting,
"Anglark Saremp"

Story 2

If there were no fractions or decimals in the world there would be many problems. For example if we were at the grocery store and we wanted half a melon we couldn't get it. Also there would be no cents. You need a decimal to show $6.38. Transportation would also be affected. You wouldn't have any gas, because you need to see a sign that says 53.9 cents per litre, but you wouldn't be able to because there is no decimal. You would not know the time at a sporting event because time wouldn't exist. There would be no distance running because you couldn't keep track of time or mileage.

No criminal punishment could be enforced because you couldn't send people to jail unless they were sentenced there for a set amount of time. Most importantly there would be poor medical care because doctors and nurses couldn't prescribe a certain drug. Also, in an emergency situation a doctor would inject too much or too little of a life-saving drug to a patient because he or she would not be able to know the proper amount. In conclusion, the world would pretty much be a wreck without fractions and decimals.

I selected these two stories because of the difference between them—the first student created a "fictional" space to write his story, and the second student wrote about the world that he currently knows and what it would look like.

This experience has caused me to think about three ways in which this type of activity would "fit" into our new curriculum. First, think about this activity as being used to help students achieve the goals of a mathematics education as stated within the *Alberta Program of Studies for K–9 Mathematics* (Alberta Education 1996). These goals indicate that students should appreciate and value mathematics (p. 2), that "students should also gain an understanding and appreciation of the contributions of mathematics, as a science and as an art, to civilization and to culture" (p. 3), and that students should "contribute to mathematical discussions, exhibit curiosity, and show some enjoyment of mathematical experiences" (p. 3).

The second way is through the Nature of Mathematics Dimension of Number. Within this dimension, the curriculum document indicates that number sense includes "a development of an appreciation of the need for numbers beyond whole numbers" (Alberta Education 1996). Although this activity limited the numbers to fractions and decimals, all students in the class, through their participation in talking about and writing about fractions and decimals, developed an appreciation for their role in our culture.

The third way is the obvious connection to the mathematical processes of Communication and Connections. This activity has students using a form of communication to think about and represent connections among the symbolic notations of decimals and fractions with their daily experiences (Alberta Education 1996, 6–7).

I'm left wondering what else I could learn about these students' mathematical thinking by using activities such as this one. Not only that, how do these students' stories relate to my life as someone who thinks about and appreciates the role of mathematics in the development of our culture. Barbara Kingsolver (1996, 256), in her most recent book *High Tide in Tucson*, says that once the reader and viewer have read and viewed a story that "the story no longer belongs to the author," the story has become a part of the reader's and viewer's life.

What do you think?

I look forward to the next time we "talk."

Florence

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One day, the British mathematician G. Hardy visited the Indian mathematician Srinivasa Ramanujan (1887–1920) with a taxi that was identified with the number 1729. "A very boring number," remarked Hardy. "Quite the opposite," replied Ramanujan. "It is a very interesting number in that it is the smallest number which can be expressed in two different ways as the sum of two cubed numbers." What are the two numbers and the two different ways?

UPCOMING CONFERENCES, NEWS RELEASES AND INFORMATION

The Alberta Advisory Committee for Educational Studies (AACES)

Reports of Research Funded by AACES

Teacher Development in Mathematical Problem-Solving

Purpose of Study

This study investigated an open-ended, experiential, problem-solving inservice program (PSI), framed in a constructivist perspective of learning, and its effects on teachers' thinking and teaching of mathematical problem-solving. Problem-solving was considered as a non-algorithmic process to solve non-routine mathematical problems, a process that requires high-level mathematical thinking.

Methodology

The participants were six elementary teachers (Grades 3–6) who disliked mathematical problem-solving. They volunteered to participate in the study because they were interested in improving their classroom processes (which they described as "traditional") when teaching mathematical problem-solving. The study was conducted as a descriptive, qualitative study. The teachers participated in a PSI program that directly and indirectly engaged them in problem-solving activities for 20 hours during their summer break. These activities included solving a variety of non-routine mathematical problems, but more important, the teachers were required to reflect on aspects of their experiences with problem-solving from different points of view: as students, as teachers and as laypeople. All PSI activities were tape-recorded and transcribed, and copies of all written work were collected. Participants were observed in their classrooms teaching problem-solving once before the PSI program and an average of three times each afterward. Each observation was followed by an in-depth, open-ended interview which was tape-recorded. The data were thoroughly examined to identify and compare patterns in the participants' thinking, attitudes and classroom behaviors prior to and following the PSI program.

Results

Prior to their participation in the PSI program, most of the teachers seldom taught problem-solving, but when they did, they focused on teaching algorithms based on key words and they guided students toward correct solutions. This process conflicted with problem-solving pedagogy that requires teachers to enable students to learn to think for themselves and to become independent problem-solvers. Following the PSI program, there were significant positive changes in the teachers' confidence in their ability to solve problems and in their teaching approaches. Positive changes were interpreted as a shift toward attitudes and classroom processes that reflected the recommendations of the National Council of Teachers of Mathematics professional standards for teaching mathematics.

Effects on Attitude

The most significant outcome, in terms of attitude, was the teachers' increased confidence in their own ability to solve both routine and non-routine problems within the context of their teaching. This shift in attitude seemed to have occurred as a result of a better understanding of problem-solving and of themselves as problem solvers. In general, the teachers' awareness resulting from this enhanced understanding seemed to free them from the traps of their past experiences and freed them to do things they thought were not allowed or valid when solving mathematics problems. It also shifted their view of problem-solving from a prescribed algorithmic process to an open-ended process in which the problem-solver had to be in control to interpret and solve the problem.

Effects on Teaching

The teachers were able to draw out particular aspects of the PSI experience to use as a basis of change in their teaching. As part of the PSI experience, they (collectively and individually) developed informal theories about teaching problem-solving, particularly with respect to teacher intervention in the students' processes. For example, collectively, they decided that teacher intervention to provide help should occur when students were stuck, off-track or lost. Although the nature of the intervention varied, the overriding principle was that the function of intervention should not be to tell students how to solve a problem, but to stimulate their thinking. The teachers implemented these theories in their classrooms in ways that enhanced their teaching of problem-solving, such as

- engaging students in problem-solving more often;
- being less dependent on a textbook;
- using more cooperative learning groups;
- having students share solutions and meanings;
- emphasizing process over final answer;
- listening more to students, focusing on the students' thinking behind "right" and "wrong" answers;
- using non-routine problems;
- considering alternative solutions; and
- asking non-leading questions and being more sensitive to the nature of intervention in the students' processes, that is, when and how they provided help.

Since the PSI program left it up to the teachers to determine how to change their teaching, each teacher decided on what was meaningful and important in a particular situation and when and how to integrate the new knowledge into it. Teachers transformed the knowledge obtained from the PSI experience to practical classroom processes that best suited each individual situation.

All of the teachers pointed out that after participation in the PSI program, they found teaching more challenging, but more interesting and rewarding, particularly because they were learning a lot from the students' thinking and because of the positive effects on their students' learning of problem-solving. They also found that the way in which they were teaching other areas of mathematics was being influenced by their problem-solving approaches.

Olive Chapman
Department of Curriculum and Instruction
Faculty of Education
The University of Calgary
2500 University Drive NW
Calgary T2N 1N4

Variables Affecting Successful Implementation of a Framework for Teaching Mathematics with Meaning

Purpose of Study

Since initial work by Sigurdson and Olson (1992), several studies by the authors have focused on a classroom teaching framework for junior and senior high school mathematics. Although the use of the Framework for Interactive Teaching (FIT) results in achievement gains for the students, teachers report variable success in their efforts to implement this teaching approach. This study set out to look at how teachers were failing in the implementation and what teacher and classroom factors accounted for these failures. Research questions addressed were as follows:

1. What implementation difficulties do teachers encounter?
2. What teaching and classroom factors relate to its successful implementation?

Methodology

Two Grade 8 mathematics teachers implemented two units each—Geometry and Data Management. A research assistant working with the teachers developed the units according to the (FIT) principles. The general principles of FIT include daily homework, adequate preparation for homework, an emphasis on the meaning of mathematics and generally organizing the lesson so that a maximum amount of time is spent on learning mathematics. Both teachers were observed a minimum of six times each.

Results

Both teachers found it difficult to adhere to the FIT principles. A categorization of the difficulties identified three areas: lesson structure, making mathematics meaningful and teaching interactively.

Lesson Structure

The structure of the FIT lesson designates specific time in the lesson for student work and lesson development. To make this possible, time for other activities such as correcting homework must be severely curtailed.

With regard to lesson structure, the following implementation concerns were identified:

1. Limiting homework correction to two to three minutes is problematic. Teachers feel that students often have good questions during homework that should be discussed.

2. Discussions of mathematical meaning too easily expand to fill more than the "half of the lesson" time they are allotted.
3. The two above items indicate that the extra time spent in opening and development meant less time for seatwork and homework.

The FIT model makes teachers very conscious of time in a lesson. Related to structure is the tendency for some teachers to incorporate seatwork time into the development. This is now taken to be a useful modification of FIT. The complexity of the classroom lesson is a major factor in implementation. However, a key feature of the framework is that more learning takes place when a teacher actively tries to accommodate for these complexities.

Many of the above findings relate to a teacher's philosophy of teaching. Individual student needs inappropriately focus classroom time and energy on non-productive teaching activities. Most teachers however are very attuned to student needs and find it difficult to avoid these interruptions. Teachers often prefer to teach in a more casual, less focused manner, giving the students more time for individuals to work alone or in groups. The research shows that, over the long term, a high involvement of teachers in the lesson leads to higher student achievement.

Mathematical Meaning

Two concerns about adding meaning in the lessons stand out. Although the inservice attempted to give possibilities for meaning, teachers miss many opportunities for meaningful discussion. In these lessons, the meaning was supposed to "fit into" mathematics learning. In today's terminology, meaning is for making connections. Adding meaning is a subtle teacher activity. It may simply be an advanced "art form." Our teachers had not had enough practice at teaching with meaning.

The second general concern with meaning is that teachers tend to treat it as "discovery" learning. This relates to its "time-consuming" nature in lessons. The distinction between discovery learning and learning with meaning needs to be fully explored by teachers.

Interaction

The FIT model emphasizes teacher-student and student-student interaction. Teachers found it easy to interact over the homework corrections but not over the actual lesson. Students play an important role in interaction and perhaps more attention has to be paid to them learning how to interact. Interaction does not consist of casual comments but focused observations. Students often focused too closely on the mathematical procedures.

Reference

Sigurdson, S. E., and A. T. Olson. "Teaching Mathematics with Meaning." *Journal of Mathematical Behavior* 11 (1992): 37–56.

A. Craig Loewen
Associate Professor
Faculty of Education
The University of Lethbridge
Lethbridge T1K 3M4

AACES Grant Guidelines

1. AACES funds are quite restricted, totaling about $20,000 a year, hence its support of educational research activities is restricted both in the size of the grant that may be provided ($5,000 is the normal ceiling) and in the number of grants that may be allotted during any one year.
2. Grant applications will be assessed by AACES with particular consideration given to the variables of relevance to the preparation of teachers, elementary and secondary education, size of the project and anticipated allocation of funds requested relative to committee guidelines.
3. Nine copies of the grant application must be received in the office of the AACES secretariat prior to 4:30 p.m. on the published deadline date. The Ethics Review Committee Report and/or Central Office Approval Report must accompany the grant application at the time of submission. Faxed copies will not be accepted.
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8. If a proposal is judged worthy of support by AACES the following matters are relevant to the allocation finally awarded:
(a) Often supported in whole or in part: research assistants' salaries; telephone and postage

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(b) Not supported: honoraria for principal investigator or investigators; capital expenses (for example, filing cabinets, tape recorders, sets of books); computer time fees; computer software; expenses associated with conference attendance for travel costs inside the researcher's community. At the discretion of the committee, allowances for travel costs outside the researcher's community may be granted.

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10. An ethics review committee approval report from the institution of the principal investigator or a central office approval report must accompany the submission to AACES.
11. Projects must be initiated and some portion of the grant must be expended within one year of receipt of the grant.
12. Projects are to be completed within two years; however, the Grants Committee may, upon application, grant an extension of not more than one year.
13. A copy of the final report including an abstract of not more than 500 words must be submitted within six months of the termination date of the project. If the final report is not received within the specified time, AACES will not consider further proposals from the principal investigator for a period of three (3) years. The appropriate dean of the faculty of education or school jurisdiction superintendent will be informed in writing that the final report was not received prior to the deadline.
14. These reports may be circulated by AACES to participating organizations and interested parties.
15. Any grant applicant who has received an AACES grant but failed to meet all obligations within two years of the termination date for that grant is automatically excluded from consideration for further allocations.
16. Any published articles resulting from research funded by AACES should be forwarded to the office of the AACES secretariat.
17. Apply to Doreen Link at The Alberta Teachers' Association, Southern Alberta Regional Office (SARO), 540 12 Avenue SW Suite 200, Calgary T2R 0H4; phone 265-2672 in Calgary or 1-800-332-1280 from elsewhere in Alberta; fax 266-6190.

The sum of two natural numbers is 90. The sum of 25 percent of the first addend and 75 percent of the second addend is exactly 30. What are the two numbers?

Improving Mathematics Achievement by Effectively Integrating Technology

Barbara Morrison

For almost two years now, I have been working with authors from across Canada to develop computer resources to match the content and intent of *The Common Curriculum Framework for K–12 Mathematics* (Alberta Education 1995). Our collective focus has been on the instructional approach and design and the implications for regular classroom use. We have concentrated our initial efforts on the Version 1.0 materials—"getting the instruction" right!

Once we had completed the writing process for the first grade level, my attention turned toward a massive classroom pilot study of the edited and programmed prepublication materials. The critical question we needed to answer was "Will using the computer materials improve student learning in math?" The expense in development costs was now approaching $2.5 million. This is a reasonable question to ask! I will share the results in the pages which follow.

Before I get ahead of myself, I will set the context. (For those of you who have seen Computer Guided Learning (CGL) demonstrations recently, you might want to skim through the following descriptive paragraphs in Section 1.) In discussing the evolution, the testing of the materials, and implementation issues and potentialities of CGL use, the next question I would like to respond to is "What would teachers be most interested in hearing about?" I have divided the following discussion into the following three major sections which I hope will spark further interest.

Section 1. What Is CGL?

CGL refers to high-quality world-class CD-ROM materials supported by print resources for teaching and learning mathematics. The primary objective of the partnership agreement between the western provinces and territories with ITP Nelson Canada Publishing is to develop resources for Grades 9–12 that will improve student performance and retention of expected mathematics outcomes.

The Learning Equation Mathematics 9 is the first level of the CGL materials to be completed under the partnership. It will be available in both Mac and Windows for teachers to use with their students in the 1997–98 school year.

Designed to match *The Common Curriculum Framework for K–12 Mathematics* of expectations for what students know and are able to do in mathematics, CGL development was initiated to support the *Western Canadian Protocol for Collaboration in Basic Education in K–12 Mathematics* (1995, 1996).

CGL is

- correlated to the curriculum,
- fully interactive,
- user friendly, and
- visually appealing.

CGL is designed to put the learner in control of his or her learning by providing superior teaching and learning strategies to match the vision set forth in the NCTM Standards.

A Commonsense Approach

The pedagogical approach is research-based. It is a commonsense approach that works for teaching and learning math. The instructional components of each lesson incorporate real-world connections to make math more relevant and meaningful for students and build on students' prior knowledge. Each lesson begins with an attention grabber and motivator to address the question "Why do we need to learn this?" This introduction is revisited in the summary where students use the mathematics they have just learned to solve the introductory problem or application.

The tutorial and wealth of examples develop the mathematical concept and lead students to build their understanding. The approach is highly interactive and incorporates diagnostic feedback. It is presented in a logical sequence within a variety of meaningful learning activities for students at different ability levels.

The language of mathematics is reinforced throughout the lessons with features such as vocabulary checks, a built-in glossary and student-response models which incorporate phrases, symbols and math vocabulary.

The practice components include Jeopardy-type board exercises and categories of problems at varying levels of difficulty and complexity. Mini-test banks provide lots of extra practice to meet the individual needs of students.

CGL Is Courseware

A unique feature of this software is that it is actually *courseware*! CGL is designed to be sensitive to the amount of instructional time available for mathematics instruction in Grade 9 programs. Each of the 56 lessons is intended to average 90 minutes in length; however, that will vary according to the individual needs and abilities of students.

The Math 9 courseware includes seven Explorer lessons that allow students to explore and build their understanding of key mathematical concepts. The Explorers are a result of the creative genius of Ron Blond who is on secondment from his position as math department head with the Edmonton Public School Board.

A second unique feature is the Student Refresher that provides additional examples and exercises to support students when they are away from the computer. This resource contains extra practice for independent study, open-ended writing activities to check for understanding and non-routine problems to apply and extend learning. This part of the CGL package is intended to complement and enrich the integration of mathematical processes through such things as group-learning activities, creating problems and working with concrete materials.

Finally, the Teacher Print Support materials are designed to assist and support teachers in the transition to a more integrated technological approach in their mathematics classrooms.

The teachers resource contains brief descriptions of each lesson on the CD-ROM as well as teacher-intervention strategies and ideas for evaluation, remediation, extension and enrichment. These strategies include learning activities and projects for individuals and groups of students.

CGL Builds Student Confidence

Students enjoy working at their own pace. They can preview, review and repeat parts of lessons to meet their individual learning needs. They receive immediate feedback as they proceed through the lessons. The feedback varies according to the type of response solicited. Diagnostic feedback has been created by experienced classroom teachers and is often based on clarifying common errors or misconceptions. Generally, feedback provides a hint leading toward a correct response.

A "Help" button is provided in instances where authors feel it is appropriate or where some students may require review or additional support in order to proceed. "Success Tips" usually offer suggestions to build students' confidence or to support independent study and learning. Hidden screens serve as motivators. A hidden picture is revealed as students work through the "Examples and Practice and Problems" components. These screens may contain related information about the lesson, build an appreciation of mathematics, relate to a real-world application, a math historian or career or even present the option to solve a more challenging problem. The "Self-Check" components of each lesson present students with an opportunity to check their mastery and measure their understanding of the lesson outcome. Students have the opportunity to review the questions, retry incorrect ones and check their answers, and/or see a Sample Solution.

Section 2. What Do Teachers and Students "Do" When They Use CGL?

The Paradigm Shift

The use of CGL as the prime mode of delivery of instruction was expected to change the role of the teacher. I believe a critical part of this change involves a conscious effort on the part of teachers to "let go" of some more traditional ways of teaching math. Moving from the "sage on the stage" to the "guide on the side" is a common shift when students are working one-on-one or two-on-two on computers—the teacher's role changes to one of coach, facilitator and mentor!

Using CGL challenges our beliefs about learning. Here are just a few questions and ideas for your personal reflection:

- What does constructivism mean?
- When students are actively engaged in meaningful learning activities, their learning is enhanced.
- Physical interaction and the use of models to demonstrate understanding make a difference in learning and retention.
- Students learn at different rates and in different ways.
- Students require both group- and individual-learning activities.
- Learning is accelerated when motivation is high.
- Time-on-task is directly correlated to mathematics achievement.

Let's turn our attention to the equally important paradigm shift for students using CGL! Students are suddenly actively engaged, talking and helping one another, and interacting in a meaningful way to develop and build their understanding of concepts.

Grade 9 students for the most part do not have a great deal of experience in note-taking, let alone reflecting on and monitoring their progress and understanding. Simply speaking, they are used to doing what they are told when they are told to do it; they copy down what they are told and so on.

The effective use of CGL parallels planning for effective teaching and, most importantly, creates a positive climate and learning environment by helping the teacher.

- Plan and establish classroom routines and procedures.
- Establish and model clear expectations.

Ask yourself:

- What do I want students to write or record in their notebooks?
- How will I integrate homework and review it?
- How will I monitor the individual learning needs of students?
- What kinds of assessment make sense and are appropriate?

What Do Pilot Teachers and Students Say?

Although the classroom pilot experience provides some very good suggestions to help other teachers integrate the technology effectively, there is no magic formula that will work for everyone. Teachers will use the resource as they do any others, as part of an approach that will work best for their students and their own comfort levels. There is no one single, right way, but we can learn from the successful practices of others. Here are some comments collected by Theresa Gross, formerly of Calgary Catholic Schools, in qualitative data from classroom interviews in the pilot of *The Learning Equation CGL Math 9* materials.

- The program provides more depth to the concepts (than a textbook). It allows for an internalization of knowledge and greater understanding in the long run.
- Teachers describe their role as manager, facilitator, helper of students (one-to-one)—no longer the principle dispenser of knowledge.
- Students in a CGL class are more actively engaged (involved) in their learning.
- Students say: "It's good to be able to slow down or repeat material I didn't get" and "I like the computer. It explains the work well and I can go at my own speed."

From a former principal's perspective, Jim Beatt, Foothills School Division, observed classes using CGL and made the following comments, comparing them to his observations of regular classes:

- Students in the CGL classes were more focused than students in regular classes. That is, more time was spent on task.
- Students using computers seemed to be more actively engaged in the task presented to them. They worked for much longer periods of time.
- The (CGL) mathematics software involves several activities (listening, reading, manipulating and viewing). If the teacher insists upon student writing as well, the combination is powerful since one actively reinforces the other.
- Students become much more responsible for their own learning as the teacher becomes more of a facilitator.

Section 3—Why Would You Want to Use CGL?

If we truly want to promote excellence in mathematics education, we must consider adapting to change and using the best tools available. Technology plays a critical role in doing this. Using CGL allows teachers to move in this direction, while being confident they have a resource correlated to the curriculum they are mandated to teach. CGL resource materials, by their very nature, contain more breadth and depth than any other print or software resource available.

Just imagine the potential computers have to improve students' achievement in math when they

- are able to click on a word or math expression when they have forgotten what it means, and get a very specific explanation with an example;
- are asked to accurately input an expression using specific math symbols and notation;
- make a slight error and immediately receive appropriate feedback to move them toward the correct response or solution;
- have more than enough examples at all levels with lots of problem-solving;
- are able to see full solutions to problems to compare with their own work
- have different explanations for a concept, idea or problem;
- see alternative solutions or methods at the click of a button;
- interact in a highly visual environment that is user-controlled; and
- are able to use computer simulations or on-screen concrete materials to manipulate and observe their effects.

A CGL-enhanced program has definite advantages. Let's look at the potential of these features:

- The computer is infinitely patient.
- Learners can know how they are doing something, while they are doing it.

- It can be fun and easy to use.
- Learning at a computer is low risk. (Mistakes are a matter between the individual and the machine.)
- The computer can repeat things over and over without raising its voice!

Why Might Educators Invest in Courseware?

The members of the Western Canadian Protocol (WCP) are now wrestling with this question. Only one answer begins to provide a solution—*demonstrable improved student performance.* Achieving this objective has been the basis of this WCP joint initiative with ITP Nelson.

Does It Work?

ITP Nelson tested the material with over 1,000 students from 41 schools in a controlled pilot study, probably the largest study of its type ever undertaken with prepublication material in Canada. Results from an independent evaluation conducted under the direction of George Fitzsimmons of the University of Alberta show students using CGL 9 demonstrated statistically significant improvements in performance when compared with students using authorized texts. Students, both male and female, who used CGL as the main delivery system performed significantly higher on a pre/post test design at all levels of math ability.

What You Need Before You Start

First, you will need a willingness to change and embrace a new teaching style. Next, you will need access to the computer hardware and a supportive school environment. Inservice training will be a valuable way to prepare yourself and build a support network. A parent-orientation session or hands-on opportunity could also contribute significantly to the success of your implementation.

With most new things, it takes time to adjust and to find your own comfort level. CGL has many flexible uses. As a teacher, you determine how you will begin, what your students' needs are and what you can manage effectively. Start slowly, taking one step, one year at a time. You can use CGL for whole-class instruction over the entire Grade 9 program or start using parts of CGL for demonstration purposes, remediation, enrichment (when students miss class) and so on. Then try it out, reflect on how it went, modify it for next time and make revisions. Teamwork is a real bonus, too, if you can swing it!

Visit the CGL Web Site

Why not visit our Web site to order your free Mac or Windows demo CD-ROM at http://cgl.nelson.com? See for yourself how CGL is

- easy to use,
- a commonsense approach to math instruction and
- covers topics in-depth.

That's why it works!

Reference

Alberta Education. *The Common Curriculum Framework for K–12 Mathematics: Western Canadian Protocol for Collaboration in Basic Education.* Edmonton: Author, 1995.

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Barnett House
11010 142 Street NW
Edmonton AB T5N 2R1