The Pyramid Question: A Problem-Solving Adventure

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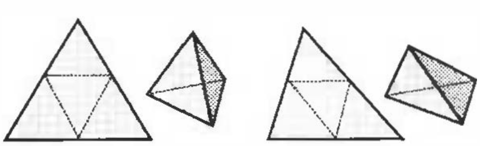
Good teachers know that a good question can launch a discovery journey through conjecture, research, serendipitous encounters, proof, answers and new questions. The importance of a problem-solving approach to investigating and understanding mathematics is underscored by the fourth grade-level standard in NCTM's *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989).

The purposes of this article are to share a question, report some discoveries and suggest ways to incorporate the adventure into classrooms.

The Question

Three-dimensional models for pyramids can be constructed from two-dimensional patterns. Figure 1 shows two patterns that produce pyramids with triangular bases when folds on the dashed segments bring matching sides of the triangles together. If the pattern is an equilateral triangle, then folds on its midlines, that is, segments connecting the midpoints of two sides, will yield a regular tetrahedron, a pyramid with four regular-triangle faces. Patterns made from nonequilateral triangles produce other types of tetrahedra.

Figure 1
Folds along the dashed segments of a triangle pattern produce a tetrahedron.



Comparing measures associated with different generating patterns and their three-dimensional models suggests a question. The triangle pattern has perimeter and area; the pyramid model has surface area, equal to the area of the pattern, and volume. Does a connection exist between the perimeter of the pattern and the volume of the model?

If the investigation focuses on tetrahedra folded from not-necessarily-regular triangles with the perimeter held constant, the suggestion of a perimeter-volume connection gives rise to more questions. Because volume is one-third the product of the base area and altitude, how can these measures be found? Where is the point of intersection for the plane of the base and the altitude? Can a formula be written that will express the volume of the pyramid as a function of the lengths of the sides of the generating triangle? Do all triangles with the required perimeter produce tetrahedra? The original open-ended question is a source of many interesting avenues for exploration.

Investigations and Discoveries

The search for answers to these questions is an appropriate activity for students at different levels. Consider the problem of finding the intersection of an altitude of a tetrahedron and the plane of the base to which it is perpendicular. Members of a basic geometry class can discover through models that the orthocenter of the generating triangle is this touchdown point. More advanced students can prove this fact (Figure 2).

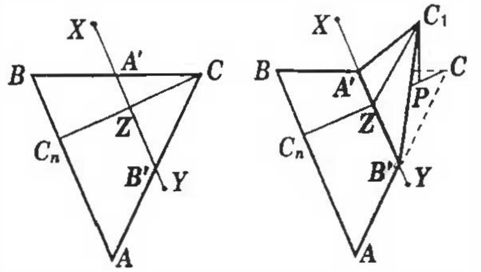
Students in precalculus classes can find coordinates for the orthocenter of a triangle with sides $2a$, $2b$ and $2c$ and study the derivation of a formula for the volume of the tetrahedron produced from this triangle. This formula, which will be discussed in more detail subsequently, yields the volume measure

$$\frac{\sqrt{2}}{12}\sqrt{(a^2+b^2-c^2)(a^2-b^2+c^2)(-a^2+b^2+c^2)}.$$

Thus students at three levels can investigate questions raised by the original problem about relating the volume of the tetrahedron to the perimeter of the triangular pattern.

Do all triangles produce tetrahedra? Geometry students who create models from a variety of triangles will discover the physical necessity of using acute triangles for the basic pattern. Only acute triangles will fold in to form tetrahedra. Right triangles fold flat; obtuse triangles present strange twists that prevent a meeting of all three pairs of matching edges.

Figure 2
The orthocenter of the triangular pattern is the foot of the altitude of the tetrahedron.



ΔABC is acute, $\overline{A'B'}$ is a midline, $\overline{CC_n}$ is an altitude, and Z is the intersection of $\overline{A'B'}$ and $\overline{CC_n}$.

On $\overline{A'B'}$ locate X and Y such that $XZ = ZY$. Since $\overline{CC_n}$ is the perpendicular bisector of $\overline{XY}$, $CX = CY$.

Folding on $\overline{A'B'}$ takes C out of the plane of ΔABC to position C_1. This motion preserves distance, so $C_1X = C_1Y$.

Let the perpendicular from C_1 to the plane of ΔABC intersect the plane at P.

By the hypotenuse-leg, $\Delta C_1PX \cong \Delta C_1PY$, so $PX = PY$. Therefore, P lies on the perpendicular bisector of $\overline{XY}$, $\overline{CC_n}$.

Any fold on a triangle's midline carries the vertex on a path over one of the altitudes. If the three vertices meet at a tetrahedon vertex, this vertex must lie directly over all three altitudes. Thus the foot of the perpendicular from this vertex to the plane of the base is the triangle's orthocenter.

Table 1
Pythagorean Inequalities and the Law of Cosines

Law of cosines: For any triangle with sides a, b, and c,
$a^2 = b^2 + c^2 - 2bc \cos(A)$

Angle A	Value of Cosine	Statement
acute	positive	$a^2 < b^2 + c^2$
obtuse	negative	$a^2 > b^2 + c^2$
right	zero	$a^2 = b^2 + c^2$

At a more advanced level, trigonometry students familiar with the law of cosines appreciate the fact that the restrictions needed to guarantee a positive radicand for the volume formula are the same inequalities that identify acute triangles. Table 1 illustrates this idea.

The perimeter-volume connection can be explored discretely at different levels. Students in a geometry class can begin with a fixed perimeter for the triangular pattern, then find all triples of integers that can represent sides of the pattern and compare the volumes of the resulting pyramids. Computer-programming students can be challenged to consider various perimeters, finding ways to list and count the number of triples, triangles and tetrahedra associated with each perimeter. An outgrowth of this investigation may include probability questions that eventually incorporate calculus concepts.

This wealth of connections suggests different ways to share these ideas with students. One approach is to introduce the problem to geometry students and then ask the same students to reconsider it as they progress through more advanced classes. Another approach is to invite classes at different levels to investigate the questions and to share insights and results. A third approach is to have students at the precalculus level tackle the basic questions together and then pursue extensions in small groups or as individual projects. Whatever approach is selected, one objective is to encourage students to continue to question and explore.

A Geometry Project

The geometry project described here was presented at the end of a unit on right triangles. The class investigated all tetrahedra that can be made from a triangular pattern with integral sides if the perimeter is fixed at 36. Students were given a list of investigative activities (Figure 3), a list of integral triples with a sum of 36 (Table 2) and an envelope marked with one of the triples that satisfied the triangle inequality. Each envelope contained an index card and three cutouts of the triangle pattern whose sides correspond to the triple written on the outside of the envelope. The three copies of the pattern were identified in the investigation list as white, blue and cardboard. The index card was used as a straightedge to facilitate folding.

Blue-paper, white-paper and cardboard triangle patterns were cut out in advance to save class time.

Centimetres were used as units for the triples; such triangles produce tetrahedra of convenient size. Fixing the perimeter of the pattern triangle at 36 produces 108 triples of integers to consider as possible sides (Table 2). These triples include 27 that form triangles, 12 of which are acute and can thus be folded to make tetrahedra. Therefore, 27 envelopes were prepared; had the class size been larger than 27, a perimeter greater than 36 would have been chosen. Eventually each student would need an envelope with one of the 12 acute triangles (see item 6 of Figure 3), so extras were readied.

Figure 3
Geometry Project Questions

Words in italic letters are in the index of your textbook.

1. Study the list of positive-integer triples with the sum of 36. Circle triples that identify triangles. Use the *triangle inequality*.
2. Find the *triangle area* for the triple listed on your envelope. Find the exact value and an approximate decimal value.
3. Test to see if your triangle is acute, right or obtuse. Verify the result with the *Pythagorean inequalities*.
4. Fold the blue-paper triangle on the *midlines*. If you can make a pyramid, tape the three edges that are not folds.
5. Share your information on the class chart.
6. Form a conjecture about triangle types and pyramids. If your triangle did not make a tetrahedron, trade your envelope for one with a triangle that will.
7. Find a formula for the *volume* of a *pyramid*.
8. How can the *midline theorem* help to find the baseline area?
9. If
$$\frac{1}{3}Bh = \frac{\sqrt{2(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)}}{12},$$
find the volume of your tetrahedron. Find both the exact value and an approximate decimal value.
10. On the cardboard triangle, record the measures for the perimeter and the area of the triangle and for the surface area and the volume of the pyramid.
11. Fold the white-paper triangle on the altitudes. Locate the *orthocenter*. Mark this point on the cardboard and use your compass to punch a hole there.
12. On one side of your cardboard triangle, write the measures from question 3. On the other side, draw midlines.
13. Match and glue the base of the blue tetrahedron to the central triangle you created by drawing the midlines in step 12.
14. Insert a toothpick through a hole at the orthocenter of the cardboard triangle and push it through, perpendicular to the cardboard. It should pass through the vertex of the pyramid.
15. Compare the surface areas and volumes of the pyramids as recorded on the class chart. What do you notice?

The first day of this three-day project was devoted to investigative activities 1 through 6. Each student was given a copy of Table 2 and one of the 27 envelopes. Immediately, students found which triples of integers with a sum of 36 satisfy the triangle inequality; those triples were at the bottoms of the columns of Table 2.

Table 2
Triples of Positive Integers with a Sum of 36

1, 1, 34	2, 2, 32	3, 3, 30	4, 4, 28	5, 5, 26
1, 2, 33	2, 3, 31	3, 4, 29	4, 5, 27	5, 6, 25
1, 3, 32	2, 4, 30	3, 5, 28	4, 6, 26	5, 7, 24
1, 4, 31	2, 5, 29	3, 6, 27	4, 7, 25	5, 8, 23
1, 5, 30	2, 6, 28	3, 7, 26	4, 8, 24	5, 9, 22
1, 6, 29	2, 7, 27	3, 8, 25	4, 9, 23	5, 10, 21
1, 7, 28	2, 8, 26	3, 9, 24	4, 10, 22	5, 11, 20
1, 8, 27	2, 9, 25	3, 10, 23	4, 11, 21	5, 12, 19
1, 9, 26	2, 10, 24	3, 11, 22	4, 12, 20	5, 13, 18
1, 10, 25	2, 11, 23	3, 12, 21	4, 13, 19	5, 14, 17
1, 11, 24	2, 12, 22	3, 13, 20	4, 14, 18	5, 15, 16
1, 12, 23	2, 13, 21	3, 14, 19	4, 15, 17	
1, 13, 22	2, 14, 20	3, 15, 18	4, 16, 16	
1, 14, 21	2, 15, 19	3, 16, 17		
1, 15, 20	2, 16, 18			
1, 16, 19	2, 17, 17			
1, 17, 18				
6, 6, 24	7, 7, 22	8, 8, 20	9, 9, 18	10, 10, 16
6, 7, 23	7, 8, 21	8, 9, 19	9, 10, 17	10, 11, 15
6, 8, 22	7, 9, 20	8, 10, 18	9, 11, 16	10, 12, 14
6, 9, 21	7, 10, 19	8, 11, 17	9, 12, 15	10, 13, 13
6, 10, 20	7, 11, 18	8, 12, 16	9, 13, 14	
6, 11, 19	7, 12, 17	8, 13, 15		
6, 12, 18	7, 13, 16	8, 14, 14		
6, 13, 17	7, 14, 15			
6, 14, 16				
6, 15, 15				
11, 11, 14	12, 12, 12			
11, 12, 13				

Next, students focused on finding the area for the triangles in their envelopes. The textbooks index included three references for triangle area: the familiar one-half the product of the base and the height; a trigonometric formula; and Heron's formula, clearly the best choice for the situation. Surprisingly, the task of finding exact values for the area presented little difficulty; a semiperimeter of 18 produced partially factored radicands with recognizable squares. The values were written on the class chart, and few incorrect values were challenged by classmates. Calculators yielded approximate decimals.

For activity 3, each student was asked to classify his or her triangle as acute, right or obtuse. The students generally did so by inspection, then confirmed their results using the Pythagorean inequalities. Activity 4 asked the student to fold the triangle and then decide if a tetrahedron could be formed. As the class shared and summarized information (Table 3), students realized that only acute triangles can fold to form pyramids. At that point, students with nonacute triangles traded for acute ones so that everyone could make a model.

Table 3
Cooperative Data-Gathering Project

Triple	Area		c^2		$a^2 + b^2$	Type of Triangle	Pyramid?	Volume	
	Exact	Decimal						Exact	Decimal
2, 17, 17	$12\sqrt{2}$	16.97	289	<	293	acute	yes	$\frac{\sqrt{287}}{12}$	1.41
3, 16, 17	$6\sqrt{15}$	23.24	289	>	265	obtuse	no		
4, 15, 17	$6\sqrt{21}$	27.50	289	>	241	obtuse	no		
4, 16, 16	$12\sqrt{7}$	31.75	256	<	272	acute	yes	$\frac{2\sqrt{62}}{3}$	5.25
5, 14, 17	$6\sqrt{26}$	30.59	289	>	221	obtuse	no		
5, 15, 16	$6\sqrt{39}$	37.47	256	>	250	obtuse	no		
6, 13, 17	$6\sqrt{30}$	32.86	289	>	205	obtuse	no		
6, 14, 16	$24\sqrt{3}$	41.57	256	>	232	obtuse	no		
6, 15, 15	$18\sqrt{6}$	44.09	225	<	261	acute	yes	$\frac{3\sqrt{207}}{4}$	10.79
7, 12, 17	$6\sqrt{33}$	34.47	289	>	193	obtuse	no		
7, 13, 16	$6\sqrt{55}$	44.50	256	>	218	obtuse	no		
7, 14, 15	$6\sqrt{66}$	48.74	225	<	245	acute	yes	$\frac{\sqrt{2015}}{4}$	11.22
8, 11, 17	$6\sqrt{35}$	35.50	289	>	185	obtuse	no		
8, 12, 16	$12\sqrt{15}$	46.48	256	>	208	obtuse	no		
8, 13, 15	$30\sqrt{3}$	51.96	225	<	233	acute	yes	$\frac{5\sqrt{11}}{2}$	8.29
8, 14, 14	$24\sqrt{5}$	53.67	196	<	260	acute	yes	$\frac{8\sqrt{41}}{3}$	17.07
9, 10, 17	36	36.00	289	>	181	obtuse	no		
9, 11, 16	$18\sqrt{7}$	47.62	256	>	202	obtuse	no		
9, 12, 15	54	54.00	225	=	225	right	no		
9, 13, 14	$18\sqrt{10}$	56.92	196	<	250	acute	yes	$\frac{9\sqrt{71}}{4}$	18.96
10, 10, 16	48	48.00	256	>	200	obtuse	no		
10, 11, 15	$12\sqrt{21}$	54.99	225	>	221	obtuse	no		
10, 12, 14	$24\sqrt{6}$	58.79	196	<	244	acute	yes	$2\sqrt{95}$	19.49
10, 13, 13	60	60.00	169	<	269	acute	yes	$\frac{25\sqrt{119}}{12}$	22.73
11, 11, 14	$42\sqrt{2}$	59.40	196	<	242	acute	yes	$\frac{49\sqrt{23}}{12}$	19.58
11, 12, 13	$6\sqrt{105}$	61.48	169	<	265	acute	yes	$2\sqrt{146}$	24.17
12, 12, 12	$36\sqrt{3}$	62.35	144	<	288	acute	yes	$18\sqrt{2}$	25.46

The second day of the project began with an unscheduled discussion about what happens to area and volume if linear measure is doubled. One student had worked ahead, calculating volumes before she was told that the variables *a, b* and *c* in the formula represented sides of the base of the pyramid, not the sides of the original triangle. The numbers she had used were double what they should have been. She suspected that her volume measures were too large when she looked at the small models, and she challenged the formula. We showed algebraically that replacing each variable with its double in the formula would introduce an extra factor of 8. Thus, to correct her calculations, she divided the volumes by 8.

One student offered a proof to show that midlines of a triangle create four congruent triangles, each with an area equal to one-fourth the area of the original triangle. Thus, if the linear measures of a triangle are doubled, the area is quadrupled. This fact allowed students to find B for the volume formula $V = 1/3Bh$. They could find one-fourth the triangle area that they had calculated for the generating triangle using Heron's formula. Several students questioned the factor 1/3 in the volume formula, and this factor was discussed. Of course, students still lacked a value for h, so they needed the formula in step 9 of Figure 3. The midline theorem referred to in step 8 follows:

> Midline Theorem. *If a segment joins the midpoints of two sides of a triangle, then it is parallel to the third side and its length is one-half the length of the third side.*

You may wish to let students seek out this theorem in their textbooks. Most work done on the second day was a group effort, and everyone managed to complete item 11.

On the third day, models were assembled, and toothpick altitudes emerged at the top vertices of the tetrahedra. Item 15 asked students to compare the surface areas and volumes of the different pyramids. This comparison was intended to evoke an optimization discovery, namely, that the equilateral triangle produced the tetrahedron with the greatest surface area and volume. However, most students responded with the observation that doubling the sides of a triangle causes the area to be four times as great and the volume to be eight times as great. The regular tetrahedron was admired for its symmetry and appreciated because it was the easiest model to make, but its maximum surface area and volume did not generate enthusiasm.

Evaluation

The assessments of students' efforts to solve this problem revealed several points. First, my concern that the time used to investigate the problem would put the geometry class behind schedule was unfounded. Three extra days were added to the unit. This time investment was justified as students reviewed important concepts, applied new knowledge of triples, previewed area and volume formulas, and discovered variation concepts usually introduced in an advanced-algebra class.

Second, students learned. On the chapter test that included this project, they showed increased proficiency in simplifying radicals. In addition to recognizing certain triples as Pythagorean, they identified triangles and classified them by angle size. On other test items, they responded as well as previous classes who had not covered the extra material.

Third, insights came from students' written responses to the question "What did you learn from this unit?" One student mentioned learning that several ways can be used to find triangular area. Another reported that he learned a use for the triangle's orthocenter. Most textbooks give reasons for locating the incenter, the circumcenter and the centroid, but the purpose of the orthocenter is unclear. He discovered that it is the place to put the toothpick altitude. Students who had received the troublesome triples mentioned learning that only acute triangles will form tetrahedra. The student who needed to divide her answers by 8 related that she learned the importance of defining variables and of estimating measures.

Fourth, the project generated new questions. Complaining that he received a nonacute triangle that was exchanged for an acute triangle whose tetrahedron ended up with the toothpick altitude on the outside, one student asked, "What are the odds of that happening?" We counted up the cases for perimeter of 36 to decide. However, if we consider the possibilities for any perimeter, we are off on another adventure. The questions did not end at item 15 in the investigation sheet.

Extensions

Continued explorations of the perimeter-volume connection are aided by computer-generated data for different perimeters. An interactive program that counts triples, triangles and tetrahedra and calculates areas and volumes helps with the planning for a geometry project involving a perimeter other than 36. Table 4 summarizes the results for a perimeter of 17. The program can also be used by students to investigate optimization questions. Readers interested in receiving this program can send to the author a stamped, self-addressed envelope and a note requesting a printout.

The derivation of the volume formula can be presented to students in analytic geometry. The steps are shown in the Appendix.

Open-ended investigations often generate questions beyond the students' current ability level, requiring teachers to decide whether to pursue or postpone extension activities. One factor in that decision should be the directive regarding core curriculum found in the *Curriculum and Evaluation Standards* (NCTM 1989). All students should be given a chance to understand the content topics and should be challenged as much as possible. As teachers, we recognize that students operate at different levels of motivation, comprehension and skill. However, we must remember that it is difficult for them to advance to higher levels if they never look beyond their present stages.

Exploring the pyramid question with the students may lead the teachers and the class into investigations that veer off in surprising directions. The problem-solving approach to mathematics is a learning adventure waiting to happen.

Table 4
Value for a Perimeter of 17

Triple	Area of Triangle	Volume of Tetrahedron
(1, 8, 8)	3.99	0.17
(2, 7, 8)	6.44	—
(3, 6, 8)	7.64	—
(3, 7, 7)	10.26	1.25
(4, 5, 8)	8.18	—
(4, 6, 7)	11.98	1.14
(5, 5, 7)	12.50	0.72
(5, 6, 6)	13.64	2.52

If integral values are chosen for the sides of triangle *ABC* with the perimeter equal to 17, then the maximum area of the triangle is 13.64, which occurs with the triple (5, 6, 6), and the maximum volume for the tetrahedron is 2.52, which occurs with the triple (5, 6, 6).

The total number of triples for a perimeter of 17 is 24. The total number of triangles for a perimeter of 17 is 8. The total number of tetrahedra for a perimeter of 17 is 5.

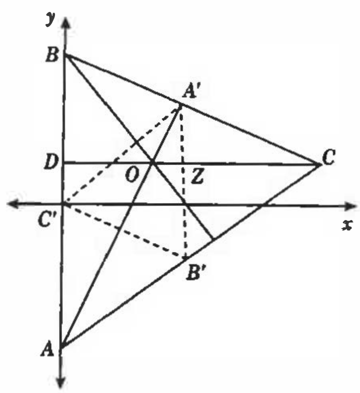
References

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

Rhode, R., G. Milauskas and R. Whipple. *Geometry for Enjoyment and Challenge.* Evanston, Ill.: McDougal, Littell, 1984.

Appendix
Derivation of Tetrahedron Volume Formula

I. Coordinates are assigned to key points.



Given:
Acute ΔABC with sides $2a$, $2b$, $2c$
$c > b > a$
B', A' and C' are midpoints of the sides.
O is the orthocenter.
$\overline{CO}$ and $\overline{A'B'}$ intersect at Z.
Area of $\Delta ABC = K$.
A: $(0, -c)$; B: $(0, c)$; C':$(0, 0)$

To find the coordinates of C: Let $C = (x, y)$. Point C lies on a circle A (radius $2b$) and on circle B (radius $2a$).

$$\left.\begin{aligned} x^2 + (y + c)^2 &= 4b^2 \\ x^2 + (y - c)^2 &= 4a^2 \end{aligned}\right\} \Rightarrow 4yc = 4(b^2 - a^2) \Rightarrow y = \frac{b^2 - a^2}{c}$$

ΔABC has a base $2c$, height x, so $x = K/c$. Therefore, C has coordinates $(K/c, (b^2 - a^2)/c)$.

To find the coordinates of O:

Equation of $\overline{CO}$:

$$y = \frac{b^2 - a^2}{c}$$

Equation of $\overline{AO}$: $y = \dfrac{-Kx}{a^2 - b^2 - c^2} - c$

$$x = \frac{Q}{-cK}$$

Equation of $\overline{BO}$: $y = \dfrac{-Kx}{a^2 - b^2 + c^2} + c$

where $Q = a^4 + b^4 - c^4 - 2a^2b^2$. Therefore, O has coordinates $(Q/-cK, (b^2 - a^2)/c)$.

To find the coordinates of Z:

Z lies on $\overline{OC}$:

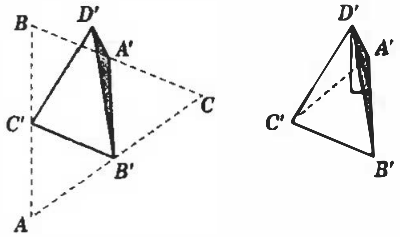
$$y = \frac{a^2 - b^2}{c}$$

Z lies on $\overline{A'B'}$:

$$x = \frac{K}{2c}$$

Therefore, Z has coordinates $(K/2c, (b^2 - a^2)/c)$.

II. The height of the tetrahedron is calculated. Applying the distance formula yields $CZ = K/2c$ and $ZO = (K^2 + 2Q)/2cK$: The folds on the midlines of the ΔABC bring the vertices to a common point D'.



The altitude of the tetrahedron, $\overline{D'O}$, is found using the Pythagorean theorem:

$$(D'O)^2 = (D'Z)^2 - (ZO)^2$$
$$= (CZ)^2 - (ZO)^2$$
$$= \left(\frac{K}{2c}\right)^2 - \left(\frac{K^2 + 2Q}{2cK}\right)^2$$
$$D'O = \frac{\sqrt{-Q(K^2 + Q)}}{Kc}$$

III. The volume formula is a function of a, b and c. Applying the formula for the volume of a pyramid to find the volume of a tetrahedron A'B'C'D' yields

$$V = \frac{1}{3}\left(\frac{K}{4}\right) \frac{\sqrt{-Q(K^2 + Q)}}{Kc}$$
$$Q = a^4 + b^4 - c^4 - 2a^2b^2.$$

Applying Heron's triangle-area formula to find K gives the area of ΔABC as

$$\frac{K}{4} = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}$$
$$= \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{4}$$

$$K = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$
$$= \sqrt{[(b+c)^2 - a^2][a^2 - (b-c)^2]},$$
$$K^2 = a^2(b+c)^2 - (b^2 - c^2)^2 - a^4 + a^2(b-c)^2$$
$$= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - b^4 - c^4 - a^4,$$
$$K^2 + Q = 2a^2c^2 + 2b^2c^2 - 2c^4$$
$$= 2c^2(a^2 + b^2 - c^2),$$
$$Q = a^4 - 2a^2b^2 + b^4 - c^4$$
$$= (a^2 - b^2)^2 - c^4,$$
$$-Q = c^4 - (a^2 - b^2)^2$$
$$= [c^2 - (a^2 - b^2)][c^2 + (a^2 - b^2)]$$
$$= [-a^2 + b^2 + c^2][a^2 - b^2 + c^2],$$
$$V = \frac{1}{12c}\sqrt{-Q(K^2 + Q)}$$
$$= \frac{1}{12c}\sqrt{(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)2c^2(a^2 + b^2 - c^2)}$$
$$= \frac{\sqrt{2}}{12}\sqrt{(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}.$$

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Sun and Moon

The distance between the sun and the earth is 387 times greater than the distance from the moon to the earth. How many times greater is the volume of the sun as compared to the moon?