Assessing Cooperative Problem Solving

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This collection of problems was originally presented at the annual meeting of the National Council of Teachers of Mathematics in San Diego, California, in April 1996 as part of a workshop on assessing cooperative problem solving.

Group Problem Solving

Solve the following problem with the other members of your group. Work together and *prepare one group paper to turn in to represent the group's solution*. Before you turn in your group's solution, you should be sure that *each person in your group* understands the solution well enough to answer questions about what the group did and could solve the problem on his or her own.

Group Problem: Triangular Arrangements

The three sides of a triangle have lengths a, b and c. Also all three lengths are whole numbers and $a \leq b \leq c$.

a. Suppose $c = 9$. Find the number of different triangles that are possible.

b. For any given value of c, find a general law that expresses the number of possible triangles.

Questions for Individuals

1. *(2 points)* In the problem your group just solved, would $c = 9$, $b = 5$, $a = 4$ be possible values for c, b and a? Explain why or why not!
2. *(4 points)* If $c = 4$, how many triangles would be possible?
3. *(4 points)* If the number of possible triangles is 36, what is the value of c?

Sample Group Solution to the Triangular Arrangements Problem

The three sides of a triangle have lengths a, b and c. Also, all three lengths are whole numbers and $a \leq b \leq c$.

a. Suppose $c = 9$. Find the number of different triangles that are possible.

b. For any given value of c, find a general law that expresses the number of possible triangles.

a. $c = 9$ and $a \leq b \leq c$

999 899 799 699 599 499 399 299 199 = 9
889 789 689 589 489 389 289 189 = 8
779 679 579 479 379 279 179 = 7
669 569 469 369 269 169 = 6
559 459 359 259 159 = 5
449 349 249 149 = 4
339 239 139 = 3
229 129 = 2
119 = 1

3 marks

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ possible triangles

b. The general way of finding all possible triangles is to add up all the whole numbers up to, and including, c.

3 marks

$1 + 2 + 3 + \ldots + c =$ total number of possible triangles

Scoring Scale for Group Problem Solving*

Understanding the Problem
- 0: Complete misunderstanding of the problem
- 3: Part of the problem misunderstood or misinterpreted
- 6: Complete understanding of the problem

Planning a Solution
- 0: No attempt, or totally inappropriate plan
- 3: Partially correct plan, or correct plan but not implemented properly
- 6: Plan does lead or could have led to a correct solution if problem had been completely understood

Getting an Answer
- 0: No answer, or wrong answer based on inappropriate plan
- 1: Copying error; computational error
- 2: Partial answer for problem with multiple answers
- 3: Correct answer and correct label for the answer; incorrect answer but the correct answer following from an incorrect plan or misunderstanding of the problem

Total possible points: 15

* Adapted from R. Charles, F. K. Lester and P. O'Daffer, *How to Evaluate Progress in Problem Solving* (Reston, Va.: National Council of Teachers of Mathematics, 1987).

Cooperative Problem Solving

The following problems might be appropriate for a junior or senior high school general mathematics class.

Group Problem: Intercom Installation

Bayview Middle School, a school with 35 teachers and 450 students, is planning to install an intercom system between all 25 classrooms and the main office. The system will permit direct conversations between any pair of classrooms, as well as between any classroom and the office. How many room-to-room and room-to-office wires will be needed?

Questions for Individuals

1. *(2 points)* In the problem as stated, how many intercom wires will be used to serve one of the rooms (for example, the art room)? Explain.
2. *(4 points)* If there had been only 6 rooms in the entire school, how many intercom wires would be needed altogether? Explain.
3. *(4 points)* If there were 28 classrooms and a main office, how many intercom wires would be needed altogether? Explain.

Group Problem: McContest

A major fastfood chain is holding a contest to promote sales. With each purchase, a customer will be given a card containing a whole number less than 100. A generous prize will be given to any person who presents cards whose numbers total 100. The company decides to print an unlimited supply of cards containing multiples of three. What other cards, and how many of each, should be printed so that there can be at most 1,000 winners throughout the country? Explain.

Questions for Individuals

1. *(2 points)* Can you find a winning combination (total of 100) using only multiples of three? Explain.
2. *(4 points)* Suppose the contest described in the group problem distributes 1,000 of each multiple of three up to 100, 25 fives and 25 additional fifteens. What is the maximum possible number of winners (people with cards totaling 100)? Explain.
3. *(4 points)* A different fastfood chain decides to have a similar contest in which winners would be persons presenting cards totaling 201. The company decides to print an unlimited supply of cards containing multiples of five. What other cards, and how many of each, should be printed so that there can be at most 2,000 winners? Explain.

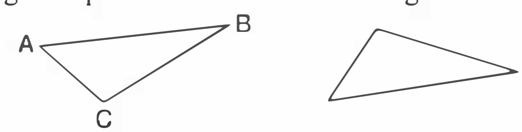
Group Problem: Uncommon Correspondences

[The following problem could be used in a geometry class for futher exploration of triangle congruence.]

Every triangle has 6 "parts": 3 sides and 3 angles. Show that it is possible for 5 of the parts of one triangle to be congruent to 5 of the parts of another triangle without the triangles being congruent to each other.

Questions for Individuals

1. *(2 points)* $\Delta ABC = \Delta DEF$. ΔABC is labeled in the diagram below. The unlabeled triangle is ΔDEF. Label this triangle to show the corresponding congruent parts between the two triangles.



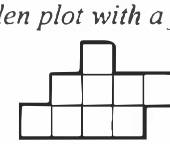
2. *(4 points)* Suppose you know the following about ΔABC and ΔDEF:
   $AB = 3$ units, $BC = 4$ units and $m\angle BAC = 60°$
   $EF = 3$ units, $DF = 4$ units and $m\angle DEF = 60°$
   Are ΔABC and ΔDEF congruent? Explain.
3. *(4 points)* Is it possible that the 5 parts of the two triangles discussed in the group problem that are congruent are the 3 sides and two of the angles? Explain your answer.

Group Problem: Don't Fence Me In

[The following problem explores the relationship between perimeter and area and might be appropriate for general mathematics, algebra or geometry students.]

Rebekah is planning her garden for the spring. She has 10 square garden plots, each of the same size. One will be for carrots, one for lettuce and so on. She wants to arrange the plots so each of them has at least one side in common with another garden plot. When she finishes arranging the plots, she plans to put a "rabbit-proof" fence around the entire plot *(a sample plot is shown below).*

Sample garden plot with a fence around it:



a. How would Rebekah arrange her garden plots so that she would use the smallest amount of fence? How would she arrange her plots in order to use the largest amount of fence?
b. In general, what happens to the smallest (and largest) amount of fence as the number of garden plots increases? Justify your answer.

Questions for Individuals

1. *(2 points)* Is there more than one way to arrange the garden plots using the smallest amount of fence? Explain your answer.
2. *(4 points)* If Rebekah had 4 garden plots of the same size, how would she arrange them so that she would use the greatest amount of fence?
3. *(4 points)* If Rebekah had 18 garden plots of the same size, how would she arrange them so that she would use the smallest amount of fence?

Group Problem: Mysterious Money

An absent-minded bank teller switched the dollars and cents when he cashed a cheque for Jana, giving her dollars instead of cents and cents instead of dollars. After buying a 5¢ stamp, Jana discovered that she had exactly twice as much left as her original cheque. What was the amount of the cheque?

Questions for Individuals

1. *(2 points)* If Jana's cheque was for \$5.43, how much money did the teller give Jana when cashing her cheque?
2. *(4 points)* Suppose the teller cashed Jana's cheque in the same manner as in the original problem. After buying a 68¢ pen, Jana discovered that she had exactly twice as much left as the cheque she had cashed. What was the amount of the cheque?
3. *(4 points)* Suppose the bank teller cashed Jana's cheque for \$11.16 in the same manner as in the original problem. After finding 63¢ in her purse, does Jana have more than twice as much money as her original cheque? Justify your answer.

Group Problem: Hiking

[The following problems are fraction applications and might be appropriate for an algebra or general mathematics class.]

Moses was hiking from Harper to Belmont along the Winding Trail, which also passed through the town of Springfield. Forty minutes after he left Harper, Moses saw a sign reading, "From Harper to here is half as far as it is from here to Springfield." Moses hiked another 11 miles and saw a second sign reading, "From here to Belmont is half as far as it is from here to Springfield." Moses hiked for one more hour and reached Belmont. If he hiked at the same pace all the way, what is the length of the Winding Trail between Harper and Belmont?

Questions for Individuals

1. *(2 points)* How long did it take Moses to hike to Springfield?
2. *(4 points)* Joni hiked from Bloomington to Nashville along the Hilly Trail, passing by Knight's Korner grocery store. Thirty minutes after she left Bloomington, she saw a sign reading, "If you came from Bloomington, you have come ¼ of the way to Knight's Korner." She hiked another 11 miles and saw a second sign reading, "From here to Knight's Korner is half as far as it from here to Nashville." Joni hiked for another two hours and reached Nashville. If she hiked at the same pace all the way, what is the length of the Hilly Trail between Bloomington and Nashville?
3. *(4 points)* If the Winding Trail, in the original problem, had been 13 miles long and Moses had walked 90 minutes from the second sign to Belmont, what would be the distance between the two signs? Note: all other information is the same as in the original problem.

Group Problem: Candles

Two candles of equal length are lit at the same time. One candle takes 9 hours to burn out and the other takes 6 hours to burn out. After how much time will the slower burning candle be exactly twice as long as the faster burning one?

Questions for Individuals

1. *(2 points)* After two hours of burning, how much longer is the slower burning candle than the faster burning one?
2. *(4 points)* Two candles of equal length are lit at the same time. One candle takes 6 hours to burn out and the other takes 3 hours to burn out. After how much time will the slower burning candle be exactly twice as long as the faster burning one?
3. *(4 points)* A blue candle is twice as long as a red candle. The blue candle takes 4 hours to burn out and the red candle takes 6 hours to burn out. After 3.5 hours, how long are each of the candles? The blue candle's length is now what fraction of the red candle's length?

Problem Solutions

Triangular Arrangements

Group Problem

a. There are 25 possible triangles.
b. If c is odd, the total number of possible triangles is the sum of the positive odd integers $\leq c$. If c is even, the total number of possible triangles is the sum of the positive even integers $\leq c$.

Individual Questions

1. No, $c = 9$, $b = 5$ and $a = 4$ are not values that can form a triangle because $a + b$ is not greater than c.
2. If $c = 4$, then $4 + 2 = 6$, so there are 6 possible triangles.

3. Since for $c = 9$, there were 25 triangles, and $36 - 25 = 11$, then c must be odd and $c = 11$ because $11 + 9 + 7 + 5 + 3 + 1 = 36$.

Intercom Installation

Group Problem

Since there are 26 rooms and each room is connected to 25 rooms, there are 26×25 connections. However, you must divide by 2 to eliminate counting the wires twice. Thus, there are 325 intercom wires.

Individual Questions

1. One room has 25 intercom wires serving it.
2. If there were six rooms, 15 intercom wires would be needed.
3. If there were 29 rooms, 406 intercom wires would be needed.

McContest

Group Problem

One answer would be to print at most 1,000 cards with 1 printed on them. Another possibility would be to print 2,000 with 50 printed on them.

Individual Questions

1. A winning combination cannot be made using only multiples of three because the sum of any multiples of three is still a multiple of three.
2. The maximum possible number of winners is 12 since it is possible for a person to total 100 using only two fives.
3. One answer would be to print at most 2,000 cards with 1 printed on them.

Uncommon Correspondences

Group Problem

The five congruent parts are the three angles and two sides. The key is that the congruent sides must be positioned differently in the two triangles.

Individual Questions

1. Label the triangle D, F, E by starting at the lower left and moving clockwise.
2. No, two triangles with angle-side-side congruence do not have to be congruent.
3. No, two triangles with side-side-side congruence are always congruent.

Don't Fence Me In

Group Problem

a. The smallest amount of fence would be used for an arrangement that is the most perfect (that is, closest to a square). For 10 plots, this would be either a $3 \times 3 + 1$ arrangement or a 2×5 arrangement. The largest amount of fence would be used for an arrangement that is the least compact. For 10 plots, this would be a 1×10 arrangement.
b. In general, as the number of plots increases by 1 plot, the amount of fence increases by 2 sides of fence. For the least compact arrangements, this is always true. For the most compact arrangements (smallest amount of fence), the amount of fence increases by two for the first plot added to a rectangular arrangement but remains at this amount up through the next rectangular arrangement. For example, for 9 plots the smallest amount of fence is used by a 3×3 arrangement (perimeter of 12). For 10 plots ($3 \times 3 + 1$ arrangement), the perimeter is 14. For 11 plots ($3 \times 3 + 2$), the perimeter is also 14, as is the amount of fence for 12 plots (3×4 arrangement). Then for 13 plots ($3 \times 4 + 1$ arrangement), the perimeter increases to 16.

Individual Questions

1. Yes. It could be for a 2×5 or a $3 \times 3 + 1$ arrangement.
2. A 1×4 arrangement would require the most fence.
3. A 3×6 arrangment would require the least amount of fence.

Mysterious Money

Group Problem

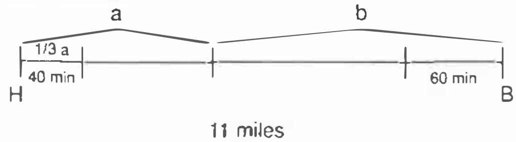
Use an equation such as $100c + d - 5 = 2(100d + c)$, where d = # of dollars and c = # of cents, and simplify to $c = (199d + 5)/98$. Then using an organized list to find possible solutions will yield \$31.63 as the amount of Jana's cheque.

Individual Questions

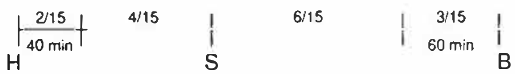
1. Switching the dollars and cents, the teller gave her \$43.05.
2. Using a solution process such as the one used in the group problem will yield \$10.21 as the amount of the cheque.
3. The teller gave Jana \$16.11, adding 63¢ gives a total of \$16.74. \$16.74 is 1.5 times as much as the original cheque of \$11.16.

Hiking

Group Problem



From H to S took 120 minutes; from S to B took 180. Thus Moses walked a total of 300 minutes.



x = total distance from H to B
$(4/15 + 6/15)x = 11$
$x = 33/2 = 16.5$

The Winding Trail is 16.5 miles between Harper and Belmont.

Individual Questions

1. Moses hiked for 300 minutes or 5 hours.
2. Using a method similar to the one in the group problem yields an answer of 22 miles for the length of the Hilly Trail between Bloomington and Nashville.
3. The distance between the two signs is $8\frac{2}{3}$ miles. This can be found by using the method illustrated above.

Candle

Group Problem

After 4.5 hours the slower burning candle will be exactly twice as long as the faster burning one.

Individual Questions

1. After 2 hours, the slower burning candle has burned 4/18—leaving 14/18 of the candle. This faster burning candle has burned 6/18—leaving 12/18 of the candle. Thus, the slower candle is 2/18 or 1/9 longer than the faster burning candle.
2. Divide the candle into sixths. After 2 hours, the slower burning candle has burned 2/6—leaving 4/6 of the candle. The faster burning candle has burned 4/6—leaving 2/6 of the candle. Thus the slower burning candle is twice as long as the faster burning candle after 2 hours.
3. Divide the red candle into 12/12 and the blue candle into 24/12. After 3.5 hours, the blue candle has burned 21/12—leaving 3/12 of the candle. The red candle has burned 7/12—leaving 5/12 of the candle. Thus the blue candle's length is now 3/5 of the red candle's length.

Number Game
Find all two-digit natural numbers, which are three times the sum of their digits.

Completely "Variable"
The variables a, b and c of the expression $a(c-b)/(b-a)$ are to be substituted with 13, 15 and 20 in such a way that the value of the expression results in a positive whole number.