Of the Fourth Dimension

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Until the beginning of the 20^{th} century, mathematicians and lay people alike looked on geometries of more than three dimensions with skepticism. It was believed that physical conditions alone precluded the existence of more than three dimensions.

The ancient Greeks devoted much time to geometry and the concept of dimensions, but concentrated solely on one, two and three dimensions. In the fourth century B.C., Aristotle wrote in his book *Heaven* that "the line has magnitude in one way, the plane in two ways, and the solid in three ways and beyond these there is no magnitude because the three are all" (Hess 1977, 1–2). Greeks calculated area and volume in geometric terms. When they studied equations and their solutions, they did so within the framework of geometry. Therefore, they regarded equations higher than cubic as unreal. Girolamo Cardano (1501–1576) stated, "The first power (of a number) refers to a line, the square to a surface, the cube to a solid, and it would be fatuous indeed for us to progress beyond for the reason that it is contrary to nature" (Eves 1969, 212).

Some progress seemed to be made by the middle of the 16^{th} century, as evidenced by Michael Stifel's statement in 1553 that in arithmetic "we set down corporeal lines and surfaces and pass beyond the cube as if there were more than three dimensions, although this is contrary to nature" (Eves 1969, 212). However, it was only with the advent of Einstein's theory of relativity that discussion of four-dimensional space began to be spoken of more realistically. It was realized that physical existence or nonexistence of a four-dimensional body in our universe has nothing to do with its existence as a mathematical entity.

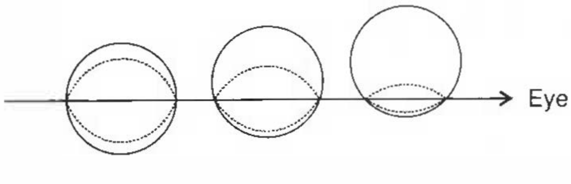
When approaching a new geometry such as four-dimensional geometry, problems that arise are almost always perceptual rather than conceptual in nature. Euclidian geometry can be used as a springboard for the study of four-dimensional space, or E_4, as it is sometimes called, to make their conceptual part less complicated.

Just as we had to assume the existence of a point not on a given plane to work with three dimensions, to enlarge our space for the study of four dimensions, we must now assume the existence of a point not belonging to our three-dimensional system. We will thereby be creating a new space in which there will exist many three-dimensional subspaces (similar to the planes in a three-dimensional space). We should logically include a name for these subspaces, and we shall accept the word "prime" as the name of our three-dimensional subspace. And just as we use parallelograms to denote planes in our study of E_3 geometry, we will represent primes by parallelepipeds:

Definition: The points of a set are said to be coprimal if and only if there is a prime which contains them all. This is not to suggest that a prime has faces like a cube or is any way limited in size. A prime extends infinitely in all directions. At this point many of us encounter a significant perceptual difficulty.

Given the definition, should there not, in effect, be only one prime in existence? Let us try to put in perspective the concept of the existence of more than one prime.

In the late 19^{th} century, Edwin Abbott put forth a way of conceiving of a fourth dimension in his fictional book *Flatland, a Romance of Many Dimensions*. He asked us to imagine "people" in a two-dimensional world. Their "universe" would be a flat plane. Now think of how impossible it would be to visualize through the eyes of those people our three-dimensional world or even to see a three-dimensional object passing through their two-dimensional world. Because they could perceive only in two dimensions, they would see a sphere passing through their world as a circle, gradually increasing in size and then gradually decreasing.



Sphere Passing Through a Plane (Abbott 1952, 73)

The inhabitants of Flatland can see only on their plane. They are not physically capable of looking "upward" or "downward" to see that there are an infinite number of planes or "universes" above and below theirs.

It would be incredibly hard for a two-dimensional being, who lived on a vast plane and who understood "forward, backward, left and right" perfectly well, to understand the concepts of "up and down." The confusion about the third dimension would be similar to the confusion we feel when thinking about the fourth.

We, as three-dimensional beings, can see the sphere and know that there are innumerable plane "universes" and we can pass freely through them all.

Now consider our three-dimensional universe as the plane of the Flatlanders. Through analogy, it stands to reason that there are other planes, or in the case of the fourth dimension, other primes, in addition to ours. We tend to dismiss this idea because we have no physical means of perceiving it or traveling to another prime.

It is true that another prime does not lie "next" to ours or "on top" of ours as planes lie on top of each other. Nevertheless, it follows that other primes, other dimensions, exist somewhere in relation to ours. If a solid can pass through planes, there should be a four-dimensional "hypersolid" that can pass through different primes, and when these hypersolids "disappear," they are actually passing out of our prime.

Because we have concluded that four-dimensional space should consist of more than one prime, we have to introduce new postulates describing their intersection properties:

Four-Dimensional Euclidean Geometry Postulates

1. Every line is a set of points and contains at least two points.
2. If X and Y are any two points, there is one and only one line which contains them.
3. Every plane is a set of points and contains at least three noncollinear points.
4. If X, Y, Z are any three noncollinear points, there is one and only one plane which contains them.
5. Every prime (three-space) contains at least four points which are neither collinear nor coplanar.
6. If two points of a line lie in a plane, then every point of the line lies in the plane.

[The remaining postulates hold true only in four dimensions]

7. If W, X, Y, Z are any four noncoplanar points, there is one and only one prime which contains them.
8. Space contains at least five points which are neither collinear, coplanar or coprimal.
9. If three noncollinear points of a plane lie in a prime, then every point of the plane lies in the prime.
10. If a plane and a prime have a point in common, their intersection is a line.

Other postulates that hold equally true in both three- and four-dimensional space include, among others, the parallel postulate and the plane separation postulate.

"The geometry of *n* dimensions is an intellectual journey that takes us through fascinating and purely mental country, and never ends" (Reid 1959, 109)

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