# delta-k

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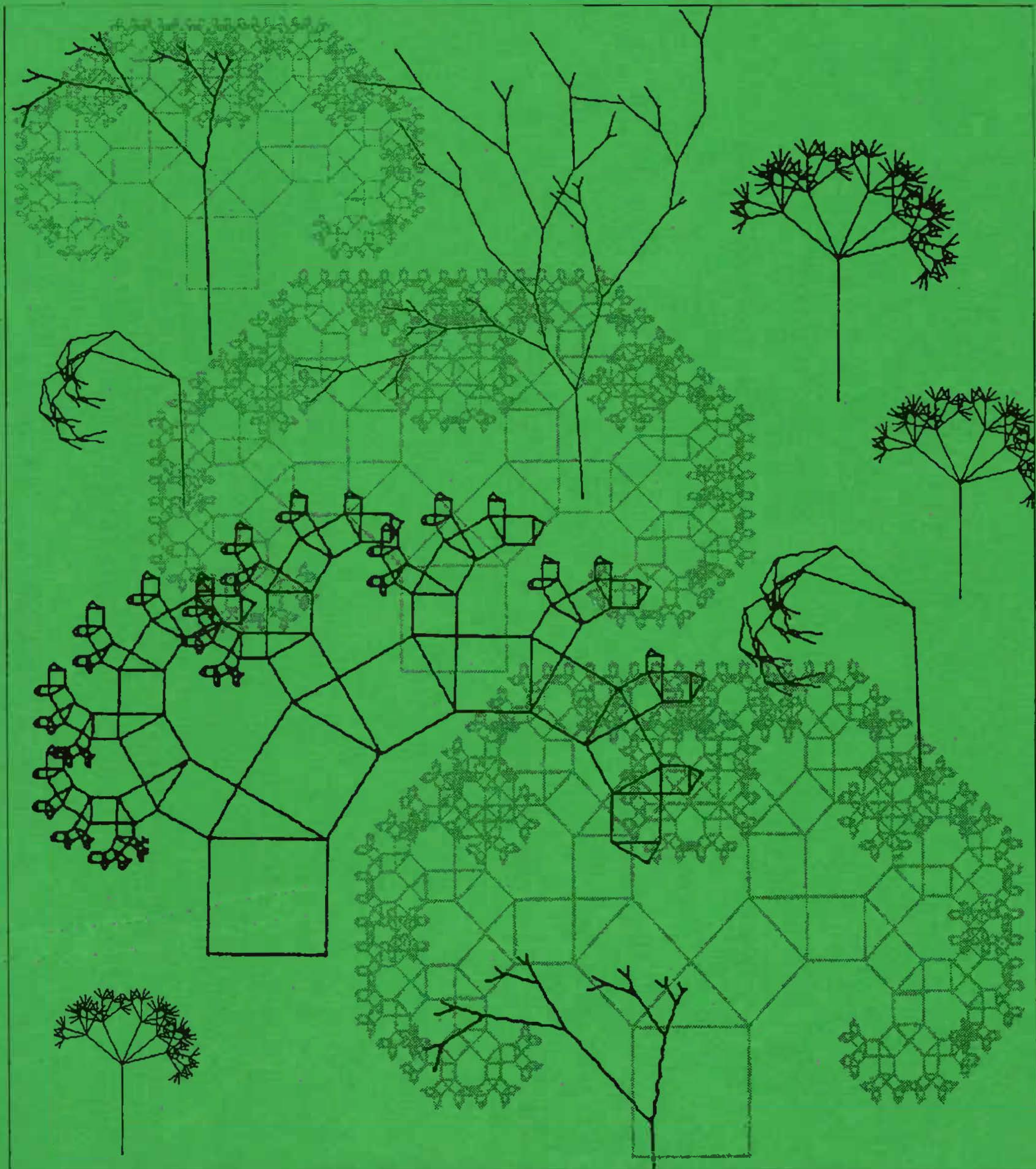

# GUIDELINES FOR MANUSCRIPTS

*delta-K* is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

## Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. Letters to the editor or reviews of curriculum materials are welcome.
7. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

## Submission Deadlines

*delta-K* is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

### MCATA Mission Statement

***Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.***

delta-k

Volume 35, Number 1 February 1998

## CONTENTS



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**FEATURE ARTICLES**

# COMMENTS ON CONTRIBUTORS

*Lynn Gordon Calvert* is an assistant professor in the Faculty of Education, Department of Elementary Education, University of Alberta, Edmonton.

*William M. Carroll* is involved in the research and development of assessment at the University of Chicago School Mathematics Project—Elementary Materials. This article is based on his work at Roosevelt High School in Chicago.

*Linda Dickerson-Frilot* teaches in an alternative education program at Obsidian Middle School, Redmond, Oregon.

*Octaviano Garcia* teaches at Cuba Middle School, Cuba, New Mexico.

*Brenda Healing* teaches at Bentley High School in Bentley.

*Art Jorgensen*, from Edson, is a retired junior high school principal and a member of the Mathematics Council of The Alberta Teachers' Association.

*Thomas E. Kieren* is a professor of education (emeritus) in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton.

*Joanna O. Masingila* is an assistant professor of mathematics and mathematics education at Syracuse University, Syracuse, New York.

*Ruth M. McClintock* has taught at the secondary and collegiate levels and is now at MacMurray College in Jacksonville, Illinois. Her interests include geometry and discrete mathematics.

*Terri-Lynn McLeod* teaches at Calling Lake High School in Calling Lake.

*Kay Melville* is a mathematics assessment specialist with the Achievement Testing Unit, Student Evaluation Branch, Alberta Education.

*Sandra M. Pulver* is a professor of mathematics at Pace University in New York.

*Michael Richards* is coordinator of the Problem Solving Task Centre Network—a worldwide support group of primary, secondary and tertiary teachers interested in establishing problem solving task centres or incorporating mathematical problem solving into their teaching. Michael also coordinates staff development at Mordialloc-Chelsea Secondary College, Mordialloc, Australia.

*Hugh Sanders* is acting assistant director, mathematics and science programs, with the Curriculum Standards Branch, Alberta Education.

*Sol E. Sigurdson* is professor of education (emeritus) in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton.

*Elaine Simmt* is an assistant professor in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton.

*Carolyn F. Talton* teaches graduate and undergraduate courses in elementary mathematics methods at Louisiana Tech University, Ruston. She is especially interested in promoting the thinking and language development necessary for problem solving.

*Gina Vivone-Vernon* is an assistant director with the Curriculum Standards Branch, Alberta Education.

# EDITORIAL

The *Common Curriculum Framework for K–12 Mathematics: Western Canadian Protocol for Collaboration in Basic Education* was implemented in September 1997 for Kindergarten through Grade 9. While this new curriculum presents many challenges to teachers of mathematics, it simultaneously offers endless opportunities for teachers to improve the teaching and learning of mathematics for students. The challenges related to the implementation of the Western Canadian Protocol flow largely from the fact that this curriculum articulates clearly defined student expectations, general outcomes and specific outcomes (see the article by H. Sanders and G. Vivone-Vernon in this issue), in contrast to the past curriculum that for the most part outlined the prescribed content of the program only. In addition, the common curriculum framework incorporates seven related mathematical processes that are intended to permeate all teaching and learning of mathematics. Three mathematical processes—communicating mathematically; connecting mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines; and relating and applying new mathematical knowledge through problem solving—have been consciously focused on in this issue of *delta-K*. The articles by R. M. McClintock, W. M. Carroll and C. F. Talton have an exclusive focus on mathematical problem solving.

The activities provided by S. E. Sigurdson, T. E. Kieren, T.-L. McLeod and B. Healing have been extracted from their booklet, "A Collection of Connections for Junior High Western Canadian Protocol Mathematics." The examples selected for this issue have a particular focus on the mathematical processes of communication and connections. The remainder of the Teaching Ideas section has a distinct focus on problem solving with helpful problem solving strategies, lesson plans and a framework for assessing cooperative problem solving.

The feature articles represent an interesting selection. S. M. Pulver presents us with an interesting discussion about the fourth dimension by providing an historical and geometric perspective. L. Gordon Calvert offers us a number of rich scenarios in which algorithms are created or re-created using a variety of technology. E. Simmt shows us how parents and children can do mathematics together and how parents can assume a more involved role in their children's education.

I hope that you find the various articles and teaching ideas presented in this issue interesting, challenging and useful in your mathematics classroom. During this time of change in mathematics, your submissions are needed more than ever. Please share your ideas, lesson plans and articles with your colleagues. They are always welcome.

*Klaus Puhlmann*

# From the President's Pen

It is difficult to believe that nearly half the school year is over. I have been thinking about what I believe about MCATA's mission statement, "Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics," and I wish to share some thoughts with you.

This statement evokes many interesting ideas for consideration and discussion. Throughout this year, I will share my thoughts about this mission statement. While I will probably leave you with more questions than answers, I hope that they will stimulate you to think about this mission statement in the context of your professional life as a mathematics educator.

First, we need to think about what it means to "encourage the continuing enhancement of teaching." So what does it mean "to teach"? I think the whole definition of "teaching" is evolving. Think about what society thought about teachers and teaching many years ago. Quite often, the teacher in the community was one of the individuals with some "schooling"—either the teacher had completed grade school and possibly higher or he or she had experienced some type of schooling outside the community. During that time, our society also equated knowing "facts" with knowledge. Is this not the same perception that a large segment of society holds today about teachers and teaching? When I think about this perception, I am reminded of many family gatherings in which I've heard the statement, "Go and ask Florence, she'll know. She's the teacher!" I think that many students who are entering teacher education programs also have this perception about teachers and teaching. So, how can we as members of MCATA help society become aware of the evolving definition of teaching?

Teaching implies that there is a learner, and our role as teachers is evolving from one that solely gives information to one in which we are enabling our students to develop an understanding of concepts. This has a significant effect on our lives as teachers who teach students mathematics. The new mathematics curriculum is showing us that even what we knew as mathematics is changing. It no longer appears to be a fixed body of knowledge that never changes. So, how does this affect our teaching? We have technology, calculators and computers that can do many of the routine procedures that we once taught and on which we spent many hours during *our* teaching and learning of mathematics. How do these tools fit into this evolving definition of teaching?

I believe that mathematics is a way to describe patterns and relationships in the world—in the same way that painting, music and stories can be used. I believe that one way we can help society come to understand teaching students mathematics differently is for our students to think about it. Have you ever asked your students what they think teaching is? Have you ever asked your students what they think mathematics is? I think that we would have interesting ideas to ponder if we could hear the thoughts of our students. I would love to hear what you discover, so write to me and share what you have learned and discovered about our profession called teaching.

*Florence Glanfield*

# The Right Angle

## Learning Technologies Branch and Alberta Distance Learning Centre Divestiture Update

The transfer of the Alberta Distance Learning Centre's (ADLC) instructional services to Pembina Hills Regional Division No. 7 was effective June 2, 1997. ADLC will continue to operate from its present locations in Barrhead and Edmonton, providing the same courses, at the same cost, with the same commitment to quality.

Alberta Education will continue its responsibility for the design, development, production and distribution of distance courses. Course design and development will be the responsibility of the newly named Learning Technologies Branch directed by Gary Popowich. (A new name is necessary because the name Alberta Distance Learning Centre is being retained by the operation which is becoming part of Pembina Hills Regional Division No. 7). The production and distribution of material will remain in the hands of the Learning Resources Distributing Centre.

The Learning Technologies Branch will share the Barrhead facility with the Alberta Distance Learning Centre. The newly named branch will maintain its current mailing address (PO Box 4000, Barrhead T7N 1P4), along with its main telephone number (403-674-5333) and main fax number (403-674-6561).

For more information, visit the Learning Technologies Branch Web site: http://ednet.edc.gov.ab.ca/ltb/

## Why Distance Learning?

What do all these scenarios have in common?

- A dropout returns to study in a storefront school.
- A student finds that a self-paced program in a city high school provides the freedom that keeps him in school.
- A parent in a home-schooling situation finds materials superbly tailored to his children.
- A youngster works on a self-study program while sailing the globe.
- A young teen receives honors grades in her studies as she and her family follow the traditional path of their trapline.
- A promising skier completes high school as she earns her way onto the Olympic ski team.

All these situations have one element in common: students experiencing distance learning with all of its flexibility and ability to be tailored to their personal situations and learning styles.

Distance learning offers adults the ability to upgrade and complete their high school education. It offers alternatives to students in schools—in the types of courses available and the flexibility of timetabling. It offers education to those who choose paths of education that are alternatives to the traditional classroom.

And today, distance learning offers more choice than ever, not just in the course selection available, but in the pathways of learning provided by the latest in technology. The learning packages allow students to pursue their learning in the ways they desire, not only through the attractive student-centred print medium but also through audiocassettes and videocassettes, through telephone and fax contact, through CD-ROM and CDI, through e-mail and the Internet. These technologies make learning choices unlimited.

*Kay Melville*

## READER REFLECTIONS

*In this section, we will share your reactions to articles and points of view on teaching and learning of mathematics. We appreciate your interest and value the views of those who write. There were no submissions received for this issue.*

## STUDENT CORNER

*Mathematics as communication is an important curriculum standard, hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.*

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternate solution to a familiar problem, poetry about mathematics or anything that is deemed to be of mathematical interest.*

*Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of The Alberta Teachers' Association to publish [insert title] in one of its publications. The student author must sign this statement or a parent if the student is under eighteen years old, indicate the student's grade level and provide an address and telephone number.*

*No submissions were received for this issue. We look forward to your submissions.*

# NCTM Standards in Action

*Klaus Puhlmann*

This year, every issue of *delta-K* will devote a section to the NCTM standards. In this issue the focus will be on Curriculum Standard 1: Mathematics as Problem Solving. The NCTM's Standards documents, *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991) and *Assessment Standards for School Mathematics* (1995), had a significant effect on the development of *The Common Curriculum Framework for K–12 Mathematics* (Alberta Education 1995).

Problem solving is a process that should pervade all mathematics instruction and should be modeled by the teacher in each lesson. However, teaching mathematics from the problem solving perspective entails more than just solving nonroutine problems. It is important that students experience mathematics through exploring, conjecturing, examining and problem solving. In short, teachers should engage students in mathematical discourse about problem solving.

When mathematics evolves naturally from real problem situations that have meaning and relevance to students and that are related to students' environment, students then see not only the interconnection of their knowledge but also the power and usefulness of mathematics in the world around them.

While problem solving is important at all levels, the *Curriculum and Evaluation Standards for School Mathematics* (1989) have made age-appropriate adjustments in the degree of emphasis between the elementary, middle school and high school levels as follows:

- From Kindergarten to Grade 4, the study of mathematics should emphasize problem solving so that students can
  - use problem solving approaches to investigate and understand mathematical content;
  - formulate problems from everyday and mathematical situations;
  - develop and apply strategies to solve a wide variety of problems;
  - verify and interpret results with respect to the original problem; and
  - acquire confidence in using mathematics meaningfully.
- In Grades 5–8, the mathematics curriculum should include numerous and varied experiences with problem solving as a method of inquiry and application so that students can
  - use problem solving approaches to investigate and understand mathematical content;
  - formulate problems from situations within and outside mathematics;
  - develop and apply a variety of strategies to solve problems, with emphasis on multistep and nonroutine problems;
  - verify and interpret results with respect to the original problem situation;
  - generalize solutions and strategies to the new problem situation; and
  - acquire confidence in using mathematics meaningfully.
- In Grades 9–12, the mathematics curriculum should include the refinement and extension of methods of mathematical problem solving so that all students can
  - use, with increasing confidence, problem solving approaches to investigate and understand mathematical content;
  - apply integrated mathematical problem-solving strategies to solve problems from within and outside mathematics;
  - recognize and formulate problems from situations within and outside mathematics; and
  - apply the process of mathematical modeling to real-world problem situations.

The three articles that follow provide useful information as well as insight into areas that can enhance the presence of problem solving in our classrooms. Ruth M. McClintock's article "The Pyramid Question: A Problem-Solving Adventure" points out that a good question can launch a discovery journey through conjecture, research, serendipitous encounters, proof, answers and new questions. This article shares a good question, reports some discoveries and suggests ways to incorporate the adventure into the classroom.

"Increasing Mathematics Confidence by Using Worked Examples" by William M. Carroll suggests that by studying already worked examples of problems, students can effectively recognize underlying similarities between problems, mathematical principles and classes of problem situations. However, he argues that worked examples are not always used effectively in instruction and he shows us how.

The third article, "Let's Solve the Problem Before We Find the Answer," by Carolyn F. Talton identifies areas that are troublesome for students when engaged in problem solving. Her useful analysis not only identifies the areas but also provides advice that allows teachers to address these problem areas.

## References


Alberta Education. *The Common Curriculum Framework for K–12 Mathematics: Western Canadian Protocol for Collaboration in Basic Education.* Edmonton: Author, 1995.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

——. *Professional Standards for Teaching Mathematics.* Reston, Va.: NCTM, 1991.

——. *Assessment Standards for School Mathematics.* Reston, Va.: NCTM, 1995.


**Mathematics from Ancient Rome**

The law in ancient Rome required that a widow was compelled to divide 3,500 denar with the child she expected in the following way: If it is a boy, the widow would receive one half the amount of the son. If it is a daughter, the widow receives twice the amount of the daughter. However, the mother gave birth to twins—a boy and a girl. How should the inherited amount of 3,500 denar be divided so that the law of Rome is not violated?

# The Pyramid Question: A Problem-Solving Adventure

*Ruth M. McClintock*

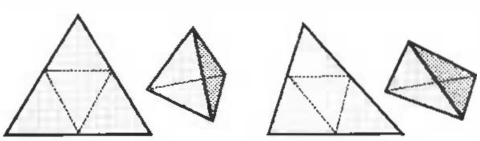
Good teachers know that a good question can launch a discovery journey through conjecture, research, serendipitous encounters, proof, answers and new questions. The importance of a problem-solving approach to investigating and understanding mathematics is underscored by the fourth grade-level standard in NCTM's *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989).

The purposes of this article are to share a question, report some discoveries and suggest ways to incorporate the adventure into classrooms.

## The Question

Three-dimensional models for pyramids can be constructed from two-dimensional patterns. Figure 1 shows two patterns that produce pyramids with triangular bases when folds on the dashed segments bring matching sides of the triangles together. If the pattern is an equilateral triangle, then folds on its midlines, that is, segments connecting the midpoints of two sides, will yield a regular tetrahedron, a pyramid with four regular-triangle faces. Patterns made from nonequilateral triangles produce other types of tetrahedra.

**Figure 1**
**Folds along the dashed segments of a triangle pattern produce a tetrahedron.**



Comparing measures associated with different generating patterns and their three-dimensional models suggests a question. The triangle pattern has perimeter and area; the pyramid model has surface area, equal to the area of the pattern, and volume. Does a connection exist between the perimeter of the pattern and the volume of the model?

If the investigation focuses on tetrahedra folded from not-necessarily-regular triangles with the perimeter held constant, the suggestion of a perimeter-volume connection gives rise to more questions. Because volume is one-third the product of the base area and altitude, how can these measures be found? Where is the point of intersection for the plane of the base and the altitude? Can a formula be written that will express the volume of the pyramid as a function of the lengths of the sides of the generating triangle? Do all triangles with the required perimeter produce tetrahedra? The original open-ended question is a source of many interesting avenues for exploration.

## Investigations and Discoveries

The search for answers to these questions is an appropriate activity for students at different levels. Consider the problem of finding the intersection of an altitude of a tetrahedron and the plane of the base to which it is perpendicular. Members of a basic geometry class can discover through models that the orthocenter of the generating triangle is this touchdown point. More advanced students can prove this fact (Figure 2).

Students in precalculus classes can find coordinates for the orthocenter of a triangle with sides $2a$, $2b$ and $2c$ and study the derivation of a formula for the volume of the tetrahedron produced from this triangle. This formula, which will be discussed in more detail subsequently, yields the volume measure

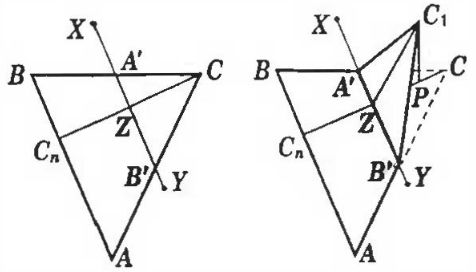
$$\frac{\sqrt{2}}{12}\sqrt{(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2)}.$$

Thus students at three levels can investigate questions raised by the original problem about relating the volume of the tetrahedron to the perimeter of the triangular pattern.

Do all triangles produce tetrahedra? Geometry students who create models from a variety of triangles will discover the physical necessity of using acute triangles for the basic pattern. Only acute triangles will fold in to form tetrahedra. Right triangles fold flat; obtuse triangles present strange twists that prevent a meeting of all three pairs of matching edges.

**Figure 2**

**The orthocenter of the triangular pattern is the foot of the altitude of the tetrahedron.**




$\Delta ABC$ is acute, $\overline{A'B'}$ is a midline, $\overline{CC_n}$ is an altitude, and $Z$ is the intersection of $\overline{A'B'}$ and $\overline{CC_n}$.

On $\overline{A'B'}$ locate $X$ and $Y$ such that $XZ = ZY$. Since $\overline{CC_n}$ is the perpendicular bisector of $\overline{XY}$, $CX = CY$.

Folding on $\overline{A'B'}$ takes $C$ out of the plane of $\Delta ABC$ to position $C_1$. This motion preserves distance, so $C_1X = C_1Y$.

Let the perpendicular from $C_1$ to the plane of $\Delta ABC$ intersect the plane at $P$.

By the hypotenuse-leg, $\Delta C_1PX \cong \Delta C_1PY$, so $PX = PY$. Therefore, $P$ lies on the perpendicular bisector of $\overline{XY}$, $\overline{CC_n}$.

Any fold on a triangle's midline carries the vertex on a path over one of the altitudes. If the three vertices meet at a tetrahedon vertex, this vertex must lie directly over all three altitudes. Thus the foot of the perpendicular from this vertex to the plane of the base is the triangle's orthocenter.

**Table 1**

**Pythagorean Inequalities and the Law of Cosines**

Law of cosines: For any triangle with sides $a$, $b$, and $c$, $a^2 = b^2 + c^2 - 2bc \cos(A)$

| Angle A | Value of Cosine | Statement |
|---|---|---|
| acute | positive | $a^2 < b^2 + c^2$ |
| obtuse | negative | $a^2 > b^2 + c^2$ |
| right | zero | $a^2 = b^2 + c^2$ |

At a more advanced level, trigonometry students familiar with the law of cosines appreciate the fact that the restrictions needed to guarantee a positive radicand for the volume formula are the same inequalities that identify acute triangles. Table 1 illustrates this idea.

The perimeter-volume connection can be explored discretely at different levels. Students in a geometry class can begin with a fixed perimeter for the triangular pattern, then find all triples of integers that can represent sides of the pattern and compare the volumes of the resulting pyramids. Computer-programming students can be challenged to consider various perimeters, finding ways to list and count the number of triples, triangles and tetrahedra associated with each perimeter. An outgrowth of this investigation may include probability questions that eventually incorporate calculus concepts.

This wealth of connections suggests different ways to share these ideas with students. One approach is to introduce the problem to geometry students and then ask the same students to reconsider it as they progress through more advanced classes. Another approach is to invite classes at different levels to investigate the questions and to share insights and results. A third approach is to have students at the precalculus level tackle the basic questions together and then pursue extensions in small groups or as individual projects. Whatever approach is selected, one objective is to encourage students to continue to question and explore.

## A Geometry Project

The geometry project described here was presented at the end of a unit on right triangles. The class investigated all tetrahedra that can be made from a triangular pattern with integral sides if the perimeter is fixed at 36. Students were given a list of investigative activities (Figure 3), a list of integral triples with a sum of 36 (Table 2) and an envelope marked with one of the triples that satisfied the triangle inequality. Each envelope contained an index card and three cutouts of the triangle pattern whose sides correspond to the triple written on the outside of the envelope. The three copies of the pattern were identified in the investigation list as white, blue and cardboard. The index card was used as a straightedge to facilitate folding.

Blue-paper, white-paper and cardboard triangle patterns were cut out in advance to save class time.

Centimetres were used as units for the triples; such triangles produce tetrahedra of convenient size. Fixing the perimeter of the pattern triangle at 36 produces 108 triples of integers to consider as possible sides (Table 2). These triples include 27 that form triangles, 12 of which are acute and can thus be folded to make tetrahedra. Therefore, 27 envelopes were prepared; had the class size been larger than 27, a perimeter greater than 36 would have been chosen. Eventually each student would need an envelope with one of the 12 acute triangles (see item 6 of Figure 3), so extras were readied.

**Figure 3**
**Geometry Project Questions**

Words in italic letters are in the index of your textbook.

1. Study the list of positive-integer triples with the sum of 36. Circle triples that identify triangles. Use the *triangle inequality*.
2. Find the *triangle area* for the triple listed on your envelope. Find the exact value and an approximate decimal value.
3. Test to see if your triangle is acute, right or obtuse. Verify the result with the *Pythagorean inequalities*.
4. Fold the blue-paper triangle on the *midlines*. If you can make a pyramid, tape the three edges that are not folds.
5. Share your information on the class chart.
6. Form a conjecture about triangle types and pyramids. If your triangle did not make a tetrahedron, trade your envelope for one with a triangle that will.
7. Find a formula for the *volume* of a *pyramid*.
8. How can the *midline theorem* help to find the baseline area?
9. If
$$\frac{1}{3}Bh = \frac{\sqrt{2(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)}}{12},$$
find the volume of your tetrahedron. Find both the exact value and an approximate decimal value.
10. On the cardboard triangle, record the measures for the perimeter and the area of the triangle and for the surface area and the volume of the pyramid.
11. Fold the white-paper triangle on the altitudes. Locate the *orthocenter*. Mark this point on the cardboard and use your compass to punch a hole there.
12. On one side of your cardboard triangle, write the measures from question 3. On the other side, draw midlines.
13. Match and glue the base of the blue tetrahedron to the central triangle you created by drawing the midlines in step 12.
14. Insert a toothpick through a hole at the orthocenter of the cardboard triangle and push it through, perpendicular to the cardboard. It should pass through the vertex of the pyramid.
15. Compare the surface areas and volumes of the pyramids as recorded on the class chart. What do you notice?

**Table 2**
**Triples of Positive Integers with a Sum of 36**

| | | | | |
|---|---|---|---|---|
| 1, 1, 34 | 2, 2, 32 | 3, 3, 30 | 4, 4, 28 | 5, 5, 26 |
| 1, 2, 33 | 2, 3, 31 | 3, 4, 29 | 4, 5, 27 | 5, 6, 25 |
| 1, 3, 32 | 2, 4, 30 | 3, 5, 28 | 4, 6, 26 | 5, 7, 24 |
| 1, 4, 31 | 2, 5, 29 | 3, 6, 27 | 4, 7, 25 | 5, 8, 23 |
| 1, 5, 30 | 2, 6, 28 | 3, 7, 26 | 4, 8, 24 | 5, 9, 22 |
| 1, 6, 29 | 2, 7, 27 | 3, 8, 25 | 4, 9, 23 | 5, 10, 21 |
| 1, 7, 28 | 2, 8, 26 | 3, 9, 24 | 4, 10, 22 | 5, 11, 20 |
| 1, 8, 27 | 2, 9, 25 | 3, 10, 23 | 4, 11, 21 | 5, 12, 19 |
| 1, 9, 26 | 2, 10, 24 | 3, 11, 22 | 4, 12, 20 | 5, 13, 18 |
| 1, 10, 25 | 2, 11, 23 | 3, 12, 21 | 4, 13, 19 | 5, 14, 17 |
| 1, 11, 24 | 2, 12, 22 | 3, 13, 20 | 4, 14, 18 | 5, 15, 16 |
| 1, 12, 23 | 2, 13, 21 | 3, 14, 19 | 4, 15, 17 | |
| 1, 13, 22 | 2, 14, 20 | 3, 15, 18 | 4, 16, 16 | |
| 1, 14, 21 | 2, 15, 19 | 3, 16, 17 | | |
| 1, 15, 20 | 2, 16, 18 | | | |
| 1, 16, 19 | 2, 17, 17 | | | |
| 1, 17, 18 | | | | |

| | | | | |
|---|---|---|---|---|
| 6, 6, 24 | 7, 7, 22 | 8, 8, 20 | 9, 9, 18 | 10, 10, 16 |
| 6, 7, 23 | 7, 8, 21 | 8, 9, 19 | 9, 10, 17 | 10, 11, 15 |
| 6, 8, 22 | 7, 9, 20 | 8, 10, 18 | 9, 11, 16 | 10, 12, 14 |
| 6, 9, 21 | 7, 10, 19 | 8, 11, 17 | 9, 12, 15 | 10, 13, 13 |
| 6, 10, 20 | 7, 11, 18 | 8, 12, 16 | 9, 13, 14 | |
| 6, 11, 19 | 7, 12, 17 | 8, 13, 15 | | |
| 6, 12, 18 | 7, 13, 16 | 8, 14, 14 | | |
| 6, 13, 17 | 7, 14, 15 | | | |
| 6, 14, 16 | | | | |
| 6, 15, 15 | | | | |

| | |
|---|---|
| 11, 11, 14 | 12, 12, 12 |
| 11, 12, 13 | |

The first day of this three-day project was devoted to investigative activities 1 through 6. Each student was given a copy of Table 2 and one of the 27 envelopes. Immediately, students found which triples of integers with a sum of 36 satisfy the triangle inequality; those triples were at the bottoms of the columns of Table 2.

Next, students focused on finding the area for the triangles in their envelopes. The textbooks index included three references for triangle area: the familiar one-half the product of the base and the height; a trigonometric formula; and Heron's formula, clearly the best choice for the situation. Surprisingly, the task of finding exact values for the area presented little difficulty; a semiperimeter of 18 produced partially factored radicands with recognizable squares. The values were written on the class chart, and few incorrect values were challenged by classmates. Calculators yielded approximate decimals.

For activity 3, each student was asked to classify his or her triangle as acute, right or obtuse. The students generally did so by inspection, then confirmed their results using the Pythagorean inequalities. Activity 4 asked the student to fold the triangle and then decide if a tetrahedron could be formed. As the class shared and summarized information (Table 3), students realized that only acute triangles can fold to form pyramids. At that point, students with nonacute triangles traded for acute ones so that everyone could make a model.

**Table 3**
**Cooperative Data-Gathering Project**

| Triple | Area Exact | Area Decimal | $c^2$ | | $a^2 + b^2$ | Type of Triangle | Pyramid? | Volume Exact | Volume Decimal |
|---|---|---|---|---|---|---|---|---|---|
| 2, 17, 17 | $12\sqrt{2}$ | 16.97 | 289 | < | 293 | acute | yes | $\frac{\sqrt{287}}{12}$ | 1.41 |
| 3, 16, 17 | $6\sqrt{15}$ | 23.24 | 289 | > | 265 | obtuse | no | | |
| 4, 15, 17 | $6\sqrt{21}$ | 27.50 | 289 | > | 241 | obtuse | no | | |
| 4, 16, 16 | $12\sqrt{7}$ | 31.75 | 256 | < | 272 | acute | yes | $\frac{2\sqrt{62}}{3}$ | 5.25 |
| 5, 14, 17 | $6\sqrt{26}$ | 30.59 | 289 | > | 221 | obtuse | no | | |
| 5, 15, 16 | $6\sqrt{39}$ | 37.47 | 256 | > | 250 | obtuse | no | | |
| 6, 13, 17 | $6\sqrt{30}$ | 32.86 | 289 | > | 205 | obtuse | no | | |
| 6, 14, 16 | $24\sqrt{3}$ | 41.57 | 256 | > | 232 | obtuse | no | | |
| 6, 15, 15 | $18\sqrt{6}$ | 44.09 | 225 | < | 261 | acute | yes | $\frac{3\sqrt{207}}{4}$ | 10.79 |
| 7, 12, 17 | $6\sqrt{33}$ | 34.47 | 289 | > | 193 | obtuse | no | | |
| 7, 13, 16 | $6\sqrt{55}$ | 44.50 | 256 | > | 218 | obtuse | no | | |
| 7, 14, 15 | $6\sqrt{66}$ | 48.74 | 225 | < | 245 | acute | yes | $\frac{\sqrt{2015}}{4}$ | 11.22 |
| 8, 11, 17 | $6\sqrt{35}$ | 35.50 | 289 | > | 185 | obtuse | no | | |
| 8, 12, 16 | $12\sqrt{15}$ | 46.48 | 256 | > | 208 | obtuse | no | | |
| 8, 13, 15 | $30\sqrt{3}$ | 51.96 | 225 | < | 233 | acute | yes | $\frac{5\sqrt{11}}{2}$ | 8.29 |
| 8, 14, 14 | $24\sqrt{5}$ | 53.67 | 196 | < | 260 | acute | yes | $\frac{8\sqrt{41}}{3}$ | 17.07 |
| 9, 10, 17 | 36 | 36.00 | 289 | > | 181 | obtuse | no | | |
| 9, 11, 16 | $18\sqrt{7}$ | 47.62 | 256 | > | 202 | obtuse | no | | |
| 9, 12, 15 | 54 | 54.00 | 225 | = | 225 | right | no | | |
| 9, 13, 14 | $18\sqrt{10}$ | 56.92 | 196 | < | 250 | acute | yes | $\frac{9\sqrt{71}}{4}$ | 18.96 |
| 10, 10, 16 | 48 | 48.00 | 256 | > | 200 | obtuse | no | | |
| 10, 11, 15 | $12\sqrt{21}$ | 54.99 | 225 | > | 221 | obtuse | no | | |
| 10, 12, 14 | $24\sqrt{6}$ | 58.79 | 196 | < | 244 | acute | yes | $2\sqrt{95}$ | 19.49 |
| 10, 13, 13 | 60 | 60.00 | 169 | < | 269 | acute | yes | $\frac{25\sqrt{119}}{12}$ | 22.73 |
| 11, 11, 14 | $42\sqrt{2}$ | 59.40 | 196 | < | 242 | acute | yes | $\frac{49\sqrt{23}}{12}$ | 19.58 |
| 11, 12, 13 | $6\sqrt{105}$ | 61.48 | 169 | < | 265 | acute | yes | $2\sqrt{146}$ | 24.17 |
| 12, 12, 12 | $36\sqrt{3}$ | 62.35 | 144 | < | 288 | acute | yes | $18\sqrt{2}$ | 25.46 |

The second day of the project began with an unscheduled discussion about what happens to area and volume if linear measure is doubled. One student had worked ahead, calculating volumes before she was told that the variables *a, b* and *c* in the formula represented sides of the base of the pyramid, not the sides of the original triangle. The numbers she had used were double what they should have been. She suspected that her volume measures were too large when she looked at the small models, and she challenged the formula. We showed algebraically that replacing each variable with its double in the formula would introduce an extra factor of 8. Thus, to correct her calculations, she divided the volumes by 8.

One student offered a proof to show that midlines of a triangle create four congruent triangles, each with an area equal to one-fourth the area of the original triangle. Thus, if the linear measures of a triangle are doubled, the area is quadrupled. This fact allowed students to find $B$ for the volume formula $V = 1/3Bh$. They could find one-fourth the triangle area that they had calculated for the generating triangle using Heron's formula. Several students questioned the factor 1/3 in the volume formula, and this factor was discussed. Of course, students still lacked a value for $h$, so they needed the formula in step 9 of Figure 3. The midline theorem referred to in step 8 follows:

> Midline Theorem. *If a segment joins the midpoints of two sides of a triangle, then it is parallel to the third side and its length is one-half the length of the third side.*

You may wish to let students seek out this theorem in their textbooks. Most work done on the second day was a group effort, and everyone managed to complete item 11.

On the third day, models were assembled, and toothpick altitudes emerged at the top vertices of the tetrahedra. Item 15 asked students to compare the surface areas and volumes of the different pyramids. This comparison was intended to evoke an optimization discovery, namely, that the equilateral triangle produced the tetrahedron with the greatest surface area and volume. However, most students responded with the observation that doubling the sides of a triangle causes the area to be four times as great and the volume to be eight times as great. The regular tetrahedron was admired for its symmetry and appreciated because it was the easiest model to make, but its maximum surface area and volume did not generate enthusiasm.

## Evaluation

The assessments of students' efforts to solve this problem revealed several points. First, my concern that the time used to investigate the problem would put the geometry class behind schedule was unfounded. Three extra days were added to the unit. This time investment was justified as students reviewed important concepts, applied new knowledge of triples, previewed area and volume formulas, and discovered variation concepts usually introduced in an advanced-algebra class.

Second, students learned. On the chapter test that included this project, they showed increased proficiency in simplifying radicals. In addition to recognizing certain triples as Pythagorean, they identified triangles and classified them by angle size. On other test items, they responded as well as previous classes who had not covered the extra material.

Third, insights came from students' written responses to the question "What did you learn from this unit?" One student mentioned learning that several ways can be used to find triangular area. Another reported that he learned a use for the triangle's orthocenter. Most textbooks give reasons for locating the incenter, the circumcenter and the centroid, but the purpose of the orthocenter is unclear. He discovered that it is the place to put the toothpick altitude. Students who had received the troublesome triples mentioned learning that only acute triangles will form tetrahedra. The student who needed to divide her answers by 8 related that she learned the importance of defining variables and of estimating measures.

Fourth, the project generated new questions. Complaining that he received a nonacute triangle that was exchanged for an acute triangle whose tetrahedron ended up with the toothpick altitude on the outside, one student asked, "What are the odds of that happening?" We counted up the cases for perimeter of 36 to decide. However, if we consider the possibilities for any perimeter, we are off on another adventure. The questions did not end at item 15 in the investigation sheet.

## Extensions

Continued explorations of the perimeter-volume connection are aided by computer-generated data for different perimeters. An interactive program that counts triples, triangles and tetrahedra and calculates areas and volumes helps with the planning for a geometry project involving a perimeter other than 36. Table 4 summarizes the results for a perimeter of 17. The program can also be used by students to investigate optimization questions. Readers interested in receiving this program can send to the author a stamped, self-addressed envelope and a note requesting a printout.

The derivation of the volume formula can be presented to students in analytic geometry. The steps are shown in the Appendix.

Open-ended investigations often generate questions beyond the students' current ability level, requiring teachers to decide whether to pursue or postpone extension activities. One factor in that decision should be the directive regarding core curriculum found in the *Curriculum and Evaluation Standards* (NCTM 1989). All students should be given a chance to understand the content topics and should be challenged as much as possible. As teachers, we recognize that students operate at different levels of motivation, comprehension and skill. However, we must remember that it is difficult for them to advance to higher levels if they never look beyond their present stages.

Exploring the pyramid question with the students may lead the teachers and the class into investigations that veer off in surprising directions. The problem-solving approach to mathematics is a learning adventure waiting to happen.

**Table 4**
**Value for a Perimeter of 17**

| Triple | Area of Triangle | Volume of Tetrahedron |
|---|---|---|
| (1, 8, 8) | 3.99 | 0.17 |
| (2, 7, 8) | 6.44 | — |
| (3, 6, 8) | 7.64 | — |
| (3, 7, 7) | 10.26 | 1.25 |
| (4, 5, 8) | 8.18 | — |
| (4, 6, 7) | 11.98 | 1.14 |
| (5, 5, 7) | 12.50 | 0.72 |
| (5, 6, 6) | 13.64 | 2.52 |

If integral values are chosen for the sides of triangle *ABC* with the perimeter equal to 17, then the maximum area of the triangle is 13.64, which occurs with the triple (5, 6, 6), and the maximum volume for the tetrahedron is 2.52, which occurs with the triple (5, 6, 6).

The total number of triples for a perimeter of 17 is 24. The total number of triangles for a perimeter of 17 is 8. The total number of tetrahedra for a perimeter of 17 is 5.

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National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

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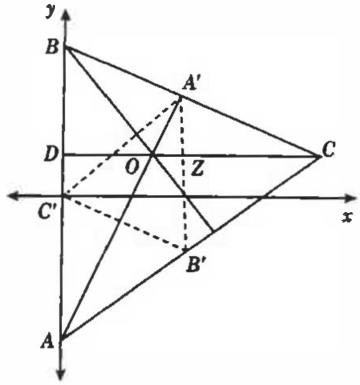
## Appendix
## Derivation of Tetrahedron Volume Formula

I. Coordinates are assigned to key points.




Given:
Acute $\Delta ABC$ with sides $2a$, $2b$, $2c$
$c > b > a$
$B'$, $A'$ and $C'$ are midpoints of the sides.
$O$ is the orthocenter.
$\overline{CO}$ and $\overline{A'B'}$ intersect at $Z$.
Area of $\Delta ABC = K$.
$A$: $(0, -c)$; $B$: $(0, c)$; $C'$:$(0, 0)$

To find the coordinates of $C$: Let $C = (x, y)$. Point $C$ lies on a circle $A$ (radius $2b$) and on circle $B$ (radius $2a$).

$$\left.\begin{aligned} x^2 + (y + c)^2 &= 4b^2 \\ x^2 + (y - c)^2 &= 4a^2 \end{aligned}\right\} \Rightarrow 4yc = 4(b^2 - a^2) \Rightarrow y = \frac{b^2 - a^2}{c}$$

$\Delta ABC$ has a base $2c$, height $x$, so $x = K/c$. Therefore, $C$ has coordinates $(K/c, (b^2 - a^2)/c)$.

To find the coordinates of $O$:

Equation of $\overline{CO}$:
$$y = \frac{b^2 - a^2}{c}$$

Equation of $\overline{AO}$: $y = \dfrac{-Kx}{a^2 - b^2 - c^2} - c$

$$x = \frac{Q}{-cK}$$

Equation of $\overline{BO}$: $y = \dfrac{-Kx}{a^2 - b^2 + c^2} + c$

where $Q = a^4 + b^4 - c^4 - 2a^2b^2$. Therefore, $O$ has coordinates $(Q/-cK, (b^2 - a^2)/c)$.

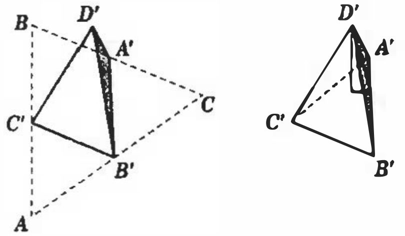
To find the coordinates of $Z$:

$Z$ lies on $\overline{OC}$:
$$y = \frac{a^2 - b^2}{c}$$

$Z$ lies on $\overline{A'B'}$:
$$x = \frac{K}{2c}$$

Therefore, $Z$ has coordinates $(K/2c, (b^2 - a^2)/c)$.

II. The height of the tetrahedron is calculated. Applying the distance formula yields $CZ = K/2c$ and $ZO = (K^2 + 2Q)/2cK$: The folds on the midlines of the $\Delta ABC$ bring the vertices to a common point $D'$.



The altitude of the tetrahedron, $\overline{D'O}$, is found using the Pythagorean theorem:

$$(D'O)^2 = (D'Z)^2 - (ZO)^2$$
$$= (CZ)^2 - (ZO)^2$$
$$= \left(\frac{K}{2c}\right)^2 - \left(\frac{K^2 + 2Q}{2cK}\right)^2$$
$$D'O = \frac{\sqrt{-Q(K^2 + Q)}}{Kc}$$

III. The volume formula is a function of $a$, $b$ and $c$. Applying the formula for the volume of a pyramid to find the volume of a tetrahedron A'B'C'D' yields

$$V = \frac{1}{3}\left(\frac{K}{4}\right) \frac{\sqrt{-Q(K^2 + Q)}}{Kc}$$
$$Q = a^4 + b^4 - c^4 - 2a^2b^2.$$

Applying Heron's triangle-area formula to find $K$ gives the area of $\Delta ABC$ as

$$\frac{K}{4} = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}$$
$$= \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{4}$$

$$K = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$
$$= \sqrt{[(b+c)^2 - a^2][a^2 - (b-c)^2]},$$
$$K^2 = a^2(b+c)^2 - (b^2 - c^2)^2 - a^4 + a^2(b-c)^2$$
$$= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - b^4 - c^4 - a^4,$$
$$K^2 + Q = 2a^2c^2 + 2b^2c^2 - 2c^4$$
$$= 2c^2(a^2 + b^2 - c^2),$$
$$Q = a^4 - 2a^2b^2 + b^4 - c^4$$
$$= (a^2 - b^2)^2 - c^4,$$
$$-Q = c^4 - (a^2 - b^2)^2$$
$$= [c^2 - (a^2 - b^2)][c^2 + (a^2 - b^2)]$$
$$= [-a^2 + b^2 + c^2][a^2 - b^2 + c^2],$$
$$V = \frac{1}{12c}\sqrt{-Q(K^2 + Q)}$$
$$= \frac{1}{12c}\sqrt{(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)2c^2(a^2 + b^2 - c^2)}$$
$$= \frac{\sqrt{2}}{12}\sqrt{(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}.$$

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**Sun and Moon**

The distance between the sun and the earth is 387 times greater than the distance from the moon to the earth. How many times greater is the volume of the sun as compared to the moon?

# Increasing Mathematics Confidence by Using Worked Examples

*William M. Carroll*

An important way to help our students learn and strengthen their knowledge of mathematics is by studying already-worked examples of problems. A considerable amount of evidence suggests that analyzing worked-out problems can be as conductive to learning as actually *solving* practice problems and that it may be more effective in helping students to recognize underlying similarities *between* problems (Sweller 1989; Ahu and Simon 1987; Sweller and Cooper 1985). Well-constructed worked examples do more than merely teach rote procedures; they illustrate mathematical principles and classes of problems.

However, worked examples are not always used effectively in instruction. Text formats vary widely, and some examples are difficult to follow (Ward and Sweller 1990). A few worked examples are generally presented a page or two ahead of practice problems, sometimes surrounded by unnecessary or difficult text. In some books, several sample problems are posed simultaneously, followed by their solutions. Because students need to search back and forth between the initial problem and the corresponding solution, this split format may interfere with learning rather than facilitate it. Whereas high-achieving students may be able to extract the important underlying mathematical principles from a few examples and then apply these principles during practice and problem solving, many others may have difficulty using a few examples isolated from problems, especially as they try to match problems to appropriate examples (Chi et al. 1989). Many students have told me that they are aware of the worked examples that precede the practice problems in their books and have difficulty relating a problem to an appropriate example. Many students have given up trying to use examples.

As a mathematics teacher, I was impressed with the results of previous studies in which worked examples were used to teach algebra to average or above-average students (Sweller 1989; Zhu and Simon 1987). As a teacher in an urban high school, I was interested in how a more extensive use of worked examples could be used to help my at-risk students. Many of the students enrolled in my algebra classes have a history of failure in mathematics and negative feelings and beliefs about their own ability. A number of students are diagnosed as having a learning disability or some other special need. Although these students might have difficulty using worked examples effectively (Chi et al. 1989), it also seemed likely that they would have much to gain from having these examples to support their learning and to help them focus on important features of a problem and its solutions. By giving sufficient meaningful examples, I hoped the students would have the necessary support to work independently at home, which in the process would lift their self-esteem concerning mathematics.

Over the past two years, I have used a worked-example format to introduce or reinforce various topics in algebra and other classes. Though not the only means of instruction, worked examples were used throughout the year on various topics, from writing and solving equations to using algebra in geometry.

## Modeling Effective Use of Worked Examples

When I first began using worked examples as an instructional tool, many of my students had difficulty using them effectively. It was clear that many viewed mathematics as merely solving a group of problems with little regard for understanding. Although I asked students to spend time studying each example until they understood it, students often skipped the examples and went directly to the practice problems that followed, even when they did not know how to solve them. If they did not know how to solve a problem immediately, many would skip it or rely on the teacher for guidance.

Because one of my goals in using worked examples was to help students become more independent learners who relied less on me and more on themselves, I found it helpful in the beginning of the year to model a strategy for using worked examples. To demonstrate such a strategy, a worked example followed by similar problem or two were placed on the overhead projector or chalkboard. I asked the class to (1) study the example for understanding in a step-by-step manner; (2) make some statements about what is taking place in the example, for instance, "Oh, I see. A variable is being used for the unknown number"; (3) proceed to the practice problem when the example is understood, referring back to the example as necessary; and (4) consider the answer in the practice problem, relating it back to the example if necessary. Not all students would *always* use this approach, but it was a place to start.

Some students did not see the point in studying a problem that was already worked out, so I also gave sets of worked-out problems in which errors occurred at various steps. Students had to identify the errors and correct the problems. Besides familiarizing students with examining worked-out problems for real understanding, I hoped that this task would also help them to remember to check their own work for errors. Two weeks of systematically modeling the use of worked examples and strategies was sufficient in my algebra classes, although some students needed occasional reminders to slow down and use the examples for understanding.

## Worked-Example Formats

A typical format would show several worked examples on a page for students to study. Two to four practice problems generally followed each example, depending on the topic's complexity. To assist students, examples were matched to similar problems initially (Figure 1). As a topic was being practiced, fewer examples were presented or they were completely eliminated. Worked examples also helped students review a mix of previous topics with minimal instruction needed.

**Figure 1**

Examples show original format, although generally more problems were given per example.
Study each worked example. Then write an equation for the problem that follows.

1. Five less than a number is seven.
   5 less than a number — is 7
   $x - 5$ $= 7$
2. Three less than a number is six.
3. Six equals a number increased by three.
   6 equals — a number increased by 3
   $6 =$ $x + 3$
4. Two equals a number increased by twelve.
5. Twice a number is negative six.
   Twice a number — is −6
   $2x$ $= -6$
6. Twice a number is fourteen.
7. Negative six increased by a number is four.
   −6 increased by a number — is 4
   $x + (-6)$ $= 4$
8. Ten increased by a number is a negative six.

Many homework assignments also followed a worked-example format, especially on difficult topics. I hoped that by having meaningful worked examples to study along with their homework, students who had difficulty at home would be more likely to persevere. This is especially important if students are to realize that they can be successful in mathematics and problem solving.

## Benefits of Using Worked Examples

Most of my students became proficient at using worked examples effectively. Students were more likely to return correct homework when they had examples as an added support. The at-risk students, who were of particular concern, profited the most. For instance, one learning-disabled student who had a high level of mathematics anxiety and a history of failure in mathematics became quite adept at solving problems, often helping other students around him, by using the examples. Another learning-disabled student showed marked improvement on classwork, homework and tests during lessons in which he had been given worked examples. The improved performance of these two students was fairly typical of at-risk students in my algebra classes. Students who were learning English seemed to learn more quickly and remember the relationship between English words and mathematical symbols better when using a worked-example format.

The improvement of the students during practice was not merely owing to rote copying of the worked examples. Most of them did much better on tests after using the examples even though they were not available during the test. Obviously more than rote learning had occurred.

Several reasons are possible why many students, especially the low achievers, gained from using these examples. First, many of them persisted longer both at home and in class because they had the worked examples for support when memory or understanding failed, which was evident by fewer problems being left undone. Second, the idea that they could study a problem, see a relationship or rule, and apply this understanding independently was highly motivating for many of them and affected their beliefs concerning mathematics and their own abilities. Many students who would typically wait for the teacher when they ran into difficulty found they could help themselves. Furthermore, it was challenging to try to understand an example and apply it rather than simply solve a set of problems just to get through the assignment. Finally, having worked examples may prohibit the development of faulty rules and procedures and help students see similarities between types of problems. In fact, some evidence exists that students who study worked examples are quicker to recognize similarities and correct solutions for classes of problems than are those who merely practise sets of problems (Sweller 1989).

When students used worked examples, they were often working in small groups, taking part in mathematical conversions as they assisted each other and relying on their own resources when they had difficulty. Because less time was generally needed for direct instruction and practice, since we usually had fewer practice problems, more time could be spent on discussions, student explanations, problem-solving activities or various mathematical activities, including finding alternative methods for solving problems. Discussions of this worked-examples strategy led to similar discussions about problem solving, alternative solutions and the use of other heuristics.

Using worked examples need not promote the idea that only one correct way is possible to solve a problem. However, it is hoped that students will be more adept at using a successful problem-solving strategy of finding and studying a good example as a useful way to understand an underlying principle or rule. A more extensive use of worked examples, integrated with various other sound instructional formats as suggested by NCTM's *Curriculum and Evaluation Standards* (1989), can be a good way to help all students achieve success and independence as they attempt to construct meaning for their mathematical tasks.

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**As Fast as the Wind**

A cyclist won a 1,000 m race. How many rotations did each wheel make around its own axis, when the diameter of the wheel is 685.8 mm?

# Let's Solve the Problem Before We Find the Answer

*Carolyn F. Talton*

Ask a roomful of elementary mathematics teachers where their students have the most trouble, and the chorus of "problem solving" is overwhelming. Teachers reply this way because it is true. Study after study continues to cite the area of problem solving as the number-one concern in the classrooms of America as was recently well documented (Suydam 1982, 56–60). Some students who can perform arithmetical computation quite well often have problems that require the application of those same computational skills. Why is this? What is so difficult about solving word problems? The answer may lie in the discrepancy that exists between the skills taught as problem-solving skills and the critical thinking skills actually needed to solve word problems.

## Problem-Solving Skills Taught in the 1960s and 1970s

In the 1960s and 1970s, many textbook authors suggested that students do the following to solve word problems:

1. Read the problem.
2. Determine what is asked in the problem.
3. Determine what facts are given.
4. Choose the operation.
5. Solve the problem.

Which step did your students find difficult? Was it step 1, "read the problem"? No, the National Assessment of Educational Progress (NAEP 1979) found through extensive research what most classroom teachers already knew, that even when problems were read aloud to students, those who were experiencing difficulties continued to find word problems difficult.

Was step 2, "determine what is asked in the problem," hard for your students? Most of us would agree that it was not. Usually the question was presented last and stated in such a way that students could readily find what must be answered in the problem. Besides, a question mark usually came with a question.

What about step 3, "determine the facts given." Couldn't your students usually pick out the facts from the problem? The main difficulty with this step in problem solving often involves finding *too many* facts as in the number "3" in the following problem:

> Jerry and Douglas went fishing 3 days last week and together they caught 12 white perch, 6 bass and 2 catfish. How many fish did they catch in all?

Do your students want to write "3 + 12 + 6 + 2 = ______" as the number sentence? Many do. Yes, students can find facts in word problems. However, they often experience difficulty eliminating unnecessary facts. Steps 2 and 3 are probably easy for most students because only recall at the knowledge level of thinking is required.

Step 4, "choose the operation," is the main area of difficulty in the preceding strategy, isn't it? Have you ever thought why this step is so difficult for many students? You have probably said, "Students won't *think* about the word problem, they just want to find an answer whether it's right or wrong."

## Problem-Solving Skills Taught in the 1980s

In the 1980s, owing largely to research by Polya (1973, 8–10) and others (Carpenter et al. 1980; Herrington 1980; Marcucci 1980; Moser 1980; NAEP 1979; Zweng, Turner and Geraghty 1979), textbooks have begun to provide a more detailed plan for problem solving that is designed to promote higher-order thinking skills. The plan is similar to the following:

1. Read.
2. Plan—make a table, think backward, apply logic, draw a diagram, work a simpler problem, choose the operation, guess and test, and so on.
3. Solve.
4. Check.

Has this plan enabled more of your students to be more successful problem solvers? The additional emphasis on making tables, drawing pictures, working backward, guessing and testing and so on, will undoubtedly enable students to view problem solving more positively, but will these strategies enable students to "determine the operation" in the traditional word problem more easily? If this skill is still a major area of difficulty for some of your students, you might like to try the procedure described in this article. The procedure involves teaching students to read word problems and then make a determination about the number(s) of groups involved. Once this determination is made, emphasis is placed on the question in the word problem to determine which of three actions is required for solving the problem. Suggestions are included to use as a guide for teaching the three actions, and diagrams of posters and bulletin boards are included to teach the process in your classroom.

## Analyzing Three Important Actions in Routine Word Problems

Word problems, both one-step and multistep, are variations of three basic actions: combine, separate and compare. In the primary grades, when two or more groups are to be combined or joined, the operation to be applied is only *addition* (see problem 1, Figure 1). In the middle and upper grades, however, before combining groups, we have to look at the types of groups to be combined. If the groups are *equal* in number, the operation of *multiplication* can be used (see problem 2, Figure 1).

**Figure 1**

**Typical routine word problems, primary through upper grade levels.**

1. May picked 3 flowers. Sue picked 5 flowers. How many flowers were picked in all? (3 + 5 = 8)
2. Mother bought 3 packages of cookies for the club meeting. If each package contained 12 cookies, how many cookies did she buy? (3 × 12 = 36)
3. Fifteen children were playing basketball on the playground. Seven of the children went home. How many stayed to play? (15 – 7 = 8)
4. Fifty-six colored eggs were hidden during the egg hunt. If 8 children each found the same number of eggs and all of them were found, how many did each child find? (56 ÷ 8 = 7)
5. Susan is 12 years old and her brother, Bill, is 9 years old. How many years older is Susan than her brother Bill? (12 – 9 = 3)
6. There were 22 cars parked in the restaurant parking lot. Five of the cars were blue, 7 were black, and 10 were red. Compare the number of cars:
   a. black to blue (7 to 5 or 7:5)
   b. blue to red (5 to 10 or 5:10) simplify to (1:2)
   c. red to total (10 to 22 or 10:22) simplify to (5:11)

When a word problem contains only one group, the only action that can be taken on the group is to separate it into parts. In the primary grades, only *subtraction* is used to separate a group into parts (see problem 3, Figure 1). By the middle grades, however, if we know that the group must be separated into equal parts, we should divide (see problem 4, Figure 1).

The third action, compare, is sometimes necessary when the word problem contains two or more groups. In the primary grades, we are asked only to compare groups to *find a difference*. Number 5 of Figure 1 is an example of this problem type, and subtraction is the choice of operation to find the answer. In the upper grades, however, word problems may involve two or more groups that must be compared to *find a ratio*. Number 6 in Figure 1 is an example of this problem type. Sometimes, as in parts b and c, *division* is necessary to simplify the ratio.

## Teaching Students to Understand the Three Actions

After a word problem has been read, students must reflect on the problem and make a very important decision regarding the question "How many groups are in the problem?" If counters such as beans or craft sticks are available, ask students to model the groups on their desks using the counters. This is a very important step in solving word problems so that children can understand the situation being presented. This step will help stop students from trying to manipulate the numbers quickly to find an answer. In this step, we are trying to get students to think about solving the situation presented in the word problem before finding an immediate answer.

After modeling the groups, reread the question in the problem to determine what action must be taken on the group or groups. If two or more groups must be joined to find a total, have students move their counters together forming one group and say, "We have *combined* the groups." If only one group of objects had been in the word problem and the action required part of the group to be taken away, have the children show the one group with counters and then take away the appropriate number as stated in the word problem. Say, "We have *separated* the group into parts." If two or more different groups of objects were present in the word problem and the action required comparing the groups to find a difference, model the groups using several colors of counters to distinguish between the groups. Allow students to match the groups being compared in one-on-one correspondence and count the number left over. Say, "We have *compared* the groups to find a difference."

Primary-grade students can readily learn the words *combine, separate* and *compare* because these terms represent real-life experiences they have all had in their environment. To develop the word meanings, try these classroom activities in the primary grades where only addition and subtraction are taught.

## Combine

1. Ask everyone wearing a red shirt to stand. Count the number and write it on the chalkboard. Then, ask everyone wearing a blue shirt to stand. Count only the students with blue shirts and write the number on the chalkboard. Ask the class to find the total number of children standing by combining or joining the two groups. For example, if three students wore red shirts and four wore blue, write $3 + 4 = 7$ on the chalkboard. The total number of children standing is seven.

2. Give each student a paper cup containing a few beans or grains of rice. Ask four students to go to the chalkboard, count the number of objects in their cups and write the numbers on the chalkboard. Ask them to place all their beans in one cup and emphasize that the objects have been combined. Write the appropriate number sentence on the chalkboard. Repeat the activity by calling on various numbers of students to combine their beans. Draw the generalization that two or more groups can be combined by addition.

## Separate

1. Place six objects on the overhead projector and call on a student to take away four of them. Say, "We separated, or took away, four objects from the group of six. How many are left?" Write $6 - 4 = 2$ on the overhead.

2. Ask all the boys in the class to stand. Count the number in the group. Ask the boys who are wearing shoes with laces in them (such as tennis shoes) to walk to the front of the room. Emphasize that the number of boys with laces in their shoes was separated from the group of boys standing. Write the appropriate number sentence on the chalkboard. In both these examples, it is important to emphasize that when we have one group in a problem situation, the only action that can be taken on the group is to separate it into parts. At the primary-grade levels, subtraction is the operation to use when separating one group into parts.

## Compare

1. Ask all the girls to walk to one side of the classroom. Count them. Ask the boys to walk to another side of the classroom and count them. The children will be able to determine quickly which of the two groups has more or fewer students. Emphasize that if we wanted to know how many more boys than girls, or vice versa, we would be comparing to find a difference between the groups. We would subtract to find the answer.

2. Place three star-shaped cutouts, five circles and seven squares on the overhead projector. Ask, "How many fewer circles are there than squares?" To demonstrate comparing, ask students to match the circles and squares in one-on-one correspondence on the overhead and count the number of squares left over. Explain that subtraction is an easier way to compare two or more groups to find a difference between groups. Write $7 - 5 = 2$ on the chalkboard or overhead and say, "Subtract to compare two or more groups to find a difference."

Students in the middle and upper grades should be encouraged to follow much the same format as suggested for primary students so that a clear understanding of the three actions can occur. However, they must consider an important step. After reading a word problem, students must again determine the number of groups in the problem, model the groups with counters and determine the action that must be taken with the group or groups. If it has been determined that the problem contains several groups to be combined, ask the students to determine if the groups are equal or unequal in size. If they are of equal size, multiplication is the choice of operation to solve the problem. If the groups are unequal, they will continue to add. If the word problem contained only one group, students must now give consideration to the size of the parts to be separated from the group. If the group is to be separated in such a way that the resulting parts will be of equal size, division is the operation choice. If the problem only required that a part be separated from the group, subtraction will continue to be used. The last problem type to be introduced in the middle and upper grades is a type of comparison problem. If it has been determined that two or more groups must be compared, we must determine if we want to find a difference using subtraction as in the primary grades or if we want to find a ratio between the groups. If a ratio is required, division is often needed to simplify, or express the ratio in lower terms. Try these classroom activities to practise the three new problem types in the middle and upper grades.

## Combine Equal Groups

1. Count the number of windows in the classroom. Then count the number of window panes in each window. Suppose six windows were in the classroom

and each had eighteen panes of glass. To find out the total number of window panes, multiplication would be the appropriate solution operation, since each window contained an equal number of panes. Write 6 × 18 = 108 on the chalkboard. Emphasize that multiplication is appropriate because each window contained and equal number of panes. Say, "Multiply to combine equal groups."

2. Use the overhead projector to illustrate setting up chairs for a concert. If twelve rows were set up and contained twenty chairs each, find the total number of chairs in each of the twelve rows. Instead of adding, we can multiply because each of the twelve groups contains an equal number of chairs.

Although the operation of addition will still find the answer, emphasize that *multiplication* is a more appropriate operation to use, as we are combining equal groups.

### Separate into Equal Parts

1. Use thirty craft sticks to represent flowers and five small boxes to represent vases. Separate the thirty "flowers" into the five "vases" so that the same number of flowers is in each vase. Since the problem states that each vase will contain the same number of flowers, this is an example of separating a group into equal parts. Write 30 ÷ 5 = 6 on the chalkboard and say, "Divide to separate one group into equal parts."

2. Set up three tables of recreational reading materials in the back of the classroom. Separate the students so that an equal number of students is at each table. More than likely, one or more students will be left over. Allow these students to read independently at their desks. Write the appropriate number sentence on the chalkboard. For example, if thirty-two students were in the classroom, write 32 ÷ 3 = 10 with a remainder of 2. Therefore, ten students would select a book from one of the three tables and two students would read at their own desks. Say, "Divide to separate a group into equal parts."

### Compare to Find a Ratio

1. Ask all the girls to stand. Count them and write the number on the chalkboard. Ask them to sit down and have the boys stand. Count them and write that number on the chalkboard. Express the numbers in a ratio. For example, if seventeen girls and fifteen boys were in the classroom, the ratio of girls to boys would be 17 to 15, or 17:15. Since 17 and 15 have no common factors greater than 1, the ratio is in its simplest form.

2. Count the number of students in the classroom who have blond hair and write the number on the chalkboard. Count the number who have brown hair and write that number on the chalkboard. If the numbers were eight with blond hair and twelve with brown hair, the ratio of blond to brown would be 8 to 12, or 8:12. Since 4 is a common factor of 8 and 12, we must simplify the ratio to 2 to 3. Say, "Divide to express a ratio in its simplest terms."

## Classroom Teaching Aids for Three Actions

Figure 2 summarizes the question model that would be appropriate to teach when addition and subtraction are the only operations that have been taught

**Figure 2**

**Primary grades question model for routine, one-step word problems**

READ the word problem.

THINK:

- Add to COMBINE two or more groups.
- Subtract to SEPARATE one group into parts.
- Subtract to COMPARE two or more groups to find a difference.

WRITE the number sentence.

SOLVE the sentence.

CHECK your answer.

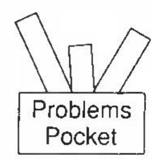
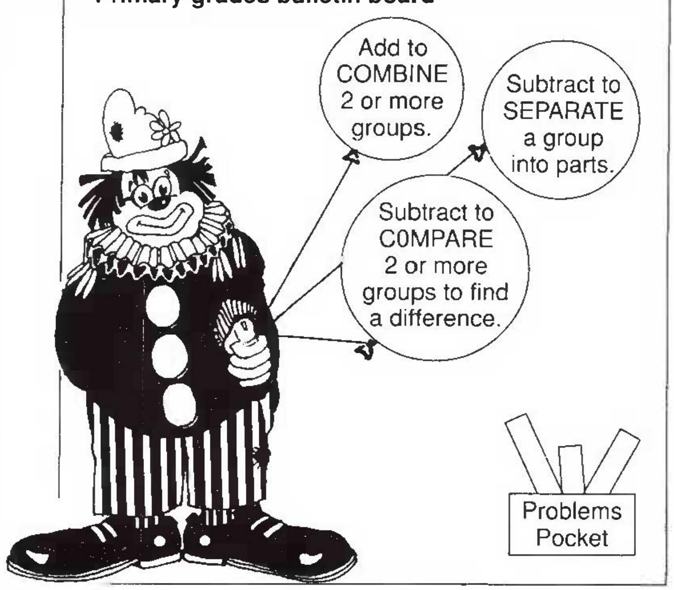



**Figure 3**

**Primary grades bulletin board**



to the students. Example problems are written on cards and placed in a "problem pocket." Students select a card and follow the algorithm outlined on the poster. After reading the problem, the student must decide the number of groups and what action is required for the groups. Then, the student writes the number sentence and solves it to find the answer to the problem. The student places the card in the appropriate pocket after solving it. This activity allows students to become actively involved with solving word problems. Young children enjoy the independence of this activity and sharing their results with the class. The answer could be written on the back of the card so that the activity is self-correcting.

Along the same lines as the poster illustrated in Figure 2, Figure 3 illustrates a bulletin board with the same directions. Examples of word problems are written on cards and placed in the appropriate balloon pocket held by the clown. Again, children become actively involved in thinking about the actions—combine, separate, compare—that are necessary to solve the problem. In a way, children are solving the problems by making the determination about actions on the groups involved before trying to find the answers.

Figure 4 is a classroom poster or bulletin-board idea appropriate in the middle and upper grades. Again, problems are written on small cards and students place them in pockets that match the problem types. As students will see from this activity, two occasions arise to use subtraction and division, but only one arises to apply the operations of addition and multiplication.

**Figure 4**

**Middle and upper grades question model for routine, one-step word problems**

READ the problem.

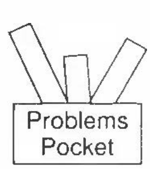
THINK:

- Add to COMBINE 2 or more unequal groups.
- Multiply to COMBINE 2 or more equal groups.
- Subtract to SEPARATE one group into unequal parts.
- Divide to SEPARATE one group into equal parts.
- Subtract to COMPARE 2 or more groups to find a difference.
- Divide to COMPARE and simplify a ratio.

WRITE the number sentence.

SOLVE the sentence.

CHECK your answer.




## Summary

Using the outlined question model and suggested activities, elementary students will improve their abilities to analyze routine, one-step word problems and make a plan for the solution. The same algorithm can be applied to multistep problems by asking the questions listed on the teaching posters. This question algorithm will enable students to decide which operation—addition, subtraction, multiplication or division—will "solve" the word problem. Students can then use their arithmetic skills to "find the answer."

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# Western Canadian Protocol: The Common Curriculum Framework (K–12 Mathematics)

*Hugh Sanders and Gina Vivone-Vernon*

## What Is Protocol?

Curriculum development in western Canada has been done in the context of two protocol agreements that have been signed by the ministers of education in Canada. These agreements provide the authority and direction for the various projects to be undertaken.

## Western Canadian Protocol

In December 1993, ministers of education from the four western provinces and two territories (Manitoba, Saskatchewan, Alberta, British Columbia, the Yukon Territory and the Northwest Territories) signed the Western Canadian Protocol for Collaboration in Basic Education, Kindergarten to Grade 12 (WCP). This agreement encourages and enables the participating jurisdictions to work more closely together because of the importance they place on

- common education goals,
- high standards in education,
- removing obstacles for student access to educational opportunities, which includes improving the ease of transfer from jurisdiction to jurisdiction, and
- optimum use of educational resources.

The WCP covers six major areas: curriculum, distance learning and technology, special education, student assessment and standards of student performance, aboriginal education, and teacher preparation and certification. There is also provision in the agreement for launching other types of cooperative projects in the future.

The WCP group has agreed to develop common curriculum frameworks of general and specific outcomes—what students will be expected to know and be able to do—for mathematics and English language arts. They will also be working together on the review of learning resources.

Alberta is playing a lead role in three of the projects: mathematics curriculum for Kindergarten to Grade 12, curriculum in languages other than English and French, and distance learning and technology. We are also participating in the language arts project for Kindergarten to Grade 12, which is being coordinated by Manitoba.

## Pan-Canadian Protocol

After discussions in 1994, a general framework for collaboration, the Pan-Canadian Protocol for Collaboration on School Curriculum, was adopted in February 1995 by the Council of Ministers of Education, Canada.

Recently, ministers of education have initiated several joint projects to develop common frameworks for school programs. Work has begun on the development of the pan-Canadian science project, which will potentially involve all Canadian provinces and territories, except Quebec.

## Framework Development for K–12 Mathematics

The development of the common curriculum framework for mathematics commenced with the first interprovincial writing in August 1994. Since then, 64 educators from six participating jurisdictions have developed the framework. It is important to note that the participants represented practising teachers (K–12, as well as some postsecondary instructors) and curriculum specialists from the various departments of education.

In Alberta, The Alberta Teachers' Association nominated teachers to the Alberta team, as did Alberta Education.

The development of the K–12 framework took place in two phases. The first phase ended in June

1995 with the distribution of the K–9 component. The second ended in June 1996 with the distribution of the 10–12 component.

## The Framework

The Common Curriculum Framework for mathematics does several things:

- It has grade-level student outcomes organized as
  - general outcomes,
  - specific outcomes and
  - illustrative examples.
- It identifies four strands within which all student outcomes are organized.
- It has a K–12 focus.
- It identifies key mathematical processes that are critical elements affecting student learning.
- It discusses the nature of mathematics.

### Student Expectations

The content of the common curriculum framework is stated in terms of outcomes. These outcomes are measurable and identify what students are required to know and do.

The outcomes are developed and based on the expectation that they are appropriate to a large majority of the students. Outcomes are stated at the level where they are expected to be "mastered." There may be some time delays between where students first encounter the learning and where they are expected to demonstrate knowledge of, or mastery in, that learning.

#### *General Outcomes*

General outcomes are general statements that identify what students are expected to know and be able to do on completion of a grade.

#### *Specific Outcomes*

Specific outcomes are statements identifying the component knowledge, skills and attitudes of a general outcome.

#### *Illustrative Examples*

Illustrative examples are sample tasks that demonstrate and elaborate on the general and specific outcomes. They are important in conveying the richness, breadth and depth intended in outcomes.

### Four Strands

All student outcomes are organized into four strands. Regardless of grade level, there are student outcomes in each of the following strands:

- Number
- Patterns and Relations
- Shape and Space
- Statistics and Probability

## K–12 Focus

The design of the common curriculum framework focuses on mathematics as a K–12 program rather than as a subset of this, which has been the practice historically.

There are two major sections of the document that show this focus. The general outcomes are presented on four two-page spreads, one for each strand. The specific outcomes in the K–9 component are also presented so that the reader can easily see the student outcomes across multiple grades.

### Mathematical Processes

The seven mathematical processes that are identified are critical elements that students must encounter in a mathematics program to achieve the goals of mathematics. Students are expected to

- communicate mathematically;
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines;
- use estimation and mental mathematics where appropriate;
- relate and apply new mathematical knowledge through problem solving;
- reason and justify their thinking;
- select and use appropriate technologies as tools to solve problems; and
- use visualization to assist in processing information, making connections and solving problems.

The common curriculum framework incorporates these seven related mathematical processes that are intended to permeate teaching and learning.

### Nature of Mathematics

By enriching our view of mathematics and the learning environment, the outcomes of the common curriculum framework can be accomplished.

The brain is constantly looking for and making connections. "Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from when learners extract understanding. . . . Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching" (Caine and Caine 1991, 5).

There are additional critical components that must be addressed in a mathematical program beyond those listed as mathematical processes. The components discussed are Pattern, Number, Shape, Change, Constancy, Dimension (size and scale), Relationships, Quantity and Uncertainty. They are used to describe mathematics in a broad way to establish the variety of connections that can be made among the various strands.

### Framework Summary

The components of the conceptual framework for K–12 mathematics, as described, dictate what should be happening in mathematics education. The components are not meant to stand alone but are to be related to enhance one another. Activities that take place in the classroom should stem from a problem-solving approach built on the *mathematical processes* and lead students to an understanding of the *nature of mathematics* through specific knowledge, skills and attitudes related to each strand.

## 1996 Annotated Bibliography of K–9 Mathematics Resources

The annotated bibliography of the K9 mathematics resources identifies the English language resources endorsed by and common to all WCP jurisdictions implementing the common curriculum framework. The bibliography also identifies authorized Alberta resources that are not listed in earlier annotated bibliographies.

The WCP resources in the annotated bibliography were selected through a collaborative review process, based on their high level of fidelity with the rationale, philosophy, mathematical processes and outcomes of the common curriculum framework for K–9 mathematics. These resources have undergone an intensive review and were found to be the most suitable of those submitted.

In addition, the annotated bibliography includes authorized K–9 resources in Alberta. These resources have been evaluated in Alberta and not by WCP evaluation process.

Learning resources for Kindergarten to Grade 6 that were listed in the 1995 elementary mathematics authorized resources annotated list have not been repeated in the 1996 bibliography. Listings of elementary learning resources authorized since publication of the 1995 list and all resources authorized for Grades 7–9 are included in the 1996 bibliography. Both the 1995 list and the 1996 bibliography are required for a complete listing of authorized resources for Kindergarten to Grade 9.

A complete list all authorized learning resources is also available on the Education in Alberta Web site at http://ednet.edc.gov.ab.ca in the Students and Learning section.

## Alberta Mathematics K–9 Program of Studies

Alberta Education has incorporated the common curriculum framework for K–9 mathematics into the program of studies for K–9 mathematics. The relationship between the common curriculum framework and the program of studies can best be described as "copy and paste." A grey background has been inserted to illustrate the required components. This amendment to the program of studies was sent to schools in June 1996.

### 1995 Kindergarten Program Statement

The 1995 Kindergarten Program Statement [was] revised in June 1997 to include the Kindergarten components of the common curriculum framework for mathematics and language arts.

## Implementation Timeline for K–12 Mathematics

In September 1996, the mathematics programs for Grades 7 and 9 were implemented, while the programs for Kindergarten to Grade 6 were available for optional implementation.

In September 1997, the Kindergarten through Grade 6 and Grade 8 programs [were] implemented.

## Differences Between Alberta's 1994 Interim K–6 and the 1996 K–9 Program of Studies

For the purposes of this article in *Early Childhood Education*, the focus will be on the differences in the K–3 portions of 1994 Interim K–6 and the 1996 K–9 programs of studies.

The organization and presentation of the documents vary. The common curriculum framework presents a multiple-grade focus. This type of presentation enhances the horizontal version of the 1994 interim program of studies. There are fewer outcomes that indicate teaching strategies; however, the illustrative examples are used to get at teaching methodology. There is more reference to technology in the 1996 program of studies as well.

In the Number strand, competency with higher numerical values is expected, and there are links between the process of estimation and mental mathematics.

In the Patterns and Relations strand, the emphasis on patterns is maintained.

In the Shape and Space strand, there is an increased emphasis on the use of language related to the concepts of *parallel, intersection, perpendicular* and *congruent*, which are introduced in Grade 3. Position language to document motion is also expected. The concept of area has its beginning in Kindergarten. There is an increased focus on standard units of measure.

## Document Acquisition

The following documents can be obtained by contacting the Learning Resources Distributing Center at 427-2767, fax 422-9750:

- *The Common Curriculum Framework for K–12 Mathematics: Western Canadian Protocol for Collaboration in Basic Education*, Alberta Education, 1995; product code 302183-01, $11.60
- *Alberta Program of Studies for K–12 Mathematics*, Alberta Education, 1996; product code 317653-01, $6.90
- *Kindergarten to Grade 9 Mathematics Resources: Annotated Bibliography*, Alberta Education, 1996; product code 319154-01, $8.40

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**Two Quadrilaterals**

Connect the midpoints of adjacent sides of a convex quadrilateral. The result is again a quadrilateral. Prove that the resulting quadrilateral is a parallelogram with area one half the size of the original convex quadrilateral. Is this true for concave quadrilaterals as well?

# A Collection of Connections for Junior High Western Canadian Protocol Mathematics

*Sol E. Sigurdson, Thomas E. Kieren, Terri-Lynn McLeod and Brenda Healing*

We have put together "A Collection of Connections" that consists of 12 uses of junior high school mathematics. These activities support the communication and connections strands of the Western Canadian Protocol mathematics curriculum. In using them, teachers may adapt them extensively. They can serve as a basis of one to three mathematics periods. We have also found that teachers need to plan if they intend to incorporate them into their teaching units. They can be used as end-of-unit activities or as focal activities in the development of a unit. We would encourage teachers to use the activities as a means of bringing mathematics to life. These contexts provide an opportunity to enhance a student's view of mathematics. As one Grade 9 girl said at the conclusion of one activity: "That just proves that mathematics is everywhere."

The following are samples from the measurement and algebra strand:

*Measurement*
Why Do Sled Dogs Curl Up?
Why Do Sled Dogs Curl Up? Student Activities
*Algebra*
Managing an Elk Herd
Managing an Elk Herd Student Activities

## Why Do Sled Dogs Curl Up?

### Intent of the Lesson

An important mathematical idea developed here is that surface area and volume are independent. The sled dog is able to change its surface area although its volume remains constant. Formulas for surface area (and volume), estimating and visualizing are used in this lesson.

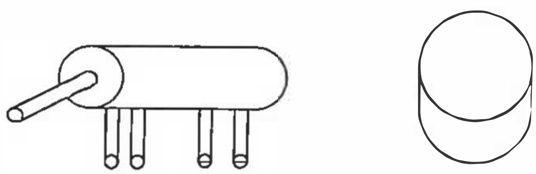
### General Question

Zoologists and veterinarians are interested in studying the behavior of animals. They found that on cold winter nights sled dogs curled up. In fact, all dogs curl up when they are cold. However, dogs sleeping in the house in front of an open fire usually sleep stretched out.

What is the reason for such behavior? It is a known fact that heat loss from an animal is related to the amount of surface area of the animal that is exposed to the surroundings. A dog would, therefore, want to decrease its surface area exposed to the cold. We should note that a dog cannot change its volume, so it must change its shape. Can we calculate the surface area of a dog when it is stretched out and when it is curled up?

#### *Discussion Questions*

A discussion of the problem at this point should raise several points. A point of mathematical interest is that a dog stretched out can be thought of as five small cylinders (legs and tail) and one large cylinder (body and head). However, when it is curled up it represents one very large but flat cylinder.



stretched-out dog    curled-up dog

Answering this question is going to require approximations. In the first place, the parts of the dog's body mentioned above are only approximately cylinders. When a dog curls up, the curled-up shape is only approximately a cylinder.

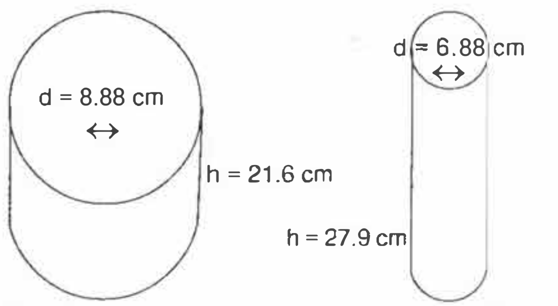
- What shape(s) does a dog represent when it is stretched out? (Long narrow cylinder.)
- What shape does a dog represent when it is curled up? (Flat wide cylinder.)
- What kind of approximations would we need to make when measuring the dog? (Ignore legs and tail.)
- What happens to the legs and tail when it is curled up? (They are tucked inside.)

- Does snow have insulating properties? (Yes.) How does this make a difference to a curled up dog? (It wants to have as large a surface area as possible in contact with the snow.)

## Preliminary Activities

### *The Paper Cylinder*

Take a sheet of paper (say 8½×11). Make a cylinder out of it lengthwise and find its volume. Then make a cylinder out of it widthwise. (See diagram.) What are the surface areas of the two cylinders? What is the volume of each of the two cylinders? The surface areas stay the same (because they are the same sheet of paper) but the volumes differ. Students may do this as a problem to solve. We should notice that when the 8½-inch side is made into a cylinder its diameter is 8½ divided by $\pi$. The volume can then be calculated with this diameter and the known height (11 inches). Students can make good use of a calculator in solving this problem.




Sample calculations (using metric measurements of 27.9 cm by 21.6 cm):

For the large flat cylinder:
circumference (c) = 27.9 cm, height (h) = 21.6 cm

diameter (d) $= \frac{c}{\pi} = \frac{27.9 \text{ cm}}{\pi} = 8.88$ cm

radius (r) $= \frac{d}{2} = \frac{8.88 \text{ cm}}{2} = 4.44$ cm

Volume (V) $= \pi r^2 h = \pi(4.44 \text{ cm})^2(21.6 \text{ cm}) = 1337.7 \text{ cm}^3$

For the long narrow cylinder:
circumference (c) = 21.6 cm, height (h) = 27.9 cm

diameter (d) $= \frac{c}{\pi} = \frac{21.6 \text{ cm}}{\pi} = 6.88$ cm

radius (r) $= \frac{d}{2} = \frac{6.88 \text{ cm}}{2} = 3.44$ cm

Volume (V) $= \pi r^2 h = \pi(3.44 \text{ cm})^2(27.9 \text{ cm}) = 1037.2 \text{ cm}^3$

We know from this investigation that the volume is independent of the surface area. That is, these two cylinders have the same surface area but their volumes differ. The taller cylinder has less volume. We might also note that, in general, a long skinny object has less volume than a short fat one. In fact, the volume of a cylinder of fixed surface area is greatest when the diameter is approximately equal to the height of the cylinder.

### *Discussion Questions*

- How can we determine the diameter of these cylinders in two ways? (Measurement and dividing the circumference by $\pi$.)
- Why are the surface areas of these two cylinders not exactly equal? (If we add the area of the top and bottom, the shorter cylinder will have slightly greater surface area.)
- How does this demonstration relate to a dog curling up? What remains constant when a dog curls up? (Volume remains constant.)
- Would we expect the [surface area] SA of a curled-up dog to be greater or less than that of a stretched-out dog? (We are going from five relatively long and skinny cylinders to one short and fat one.)

### *The Snake*

A snake curls up for much the same reason as the dog—to conserve its heat in the cold desert nights. In fact, a snake has to warm up every morning before it can begin to move. Because of this, it is important to the snake not to lose too much heat. Mathematically, the snake is much nicer than the dog. The snake is an obvious cylinder, both stretched out and curled up. Stretched out it is a very long cylinder and curled up it is a large flat cylinder. (For this experiment, a snake can be simulated by a piece of rope with one-inch diameter or a child's play snake or [use] a real snake if you happen to have one.)

In addition to finding the surface areas we should find the volumes of the stretched out shape and the curled-up one. These volumes should be approximately the same. This calculation is an important check on our approximate measurements. Note: because the snake is curled up we cannot find its diameter by dividing its length by $\pi$.

### *Teaching Suggestion*

Although the lesson is centred around the behavior of sled dogs, the preliminary activities lend themselves to much better calculations. In this lesson, the preliminary activities are more important than "answering the general question." This latter section, because of the difficulty in dealing with a real dog, is more a matter of estimation and discussion. In some ways the teacher can treat the sled dog question as the motivational device while the preliminary activities become the mathematical aspects of the lesson. With respect to the approximations in the next section of the lesson, we have found that students are eager to participate in these approximations. They do not treat the lesson as less significant just because it contains a lot of approximations.

## Answering the General Question

Having a real husky dog and taking actual measurements would be ideal. A student in the class might have a suitable pet and could be encouraged to provide measurements for the class. Approximation is fundamental to this activity. We need realistic measurements to begin with but approximation is still necessary. One approximation we might make is simply to measure the body and head of the dog, ignoring the legs and the tail. The outside of the fur should be the surface area rather than the actual skin surface because it is the outside that is losing the heat. A rolled-up blanket, a cylindrical pillow, a flexible toy dog or a slinky toy could be used in making approximate measurements.

### *Discussion Questions*

- Why is the diameter of the dog's body when it is stretched out equal to the height of the flat cylinder of the curled-up dog?
- By ignoring the legs and tail in our calculations, on which side of the comparison of surface areas are we erring? (The legs and tail contribute a large surface area to the stretched out dog and essentially no surface area to the curled-up dog.)
- What happens to the legs and tail of the curled-up dog? Do they lose more heat in the stretched out position or in a curled-up position?
- What is the effect of the snow insulating the dog in both positions? In other words, which position makes best use of the insulating properties of snow?
- Which shape (long or short) of dog benefits most from the curling up to conserve heat?
- Does your knowledge of surface area increase your understanding of the behavior of dogs?

[We might consider the curled-up dog to be a sphere. A discussion among students could settle this issue and calculations for the sphere could be made for comparison with a sphere. That is, what happens to surface area if we consider it a sphere?

Volume of cylinder = $\pi r^2 h$

Surface area of cylinder = $2\pi rh + 2\pi r^2$

Volume of sphere = $(4/3)\pi r^3$

Surface area of sphere = $4\pi r^2$]

## Materials

Colored sheets of 8½×11 paper, transparent tape, toy snake or 1-inch rope (4 feet long), long cushion (to simulate a dog), stuffed toys and a slinky toy. Although we have not tried it, a piece of "dryer venting" hose could be used to simulate a dog.

## Modifications

### *Other Animals*

Once the situation of the dog has been discussed, students can bring stuffed toys from home that could be measured in the stretched-out position and curled-up position. As mentioned earlier, a Slinky toy has possibilities.

When a fat teddy bear is curled up, its shape becomes a sphere rather than a cylinder. Some stuffed toys have very large legs that cannot be ignored in the calculation of the surface area before the toy is curled up. The legs can be treated as cylinders with one end.

The idea of the proportion of the surface in contact with the snow being much larger in the curled-up position than the stretched-out position is worth discussing. In general, the snow is a better insulator than the air, especially because of the wind blowing (wind chill) on the exposed side. In fact, the dog loses only about one-tenth as much heat through the snow as through the air. This makes the curling-up behavior even more understandable. If the dog is considered to be a flat cylinder, using your measurements, what percent of the surface is next to the snow? In curling up, not only does a dog reduce his or her surface area but also a good percentage of this smaller surface area is better insulated.

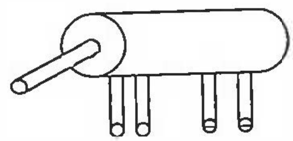
A survival suggestion for humans in cold water is to curl up into the fetal position. Students can approximate the surface area of a classmate in stretched-out and curled-up positions.

# Why Do Sled Dogs Curl Up? Student Activities

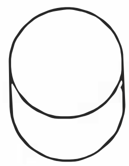
### *General Question*

Zoologists and veterinarians are interested in studying the behavior of animals. They found that on cold winter nights sled dogs curled up. In fact, all dogs curl up when they are cold. However, dogs sleeping in the house in front of an open fire usually sleep stretched out.

What is the reason for such behavior? It is a known fact that heat loss from an animal is related to the amount of surface area of the animal that is exposed to the surroundings. A dog would, therefore, want to decrease its surface area exposed to the cold. We should note that a dog cannot change its volume, so it must change its shape. How much more is the surface area of a dog when it is stretched out than when it is curled up?



stretched-out dog



curled-up dog

### Activities

1. (a) With an 8½×11 sheet of paper, we can make a cylinder in two ways. Once the cylinder is made, we can find the diameter of the cylinder in two ways. What are they?
   (b) How do you convert inches to centimetres? What is 8½ inches in centimetres?
2. (a) Why are the curved surfaces of both cylinders that you made with the paper (question 1) equal?
   (b) To what other surface area are they both equal?
   (c) Why are the *total* surface areas of both cylinders not the same? By how much are they different?
3. (a) A tall cylinder has less volume than a short cylinder of the same surface area. However, if a tall cylinder and a short cylinder have the same volume, which will have the greater surface area?
   (b) How does this conclusion relate to the stretched-out dog when he curls up?
4. (a) Using your measurements for a "snake," show the calculations for the snake stretched out and the snake curled up.
   (b) How much surface area does the snake lose by curling up?
   (c) What percent is this loss of the stretched out surface area?
   (d) What percent surface area does the snake gain when he goes from being curled up to stretching out?
   (e) Why are these percentages not the same? If you wanted to impress someone with the importance of the reduction of surface area, which of these percentages would you use?
5. (a) Take approximate measurements from a real dog, a toy animal or a piece of tubing and determine the surface area of the "animal" in both positions.
   (b) Using the same measurements, make the calculation to determine the volume of the "animal" in both positions. Are the volumes nearly equal? Why?
   (c) Again with approximate measurement, determine the surface areas of the legs and tail. Go back to question 5(a) and add these surface areas to the surface of the stretched-out animal. How much does this increase the differences in surface areas?
   (d) Why do we not add the surface area of the legs and tail to the curled up surface area? What assumptions are we making?
6. (a) What shape do you think polar bears become when they curl up?
   (b) Why do you think the polar bear's fur is much thicker on the back and sides than on the belly?

## Managing an Elk Herd

### Intent of the Lesson

The linear equation is used to simulate the growth of a population, a common problem in wildlife management. The mathematics includes linear equations, percentages, probability and graphing. It is possible to use a spreadsheet to illustrate the solution.

### General Question

All wildlife areas, even in a country as large as Canada, are limited in size. Because of this limitation, the wildlife that the area will support is limited. This is especially true in an area such as Jasper National Park. Even in larger areas where hunting is allowed, wildlife managers need information about herd sizes. Each year, decisions are made about how many animals may be taken by hunters. In making this decision, managers need to be able to predict the growth of herds.

Consider a typical problem for a wildlife manager—a herd of animals is 10,000. How large will that herd be in 5 years or 10 years? The answer to the problem can be found with the help of mathematics. First, information about survival (death) rates of the animals from year to year must be known, as well as birth rates. Very simply, if the death rate is 20 percent and the birth rate is 40 percent, the herd size will grow. Under such conditions, the herd will soon become too large for the area. Animals will need to be moved to new areas and/or hunted. The question is can we predict the size of the herd from year to year?

#### *Teaching Suggestions*

The teacher should be prepared to fully discuss the context of this problem. This may include discussion of biological processes, ethical considerations of managing any wildlife population or hunting.

#### *Discussion Questions*

- What do you think survival rate means?
- Will the survival rate be different for males and females? For fawns? (will be higher for males than females and higher for adults than for fawns)

- If the herd is growing too fast and a harvest is needed, what options do wildlife managers have? (Allow hunting or capturing and moving animals to other areas.)
- What kind of herd growth is desirable in most wildlife areas? (no population growth)

## Preliminary Activity

### *Exam Marks*

Interestingly, the mathematics needed for the Elk Herd problem is the same mathematics that students needs to figure out final marks:

> If the midterm mark (M) counts for 25 percent of the course mark (C) and the final exam (F) counts for 75 percent, what will the course mark be if your midterm mark was 72 percent and your final exam mark was 92 percent?

To solve this we take 25 percent of 72 which is 18 and 75 percent of 92 which is 69 and add them together for a final score of 87. We should observe that the 87 is much closer to 92 than to 72. Why is that? The reason is that the final exam mark is much more important than the midterm mark and is worth more. Now, what is the mathematical equation to determine the course mark?

$0.25M + 0.75F = C$

If we think about these as fractions, ¼ of the midterm and ¾ of the final exam are added together.

In another class, the course mark (C) is determined by taking 15 percent of assignments (A), 25 percent of midterm (M) and 60 percent of the final exam (F). What is the equation to determine the mark in this class?

$0.15A + 0.25M + 0.60F = C$

In this class the final exam is four times more important than the mark in assignments.

Ask students to do the calculation for given A, M and F.

## Answering the General Question

### *The Management of an Elk Herd*

A wildlife officer has the following information on an elk herd in her region:

| | | |
|---|---|---|
| Survival rates for | males | 95% |
| | females | 90% |
| | male fawns | 50% |
| | female fawns | 45% |
| Birth rate for | male fawns | 48% of adult females |
| | female fawns | 42% of adult females |

### *Discussion Questions*

- How do we get these numbers? (Biologists and wildlife managers)
- Why are survival rates for male and female fawns different? (Male elk are bigger.)
- Why is birthrate only dependent on adult females? (In a herd of 100 males and 10 females, what is the maximum number of fawns that can be born?)
- What does the male fawn survival rate of 50 percent mean? (One half live until next year.)

Using these rates, we can predict from year to year how large a herd will be. Very simply, a herd in year one with 100 males and 100 females will have 95 males and 95 females surviving in year two. The number of fawns will be 48 male fawns and 42 female fawns. The total herd will be 95 + 95 + 48 + 42 = 280. This number is based on a birth rate determined from the number of females in the previous year.

The answer will be different if only the surviving females give birth, that is, if the birthrate is based on the number of females surviving from the year before. Initially, let's consider that the birth rate is based on the number in the herd in year one. Now we should write the equation, just as we did for the course mark:

$0.95M$ (males) + $0.90F$ (females) + $0.48F + 0.42F$ = Herd in year two

(Using "year one" and "year two" terminology leads to less confusion in the lesson.)

Now we introduce a harvest of the elk herd. The harvest may be gathering them up and moving them or it may mean hunting them. If, in the case above, we wanted to keep the herd at 200, we would need to harvest about 80 animals in year two. (In this equation we are assuming that the count is made after the harvest. In other words, the count is the last thing that happens in the year.) The equation would look like this:

$0.95M + 0.90F + 0.48F + 0.42F - H$ (harvest) = Herd in year two

Now we can proceed to figure out the herd size in year three. Notice that we have four types of elk:

| | |
|---|---|
| M(ale) adults - | 95% survival rate |
| F(emale) adults - | 90% survival rate |
| m(ale) fawns - | 50% survival rate |
| f(emale) fawns - | 45% survival rate |

To do our calculations for year two, we really need to know, not the total size of the herd, but how many of each type of animal we have. Also, because females are more important to the herd survival, we will want a system that harvests more males than females. Therefore, we will have a male harvest (Hm) and a female harvest (Hf). Of course, only adults are harvested. Why is this? We need several equations, one for each type of animal. What, for example, is the equation to determine the number of males in any year? Since the adult male population is made up of surviving adults plus surviving fawns less the harvest, the equation for males in any year is

0.95M + 0.50m – Hm = Number of males

Because we have so many elements to attend to, a table is convenient.

### *Using a Table*

The use of a table will simplify our calculations.

| Column A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| Year | Males (adult) | Females (adult) | males (fawns) | females (fawns) | Total | Hm = Hf = |
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |

We need equations to go from year to year:

| | | |
|---|---|---|
| Male adults: | 0.95M + 0.5m – Hm | where M and m are males from previous year |
| Female adults: | 0.90F + 0.45f – Hf | where F and f are females from previous year |
| males: | 0.48F | where F is female adults from previous year |
| females: | 0.42F | |

Harvests (Hm and Hf) are constants from year to year which are subtracted. We should note the assumptions that are being made about the how the herd functions.

### *Discussion Questions*

- Why does 0.5m become part of the adult males? (We are assuming that fawns become adults after one year.)
- What does 0.48F mean? (For every 100 females that were alive the year before, 48 male calves were born.)
- If we assume that only surviving females give birth, what does the 0.48F become? (It becomes 0.48 (0.90F) because 90 percent of the females survive and 48 percent of those have male fawns.)
- What does the 0.48F become if first we count our F, then a harvest occurs (Hf), then 90 percent survive and then 48 percent of these have male calves? (The 0.48F becomes (F – Hf) × 0.90 × 0.45.)

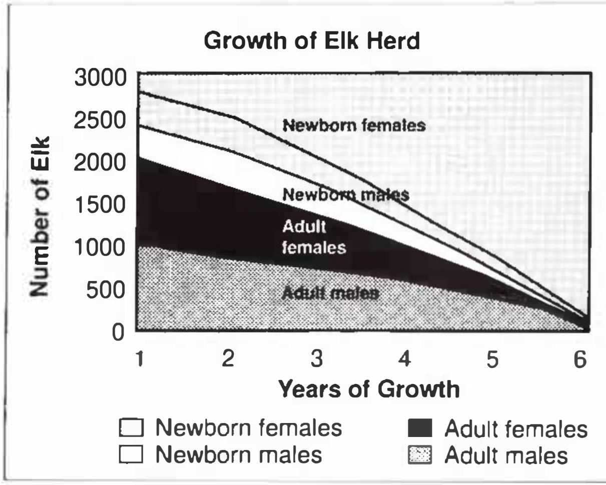
The point of these questions is that our model is a simplified version of what probably happens in an elk herd. Once we understand how the simplified version works, we can make it more complicated. A student assignment can be to determine how equations differ under different assumptions about herd growth.

Although we can begin the herd with any numbers, let us start with 400 male and 400 female fawns, 1,000 each of males and females and a harvest of 300 males and 300 females. Note: our equations determine the number of adult males, for example, in any year by calculating survivors (both adults and fawns) subtracting the harvest. So, in the example below the 1,000 males and females is a count after the harvest. Realistically, the count could have been taken around Christmas, after the harvest.

| 1 | A | B | C | D | E | F | G | H | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | Year | Adult males | Adult females | Newborn males | Newborn females | Total herd | Harvest males | Harvest females | | |
| 3 | 1 | 1,000 | 1,000 | 400 | 400 | 2,800 | 300 | 300 | Male survival rate | 95% |
| 4 | 2 | 850 | 780 | 480 | 420 | 2,530 | 300 | 300 | Female survival rate | 90% |
| 5 | 3 | 748 | 591 | 374 | 328 | 2,041 | 300 | 300 | Male birthrate | 48% |
| 6 | 4 | 597 | 379 | 284 | 248 | 1,509 | 300 | 300 | Female birthrate | 42% |
| 7 | 5 | 409 | 153 | 182 | 159 | 904 | 300 | 300 | Newborn male survival rate | 50% |
| 8 | 6 | 180 | –91 | 73 | 64 | 227 | 300 | 300 | Newborn female survival rate | 45% |
| 9 | | | | | | | | | Harvest males | 300 |
| 10 | | | | | | | | | Harvest females | 300 |

Year 2 Males = 0.95(1000) + 0.50(400) – 300 = 850
Year 2 Females = 0.90(1000) + 0.45(400) – 300 = 780
Year 2 male fawns = 0.48(1000) = 480
Year 2 female fawns = 0.42(1000) = 420

The graph for each category for each year is shown below:




### *Teaching Suggestion*

The simplified system of predicting the herd is the most appropriate for the curriculum because of the clarity of the resulting weighted variables in the equations. That is, in any given year, the increase in the total male population, for example, is the result of a certain percentage of adults surviving, a certain percentage of fawns surviving and a certain birthrate. The growth of the total male population is a linear equation with three variables and a constant term.

Students may not be concerned with the actual seasonal happening of the herd. Realistically, the sequence of events is probably a fall herd being harvested, the count being taken, the remaining animals surviving and then fawns being born to surviving females. Some of these elements may be introduced as exercises.

### *Using Our System*

Ten groups of students could be given a common herd size, say 1,000, but different harvest rates. For example, if, after five years, we want the herd to be about 1,500, what harvesting policy should we have? That is, which group ends up closest to 1,500 after five years? Because females are more important to herd growth assign different harvest rates for males and females. We can see how the wildlife manager would be able to make predictions.

Another possibility is to have a harvesting rate that varies from year to year.

### *Teaching Suggestion*

A table on the blackboard can be used to record the results from the different groups. Each group would record its harvest rate and the herd size in each of the five years. Based on an examination of the pattern, an estimate of the appropriate harvest (for a herd size after five years of 1,500) can be discussed.

## Materials

Although no special materials are needed for this lesson, a spreadsheet program may be useful.

## Modifications

### *Using a Spreadsheet*

This problem lends itself well to use of the spreadsheet. Equations can be entered in the cells and the harvest can be varied for different runs. Graphs which show adult males, adult females, male fawns and female fawns in different colors are quite dramatic.

Students can easily be taught to write equations for a spreadsheet. Referring back to the table, the equations for cell $B_4$ is $B_3(0.95) + D_3(0.45) - Hm$. Similarly, the equation for $B_5$ is $B_4(0.95) + D_4(0.45) - Hm$ and so on. Once students understand the problem, filling in cells in a spreadsheet is simple and the graphing results are very motivating.

On page 36 are a spreadsheet and graph of a harvest policy that keeps the herd constant for 20 years.

# Managing an Elk Herd Student Activities

## General Question

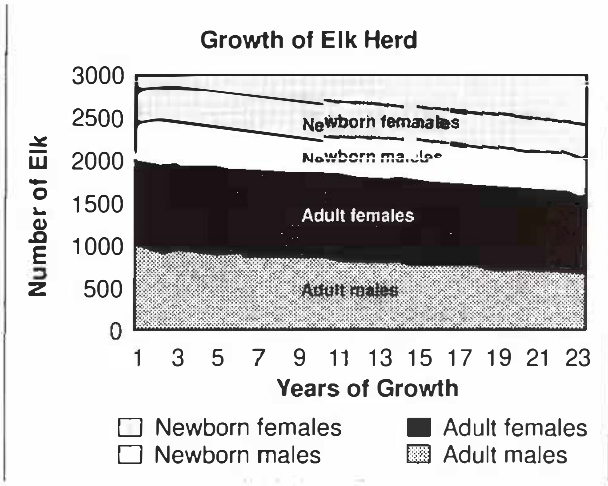
All wildlife areas, even in a country as large as Canada, are limited in size. Because of this limitation, the wildlife that the area will support is limited. This is especially true in an area such as Jasper National Park. Even in larger areas where hunting is allowed, wildlife managers need information about herd sizes. Each year, decisions are made about how many animals may be taken by hunters. In making this decision, managers need to be able to predict the growth of herds.

Consider a typical problem for a wildlife manager. A herd of 10,000 animals has 5,000 males and 5,000 females. Given the following information, how large will that herd be in 5 years or 10 years?

| | | |
|---|---|---|
| Survival rates for | males | 95% |
| | females | 90% |
| | male fawns | 50% |
| | female fawns | 45% |
| Birth rate for | male fawns | 48% of adult females |
| | female fawns | 42% of adult females |

The answer to the problem can be found with the help of mathematics. First, information about survival (death) rates of the animals from year to year must be known, as well as birth rates. Very simply, if the

| 1 | A | B | C | D | E | F | G | H | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | Year | Adult males | Adult females | Newborn males | Newborn females | Total herd | Harvest males | Harvest females | | |
| 3 | 1 | 1,000 | 1,000 | 400 | 400 | 2,800 | 200 | 90 | Male survival rate | 95% |
| 4 | 2 | 950 | 990 | 480 | 420 | 2,840 | 200 | 90 | Female survival rate | 90% |
| 5 | 3 | 943 | 990 | 475 | 416 | 2,824 | 200 | 90 | Male birthrate | 48% |
| 6 | 4 | 933 | 988 | 475 | 416 | 2,812 | 200 | 90 | Female birthrate | 42% |
| 7 | 5 | 924 | 986 | 474 | 415 | 2,800 | 200 | 90 | Newborn male survival rate | 50% |
| 8 | 6 | 915 | 985 | 473 | 414 | 2,787 | 200 | 90 | Newborn female survival rate | 45% |
| 9 | 7 | 906 | 983 | 473 | 413 | 2,774 | 200 | 90 | Harvest males | 200 |
| 10 | 8 | 897 | 980 | 472 | 413 | 2,761 | 200 | 90 | Harvest females | 90 |
| 11 | 9 | 888 | 978 | 471 | 412 | 2,748 | 200 | 90 | | |
| 12 | 10 | 879 | 975 | 469 | 411 | 2,734 | 200 | 90 | | |
| 13 | 11 | 869 | 973 | 468 | 410 | 2,720 | 200 | 90 | | |
| 14 | 12 | 860 | 970 | 467 | 409 | 2,705 | 200 | 90 | | |
| 15 | 13 | 851 | 967 | 466 | 407 | 2,690 | 200 | 90 | | |
| 16 | 14 | 841 | 963 | 464 | 406 | 2,674 | 200 | 90 | | |
| 17 | 15 | 831 | 960 | 462 | 405 | 2,658 | 200 | 90 | | |
| 18 | 16 | 820 | 956 | 461 | 403 | 2,640 | 200 | 90 | | |
| 19 | 17 | 810 | 952 | 459 | 401 | 2,622 | 200 | 90 | | |
| 20 | 18 | 799 | 947 | 457 | 400 | 2,602 | 200 | 90 | | |
| 21 | 19 | 787 | 942 | 455 | 398 | 2,582 | 200 | 90 | | |
| 22 | 20 | 775 | 937 | 452 | 396 | 2,560 | 200 | 90 | | |
| 23 | 21 | 762 | 931 | 450 | 394 | 2,537 | 200 | 90 | | |
| 24 | 22 | 749 | 925 | 447 | 391 | 2,513 | 200 | 90 | | |




death rate is 20 percent and the birth rate is 40 percent, the herd size will grow. Under such conditions, the herd will soon become too large for the area. Animals will need to be moved to new areas and/or hunted. How can we predict the size of the herd from year to year?

## Activities

1. (a) A teacher wants to determine your final grade by giving equal weight to your assignments and midterm test and double that to your final. What would be the equation that she could use?
   (b) What would your mark be if you got 80 percent on both the assignments and the midterm and 60 percent on your final?
2. (a) When we use the equation $0.95F + 0.45f - Hf$ to determine the number of adult females in a certain year, what sequence of events are we assuming? (Remember that we have four events: the count, the survival, the birth and the harvest.)
   (b) What would this equation be if adult female survival was only 80 percent and the survival from fawn to adult for females was 30 percent?
3. (a) If the survival rate for the adult elk is applied after the yearly harvest, what does the equation for adult males become? In this scenario, the young are born, then the count is made

giving the M and F values for the year, then the adults are harvested and then 95 percent of them survive the winter.

(b) In the scenario in question 3(a) what would be the equation for the number of females able to have calves during the next summer?

4. (a) In our mathematical system, the number of fawns depends only on the number of females. Suppose that a biologist tells you that a herd cannot grow normally unless at least 20 of the adults are male. How would you change the mathematical system to take this into consideration?

(b) Another biologist tells you that the rule is 20 percent of the adults are male. How would you account for that?

5. (a) Make up what you think is the most likely sequence of events in the growth of a herd. Assume the year goes from January to January; in other words, the *count* is made in January. Remember four things happen: births, survival, harvest and count.

(b) Make the equation for adult males for your sequence from question 5(a).

(c) Make the equation for female birthrates for your sequence from question 5(a).

6. If you know the survival rates and birth rate of a herd, how could you estimate what the harvest should be to keep the herd size stable?

7. How would the equations have to be changed for a population of wolves who pair with a mate for life? Which equation(s) would have to change?

8. Find a spreadsheet and write up the first four lines of it for the elk herd.

## Authors' Note

Those readers interested in the entire volume of "A Collection of Connections" may contact Sol E. Sigurdson, Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton T6G 2G5; phone 492-0753.

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**Wrapping a Cardboard Tube**

Take a ribbon, 25 m long and 0.1 mm thick, and wrap it tightly around a cardboard tube. The cylinder has now a diameter of 1 dm. What is the diameter of the cardboard tube?

# Calendar Math

*Art Jorgensen*

## March 1998

1. When you write all the numbers from 1 to 50, how many times do you write the digit 3?
2. Choose a number between 1 and 50. Have your classmates determine the number by asking questions that can be answered by "yes" or "no." For example: Is the number less than 25? The winner gets to choose the next number.
3. Think of a number. Add 2 to your number. Double the amount you now have. Add 6. Divide by 2. Subtract your original number. What is the result? Does this always work?
4. In a bag there are twice as many red blocks as blue blocks, and twice as many yellow blocks as red blocks. Altogether there are 14 blocks. How many of each color are there in the bag?
5. Five boys ran a 400-metre dash. A came in first. B came in last. If D was ahead of C, and E was just behind him, who came in second?
6. Six girls try to guess the number of pennies in a jar. The six guesses are 52, 59, 62, 65, 49 and 42. One guess is 12 away, and the other guesses are 1, 4, 6, 9 and 11 away. How many pennies are in the jar?
7. How many different 4-digit numbers can you form using the digits 1, 9, 3 and 9?
8. Use the numbers 3, 4, 5, 6, 7 and 10 exactly once and any of the four basic operations in an equation to make the number 1.
9. Within the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 there is a mystery number. By doing the following exercises, determine the mystery number. Use each number only once.
   Find 2 numbers whose sum is 3.
   Find 2 numbers whose sum is 8.
   Find 2 numbers whose sum is 12.
   Find 2 numbers whose sum is 15.
   The number left is the mystery number.
10. With one straight line, slice a circle into two pieces. With two cuts how many pieces do you get? What is the largest number of pieces you can get with three cuts?
11. The ratio of blue marbles to red marbles in a box is 3 to 5. If 30 red marbles are in the box, how many blue marbles are there?
12. Using only the number 4 and the four basic operations (+, −, ×, ÷), write the numbers from 1 to 16. For example: 12 = 4 + 4 + 4.
13. A perfect number is a number that is the sum of its factors. Find the smallest perfect number.
14. A certain magical plant doubles in height every day. If it is 1 cm tall today, how tall will it be in 5 days?
15. What is $10^8$ divided by one million?
16. Cyprian's shadow is 10 m long at the same time that the shadow of a tree is 40 m long. If Cyprian is 2 m tall, how tall is the tree?
17. If March 1 is on a Tuesday, on what day of the week will April Fools' Day fall?
18. In a class of 25 students, 12 wear glasses and 11 wear braces. If 7 wear both glasses and braces, how many students wear neither glasses nor braces?
19. Tom saved $1 the first week and in each subsequent week he saved three times as much as the previous week. How much had he saved after five weeks?
20. Which of these is the largest?
    a. One more than ½ of 16.
    b. One plus 2/3 of a dozen.
    c. Five more than 1/5 of 20.
    d. Five more than twice 2.
    e. They are all the same.
21. Sibongile bought a pair of shorts, a T-shirt and a pair of socks for $20. The shorts cost $9 more than the T-shirt, and both together cost $16 more than the socks. How much did Sibongile pay for each item?
22. Mary wants to make a pen for her cat. She has 22 m of wire. What is the largest area she can enclose?
23. What is the smallest number that is equal to four times the sum of its digits? Five times the sum of its digits? Six times the sum of its digits? Seven times the sum of its digits? Eight times the sum of its digits? Nine times the sum of its digits?
24. How many whole numbers between 1 and 100 have their digits in decreasing order? For example: 32.
25. Write an expression equal to 15 using six 1s and the addition sign. No other symbols may be used.

26. Find the missing numbers in the number sequence.
   1, 2, 4, 7, 11, __, __, 29.
   1, 1, 2, 3, 5, 8, __, 21, __, __.
27. It's greater than 39.
   It's not a multiple of 9.
   It's less than 60.
   It's a multiple of 6.
   The sum of its digits is 12.
   What is the number?
28. Jason is 23 years younger than his father. In 5 years, the sum of their ages will be 41 years. How old is Jason today?
29. A salesman traveled at 60 km/h while making a 120-km trip to a client, then returned home at 40 km/h. What was his average speed for the round trip?
30. The price of 3 shirts is $n$ dollars. At that price, how many shirts can be bought for $40?
31. Paul can drill the holes he needs in 5 minutes with a power drill, or in 20 minutes with a hand drill. He starts with the power drill, but after 2 minutes it stops working and he finishes with the hand drill. How long does he work with the hand drill?

## Answers

1. 15
2. No specific answer
3. 5
4. 2 blue, 4 red, 8 yellow
5. D
6. 53 pennies
7. 12
8. 3 + 4 – 7 + 5 + 6 – 10 = 1
9. 1 + 2, 3 + 5, 4 + 8, 6 + 9. Seven is the mystery number.
10. 4 pieces, 7 pieces
11. 18 blue marbles
12. 4/4 = 1, 4/4 + 4/4 = 2, 4 – 4/4 = 3, 4 = 4, 4 + 4/4 = 5, 4 + 4/4 + 4/4 = 6,
    4 + 4 – 4/4 = 7, 4 + 4 = 8, 4 + 4 + 4/4 = 9, 4 + 4 + 4/4 + 4/4 = 10,
    4 + 4 + 4 – 4/4 = 11, 4 + 4 + 4 = 12, 4 + 4 + 4 + 4/4 = 13,
    4 + 4 + 4 + 4/4 + 4/4 = 14, 4 × 4 – 4/4 = 15, 4 × 4 = 16.
    There are many more solutions.
13. 6 = 1 + 2 + 3
14. 16 cm
15. 100
16. 8 m
17. Friday
18. 9
19. $121
20. e. They are all the same
21. Shorts cost $13.50, T-shirts cost $4.50, socks cost $2.00.
22. 38.54 $m^2$
23. 12, 45, 54, 21, 72, 81
24. 45
25. 11 + 1 + 1 + 1 + 1 = 15
26. a. 16, 22 b. 13, 34, 55
27. 48
28. Jason is 4 years old.
29. 48 km/h
30. 120/$n$
31. 12 minutes

**Strange "30"**

The sum of two natural numbers is 90. The sum of 25 percent of the first addend and 75 percent of the second addend is exactly 30. What are the two natural numbers?

# Problem Solving Strategies

*Michael Richards*

If you are having some trouble solving a maths problem, these ideas might get you going.

## Make Sure You Understand the Problem

This may seem obvious but it is easy to jump straight into solving a problem before you really understand it. So, sure, have a bit of a play around with it at first, if you like, but then read the problem carefully two or three times if necessary. If you are still not sure about the problem, try

- talking it over with someone else; or
- writing out the problem *in your own words* and having someone else check that what you have written is the same as the original problem.

Another way of getting going with problem solving is **making a start**. Don't be scared of having a go! Draw some sketches, try some possible answers, talk the problem over with a friend.

## Making Mistakes!

It's true: good problem solvers make plenty of mistakes. You may have heard the expression: *learn from your mistakes*. Well, this statement is true. Try things out, make mistakes, then try some other way of attacking the problem.

## Do Not Get Frustrated

I know that is easy to say, but problem solving is all about coming up against something you don't quite understand. *That is why it is a problem*. So accept that fact and take it easy; if you worry about it too much, you might lose concentration. So stay cool.

## Keep a Record

If you do not keep a record of what you have done (that is, all your rough work and notes), you might end up repeating some of your earlier work without realizing it. This is particularly true if you are going to leave your problem for a while before coming back to it. Make sure you write down exactly where you are up to, so it will be easy to get back into at that later date. I know it seems like a real nuisance writing things down sometimes, but it is worth it if you save time and get closer to a problem's solution sooner.

## Draw a Diagram or Do a Sketch

These mean the same thing, and it is often the best thing to do to start.

## Make a List, Then Look for a Pattern

Often in mathematical problems, there are patterns to be looked for that will help in their solution. So it is a good idea to start off a list that you can easily check for patterns. Make sure you include the simpler parts of the problem in your list.

## Start with the Easier Parts of the Problem or Make the Problem Simpler

Understanding a simple version of a problem often is the first step to understanding a whole lot more of it. So don't be scared of looking at a simple version of a problem, then gradually extending your investigation to the more complicated parts.

## Check Your Answers

You might think you've got it but you'd better check, just in case. This is the vital and final (one hopes) part of solving a problem. And if it is not the final part, it is just as well you checked—isn't it? So always check your answer.

## Finally

Best wishes with your problem solving. Keep trying different ways of solving the problems and don't get worried if you run into difficulties. Be positive. I hope the pointers I have provided here are of some help.

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*Reprinted with the author's permission from* Problem Solving and Maths Tools, *published in 1996 by JAMI Educational Services, 20 Shandon Street, Mornington, Victoria 3931, Australia. The book is a guide for teaching and assessing mathematical problem solving; those interested in the book can e-mail the author at jami@cx.net. Minor changes have been made to spelling and punctuation to fit ATA style.*

# Problem Solving—A Part of Everyday Thinking

*Octaviano Garcia*

## Lesson Plan for Grades 4–12

## Overview

Many students view problem solving, its process and necessary skills as a task assigned by the teacher Monday between 9:15 and 10:15 am that must be completed within a set time or else they will suffer the consequences----gradewise. These students have not acquired the habit of critically reviewing every recommendation and decision they make before acting on it. Therefore, the process of critically thinking through every significant idea that comes to mind is viewed as playing a very minor role in their daily lives.

## Purpose

As a course introduction or an introduction to any unit of study, this activity helps students master the process of applying critical thinking to each and every problem or task that confronts them. Further, this activity can serve as a base reference and model for every problem or task assigned or any problem that students bring up.

## Objectives

As a result of this activity, each student will

1. demonstrate knowledge of six basic steps to problem solving by listing them or reciting them orally;
2. demonstrate mastery of the six basic steps to problem solving by actively applying them in problem solving when the task lends itself to such a process; and
3. demonstrate ability to apply these basic steps to problem solving by guiding other students in the solution of a given problem or set of problems.

This activity works well as an introduction to a course and its objectives, requirements and expectations. The activity can also help students achieve the following secondary learning objectives, given that the necessary follow-through is provided during the school year:

1. Demonstrate a high awareness of the need for the application of critical thinking skills to everyday problem solving.
2. Demonstrate an observable and measurable improvement on problem-solving skills by way of constantly and consistently applying this modeled process.
3. Show understanding and mastery of the problem-solving process by way of written or oral explanation of the critical steps in problem solving learned through this mode.
4. Show a marked improvement in the quality and completeness of written and oral responses to other assigned, as well as, routine personal problems.

## Resources/Materials

No resources other than pencils and paper are required if solution to the problem is assigned individually. Chart paper may be used for group work if the problem is assigned to small groups. Overhead transparencies may be used for easy viewing if the teacher works the solution through with the whole class.

## Activities and Procedures

Set the stage by explaining the purpose of the following story problem. Orient students to the explanations. Review with students the basic strengths of a "good" problem solver. Emphasize a student's ability to think critically; to identify, group and classify information in an order and form that make it relevant and applicable to a given solution. Alert students to the fact that one's ability to solve daily problems, simulated or real, depends on the ability to separate useful from useless information and necessary from unnecessary information and then apply the pertinent information to the problem or task at hand. Remind them that their success in the course or unit of study depends on their attention and critical thinking skills/habits. Tune-in, tune-out habits will result in low-quality solutions to assigned tasks, while critical listening, critical thinking and assertive problem-solving processes will result in high-quality solutions to any problem students solve. The story problem you are about to share contains many fabricated distractors, together with the pertinent and necessary information for solving the problem that the story poses.

## The Story: The Shepherd and the Harvard Boys

A few years ago, not counting those that came later, two Harvard sophomore students decided that they would spend their summer break traveling across the United States, the same country that they had studied in for years. In preparation for this long journey, they were careful to pack the necessary credit cards, which their generous parents provided, maps, some light casual traveling clothes and the friendliest Harvard smiles they could muster. They were to travel in an old 1961 Chevy panel truck that one boy had gotten as a high school graduation gift from an uncle. The vehicle had made many trips between their hometown Newport, Maine, and the Harvard campus in Cambridge, Massachusetts. Therefore, the two agreed that the panel truck would make it to the west coast and back. All due caution was taken in preparing the vehicle for the trip.

Traveling through the midwestern and southeastern states provided much entertainment and challenge for their trained minds. The real fun began, however, when they reached California. They spent four weeks in Hollywood, two weeks at the San Diego Zoo, a few days in San Francisco and many days in Disneyland. All too soon, it appeared, their summer had come to an end and it was time for them to start back to more familiar territory. After picking up some supplies in Sacramento, they headed east on Interstate 80. On the second day, the panel truck made only 150 miles during a 10-hour day. Repairs had taken up most of that day. On the third day, after six stops for repairs over a 60-mile stretch, the two abandoned the vehicle and hitched a ride home on I-80.

The hour was marching on to four o'clock in the afternoon when they bade their panel truck good-bye. Both boys were silent for the first hour of walking but each was thinking of the night ahead and having to round up a meal or go without. The more timid of the two was re-experiencing fears that he had thought he left behind at age 12 when he was in Grade 7. He began envisioning attacks by wild lions, tigers, panthers and the like. When he spoke up, his first question was, "Where are we going to sleep tonight and what are we going to eat?" "Fear not, my good friend," responded his partner, "I shall teach you how the pioneers of the frontier survived in these desolate plains long before there was even a highway through here. Just keep your eyes alert for sheep. When you see some and their shepherd, let me do the talking. For I fear that in your condition you might jeopardize our chances for an evening meal and perhaps even a bed to sleep in."

Our timid friend did not speak up, but he did not fancy the idea of chasing and wrestling a sheep down for their dinner, not to mention having to butcher and prepare it over an open fire. He was deep in his thoughts when his traveling companion's shout of glee brought him back to reality. "There, there, by those trees on that other slope!" he shouted as he pointed with excitement at some white spots that, in Mr. Timid's eyes, looked like rocks. "Those are nothing but rocks," he returned with the air of certainty that he often used on the Harvard campus. Nonetheless, he was happy to accept his error when they approached the white spots, which turned out to be sheep.

No sooner had they arrived at the herd's northernmost edge when out of the scrub oak thicket came two Australian sheep dogs. The young men stopped and searched the valley for the whereabouts of the shepherd and, sure enough, from under the tallest pine tree, there emerged what looked like a person. As they got closer, they saw that he had needed a shave, a haircut and probably a bath for several weeks. But they left all that aside and decided that there was a good opportunity to cash in on that evening meal they so badly needed and perhaps even a place to spend the night, if they applied their best manners and savvy.

While still about a dozen or so yards from the shepherd, Mr. Timid's partner took a quick glance over the entire herd and mentally made the best estimate of the number of sheep in the herd that his bright mind could compute. He greeted the shepherd, saying "A very good afternoon to you, Mr. Shepherd of 2,000 sheep," not knowing his name. "Your greetings are kindly accepted, my traveling friends, but you err, Mr. Bright Boy. I am not the shepherd of 2,000 sheep. If I had that many sheep out there plus another herd as large as that, then again half as many as I have out there, I should be the shepherd of 2,000 sheep."

Mr. Timid immediately set his mathematical mind to the problem and, by the time they had arrived at the shepherd's tent a few hundred yards away, he had figured out how many sheep the shepherd actually had in that herd.

## Problem and Solution

### Question to students

How many sheep were in this shepherd's herd?

1. Direct the students to apply the six basic steps of problem solving. To do so, students must list each step, and next to it or immediately following, they must list the information from this story that applies and is pertinent to that particular step. Remind students that you began by explaining that the point of the story was to see how well they can separate useful information from useless information given a particular task or problem to solve.
2. You may wish to accept a solution that is arrived at by guess-and-test (trial-and-error) method, or you may direct the students to apply their algebra skills and produce an algebraic formula/equation: for example, $1x + \frac{1}{2}x = 2{,}000$ sheep.
3. You may wish to have students attempt the solution to this problem individually or in small groups. If you feel the group is very unfamiliar with the six basic steps to problem solving, you may want to use this story problem to establish familiarity with these steps and do a whole-group problem solving exercise. This problem lends itself well to any of these approaches.
4. A reward may be offered to the person or group that produces the most complete and well-formulated solution.

## Tying It Together

1. Review by having students re-state and review the purpose (objective) associated with this story problem.
2. Be sure to review the best solution (problem-solving process) with the students before this story problem is set aside for the day or the week.
3. Remind students that constant reference will be made to the components of this problem and especially the steps followed in solving it, as problems of a similar nature come up during the year or unit.
4. Encourage students to share the results of this activity with their parents or guardians.
5. Use this story problem to introduce or review the problem-solving process with any lesson unit or course. Remember, you can vary it for different grade levels; for example, the number of sheep may be 20, 200 or any other number.

**Divisibility of Numbers**

The number $3^{105} + 4^{105}$ is divisible by 13, 49, 181 and 379, but not by 5 and 11. How can you prove this?

**Equation**

Find the solution of the following equation: $(x^2 + x + 1)(2x^2 + 2x - 3) = -3(1 - x - x^2)$

# Cube Coloring Problem

*Linda Dickerson-Frilot*

## Lesson Plan for Grades 5–12

## Overview

Investigate what happens when differently sized cubes are constructed from unit cubes, the surface areas are painted and the large cubes are taken apart. How many of the 1 × 1 × 1 unit cubes are painted on three faces, two faces, one face, no faces?

## Objective(s)

Students will be able to do the following:

1. Work in groups to solve a problem
2. Determine a pattern from the problem
3. Write exponents for the patterns
4. Predict the pattern for larger cubes
5. Graph the growth patterns
6. Extend to algebra

## Resources/Materials

A large quantity of unit cubes, graph paper, colored pencils or markers

## Activities and Procedures

1. Hold up a unit cube. Tell students that this is a cube on its first birthday. Ask students to describe the cube (eight corners, six faces, twelve edges).
2. Ask students to build a cube on its second birthday and describe it in writing.
3. Ask students how many unit cubes it will take to build a cube on its third birthday, fourth, fifth. . . .
4. Pose this coloring problem: The cube is 10 years old. It is dipped into a bucket of paint. After it dries, the 10-year-old cube is taken apart into the unit cubes. How many faces are painted on three faces, two faces, one face, no face?
5. Have the students chart their findings for each age cube up to 10 and look for patterns.
6. Have students write exponents for the number of cubes needed. Expand this to the number of cubes painted on three faces, two faces, one face, no faces.
7. Have students graph the findings for each dimension of cube up to 10 and look for the graph patterns.
8. For further extension, see NCTM Addenda series/ Grades 6–8.

## Tying It All Together

The students will have a chance to estimate, explore, use manipulatives, predict, and explain in writing and orally. They will note that the three painted faces are always the corners—eight on a cube. The unit cubes with two faces painted occur on the edges between the corners and increase by 12 each time. The unit cubes with one face painted occur as squares on the six faces of the original cube. The cubes with no faces painted are the cube(s) within the cube. This is an excellent way for students to become involved in exploring a problem of cubic growth.

# Assessing Cooperative Problem Solving

*Joanna O. Masingila*

*This collection of problems was originally presented at the annual meeting of the National Council of Teachers of Mathematics in San Diego, California, in April 1996 as part of a workshop on assessing cooperative problem solving.*

## Group Problem Solving

Solve the following problem with the other members of your group. Work together and *prepare one group paper to turn in to represent the group's solution*. Before you turn in your group's solution, you should be sure that *each person in your group* understands the solution well enough to answer questions about what the group did and could solve the problem on his or her own.

### Group Problem: Triangular Arrangements

The three sides of a triangle have lengths *a, b* and *c*. Also all three lengths are whole numbers and $a \le b \le c$.

a. Suppose $c = 9$. Find the number of different triangles that are possible.

b. For any given value of *c*, find a general law that expresses the number of possible triangles.

***Questions for Individuals***

1. *(2 points)* In the problem your group just solved, would $c = 9$, $b = 5$, $a = 4$ be possible values for *c, b* and *a*? Explain why or why not!
2. *(4 points)* If $c = 4$, how many triangles would be possible?
3. *(4 points)* If the number of possible triangles is 36, what is the value of *c*?

### Sample Group Solution to the Triangular Arrangements Problem

The three sides of a triangle have lengths *a, b* and *c*. Also, all three lengths are whole numbers and $a \le b \le c$.

a. Suppose $c = 9$. Find the number of different triangles that are possible.

b. For any given value of *c*, find a general law that expresses the number of possible triangles.

a. $c = 9$ and $a \le b \le c$
999 899 799 699 599 499 399 299 199 = 9
889 789 689 589 489 389 289 189 = 8
779 679 579 479 379 279 179 = 7
669 569 469 369 269 169 = 6
559 459 359 259 159 = 5
449 349 249 149 = 4 (3 marks)
339 239 139 = 3
229 129 = 2
119 = 1
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 possible triangles

b. The general way of finding all possible triangles is to add up all the whole numbers up to, and including, c. (3 marks)
1 + 2 + 3 + . . . + c = total number of possible triangles

## Scoring Scale for Group Problem Solving*

Understanding the Problem
- 0: Complete misunderstanding of the problem
- 3: Part of the problem misunderstood or misinterpreted
- 6: Complete understanding of the problem

Planning a Solution
- 0: No attempt, or totally inappropriate plan
- 3: Partially correct plan, or correct plan but not implemented properly
- 6: Plan does lead or could have led to a correct solution if problem had been completely understood

Getting an Answer
- 0: No answer, or wrong answer based on inappropriate plan
- 1: Copying error; computational error
- 2: Partial answer for problem with multiple answers
- 3: Correct answer and correct label for the answer; incorrect answer but the correct answer following from an incorrect plan or misunderstanding of the problem

Total possible points: 15

* Adapted from R. Charles, F. K. Lester and P. O'Daffer, *How to Evaluate Progress in Problem Solving* (Reston, Va.: National Council of Teachers of Mathematics, 1987).

# Cooperative Problem Solving

The following problems might be appropriate for a junior or senior high school general mathematics class.

## Group Problem: Intercom Installation

Bayview Middle School, a school with 35 teachers and 450 students, is planning to install an intercom system between all 25 classrooms and the main office. The system will permit direct conversations between any pair of classrooms, as well as between any classroom and the office. How many room-to-room and room-to-office wires will be needed?

### *Questions for Individuals*

1. *(2 points)* In the problem as stated, how many intercom wires will be used to serve one of the rooms (for example, the art room)? Explain.
2. *(4 points)* If there had been only 6 rooms in the entire school, how many intercom wires would be needed altogether? Explain.
3. *(4 points)* If there were 28 classrooms and a main office, how many intercom wires would be needed altogether? Explain.

## Group Problem: McContest

A major fastfood chain is holding a contest to promote sales. With each purchase, a customer will be given a card containing a whole number less than 100. A generous prize will be given to any person who presents cards whose numbers total 100. The company decides to print an unlimited supply of cards containing multiples of three. What other cards, and how many of each, should be printed so that there can be at most 1,000 winners throughout the country? Explain.

### *Questions for Individuals*

1. *(2 points)* Can you find a winning combination (total of 100) using only multiples of three? Explain.
2. *(4 points)* Suppose the contest described in the group problem distributes 1,000 of each multiple of three up to 100, 25 fives and 25 additional fifteens. What is the maximum possible number of winners (people with cards totaling 100)? Explain.
3. *(4 points)* A different fastfood chain decides to have a similar contest in which winners would be persons presenting cards totaling 201. The company decides to print an unlimited supply of cards containing multiples of five. What other cards, and how many of each, should be printed so that there can be at most 2,000 winners? Explain.

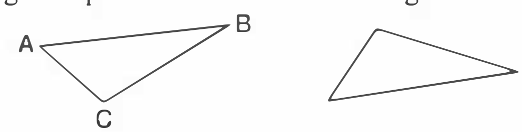
## Group Problem: Uncommon Correspondences

[The following problem could be used in a geometry class for futher exploration of triangle congruence.]

Every triangle has 6 "parts": 3 sides and 3 angles. Show that it is possible for 5 of the parts of one triangle to be congruent to 5 of the parts of another triangle without the triangles being congruent to each other.

### *Questions for Individuals*

1. *(2 points)* ΔABC = ΔDEF. ΔABC is labeled in the diagram below. The unlabeled triangle is ΔDEF. Label this triangle to show the corresponding congruent parts between the two triangles.




2. *(4 points)* Suppose you know the following about ΔABC and ΔDEF:

   AB = 3 units, BC = 4 units and m<BAC = 60°
   EF = 3 units, DF = 4 units and m<DEF = 60°

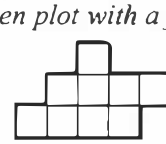
   Are ΔABC and ΔDEF congruent? Explain.
3. *(4 points)* Is it possible that the 5 parts of the two triangles discussed in the group problem that are congruent are the 3 sides and two of the angles? Explain your answer.

## Group Problem: Don't Fence Me In

[The following problem explores the relationship between perimeter and area and might be appropriate for general mathematics, algebra or geometry students.]

Rebekah is planning her garden for the spring. She has 10 square garden plots, each of the same size. One will be for carrots, one for lettuce and so on. She wants to arrange the plots so each of them has at least one side in common with another garden plot. When she finishes arranging the plots, she plans to put a "rabbit-proof" fence around the entire plot *(a sample plot is shown below).*

*Sample garden plot with a fence around it:*



a. How would Rebekah arrange her garden plots so that she would use the smallest amount of fence? How would she arrange her plots in order to use the largest amount of fence?
b. In general, what happens to the smallest (and largest) amount of fence as the number of garden plots increases? Justify your answer.

*Questions for Individuals*

1. *(2 points)* Is there more than one way to arrange the garden plots using the smallest amount of fence? Explain your answer.
2. *(4 points)* If Rebekah had 4 garden plots of the same size, how would she arrange them so that she would use the greatest amount of fence?
3. *(4 points)* If Rebekah had 18 garden plots of the same size, how would she arrange them so that she would use the smallest amount of fence?

## Group Problem: Mysterious Money

An absent-minded bank teller switched the dollars and cents when he cashed a cheque for Jana, giving her dollars instead of cents and cents instead of dollars. After buying a 5¢ stamp, Jana discovered that she had exactly twice as much left as her original cheque. What was the amount of the cheque?

*Questions for Individuals*

1. *(2 points)* If Jana's cheque was for $5.43, how much money did the teller give Jana when cashing her cheque?
2. *(4 points)* Suppose the teller cashed Jana's cheque in the same manner as in the original problem. After buying a 68¢ pen, Jana discovered that she had exactly twice as much left as the cheque she had cashed. What was the amount of the cheque?
3. *(4 points)* Suppose the bank teller cashed Jana's cheque for $11.16 in the same manner as in the original problem. After finding 63¢ in her purse, does Jana have more than twice as much money as her original cheque? Justify your answer.

## Group Problem: Hiking

[The following problems are fraction applications and might be appropriate for an algebra or general mathematics class.]

Moses was hiking from Harper to Belmont along the Winding Trail, which also passed through the town of Springfield. Forty minutes after he left Harper, Moses saw a sign reading, "From Harper to here is half as far as it is from here to Springfield." Moses hiked another 11 miles and saw a second sign reading, "From here to Belmont is half as far as it is from here to Springfield." Moses hiked for one more hour and reached Belmont. If he hiked at the same pace all the way, what is the length of the Winding Trail between Harper and Belmont?

*Questions for Individuals*

1. *(2 points)* How long did it take Moses to hike to Springfield?
2. *(4 points)* Joni hiked from Bloomington to Nashville along the Hilly Trail, passing by Knight's Korner grocery store. Thirty minutes after she left Bloomington, she saw a sign reading, "If you came from Bloomington, you have come ¼ of the way to Knight's Korner." She hiked another 11 miles and saw a second sign reading, "From here to Knight's Korner is half as far as it from here to Nashville." Joni hiked for another two hours and reached Nashville. If she hiked at the same pace all the way, what is the length of the Hilly Trail between Bloomington and Nashville?
3. *(4 points)* If the Winding Trail, in the original problem, had been 13 miles long and Moses had walked 90 minutes from the second sign to Belmont, what would be the distance between the two signs? Note: all other information is the same as in the original problem.

## Group Problem: Candles

Two candles of equal length are lit at the same time. One candle takes 9 hours to burn out and the other takes 6 hours to burn out. After how much time will the slower burning candle be exactly twice as long as the faster burning one?

*Questions for Individuals*

1. *(2 points)* After two hours of burning, how much longer is the slower burning candle than the faster burning one?
2. *(4 points)* Two candles of equal length are lit at the same time. One candle takes 6 hours to burn out and the other takes 3 hours to burn out. After how much time will the slower burning candle be exactly twice as long as the faster burning one?
3. *(4 points)* A blue candle is twice as long as a red candle. The blue candle takes 4 hours to burn out and the red candle takes 6 hours to burn out. After 3.5 hours, how long are each of the candles? The blue candle's length is now what fraction of the red candle's length?

# Problem Solutions

## Triangular Arrangements

*Group Problem*

a. There are 25 possible triangles.

b. If $c$ is odd, the total number of possible triangles is the sum of the positive odd integers $\leq c$. If $c$ is even, the total number of possible triangles is the sum of the positive even integers $\leq c$.

*Individual Questions*

1. No, $c = 9$, $b = 5$ and $a = 4$ are not values that can form a triangle because $a + b$ is not greater than $c$.
2. If $c = 4$, then $4 + 2 = 6$, so there are 6 possible triangles.

3. Since for $c = 9$, there were 25 triangles, and $36 - 25 = 11$, then c must be odd and $c = 11$ because $11 + 9 + 7 + 5 + 3 + 1 = 36$.

## Intercom Installation

### *Group Problem*

Since there are 26 rooms and each room is connected to 25 rooms, there are 26 × 25 connections. However, you must divide by 2 to eliminate counting the wires twice. Thus, there are 325 intercom wires.

### *Individual Questions*

1. One room has 25 intercom wires serving it.
2. If there were six rooms, 15 intercom wires would be needed.
3. If there were 29 rooms, 406 intercom wires would be needed.

## McContest

### *Group Problem*

One answer would be to print at most 1,000 cards with 1 printed on them. Another possibility would be to print 2,000 with 50 printed on them.

### *Individual Questions*

1. A winning combination cannot be made using only multiples of three because the sum of any multiples of three is still a multiple of three.
2. The maximum possible number of winners is 12 since it is possible for a person to total 100 using only two fives.
3. One answer would be to print at most 2,000 cards with 1 printed on them.

## Uncommon Correspondences

### *Group Problem*

The five congruent parts are the three angles and two sides. The key is that the congruent sides must be positioned differently in the two triangles.

### *Individual Questions*

1. Label the triangle D, F, E by starting at the lower left and moving clockwise.
2. No, two triangles with angle-side-side congruence do not have to be congruent.
3. No, two triangles with side-side-side congruence are always congruent.

## Don't Fence Me In

### *Group Problem*

a. The smallest amount of fence would be used for an arrangement that is the most perfect (that is, closest to a square). For 10 plots, this would be either a 3 × 3 + 1 arrangement or a 2 × 5 arrangement. The largest amount of fence would be used for an arrangement that is the least compact. For 10 plots, this would be a 1 × 10 arrangement.

b. In general, as the number of plots increases by 1 plot, the amount of fence increases by 2 sides of fence. For the least compact arrangements, this is always true. For the most compact arrangements (smallest amount of fence), the amount of fence increases by two for the first plot added to a rectangular arrangement but remains at this amount up through the next rectangular arrangement. For example, for 9 plots the smallest amount of fence is used by a 3 × 3 arrangement (perimeter of 12). For 10 plots (3 × 3 + 1 arrangement), the perimeter is 14. For 11 plots (3 × 3 + 2), the perimeter is also 14, as is the amount of fence for 12 plots (3 × 4 arrangement). Then for 13 plots (3 × 4 + 1 arrangement), the perimeter increases to 16.

### *Individual Questions*

1. Yes. It could be for a 2 × 5 or a 3 × 3 + 1 arrangement.
2. A 1 × 4 arrangement would require the most fence.
3. A 3 × 6 arrangment would require the least amount of fence.

## Mysterious Money

### *Group Problem*

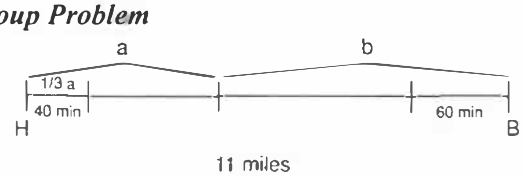
Use an equation such as $100c + d - 5 = 2(100d + c)$, where $d$ = # of dollars and $c$ = # of cents, and simplify to $c = (199d + 5)/98$. Then using an organized list to find possible solutions will yield $31.63 as the amount of Jana's cheque.

### *Individual Questions*

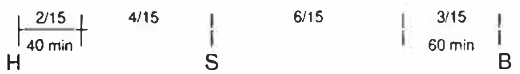
1. Switching the dollars and cents, the teller gave her $43.05.
2. Using a solution process such as the one used in the group problem will yield $10.21 as the amount of the cheque.
3. The teller gave Jana $16.11, adding 63¢ gives a total of $16.74. $16.74 is 1.5 times as much as the original cheque of $11.16.

## Hiking

### *Group Problem*




From H to S took 120 minutes; from S to B took 180. Thus Moses walked a total of 300 minutes.




$x$ = total distance from H to B
$(4/15 + 6/15)x = 11$
$x = 33/2 = 16.5$

The Winding Trail is 16.5 miles between Harper and Belmont.

### *Individual Questions*

1. Moses hiked for 300 minutes or 5 hours.
2. Using a method similar to the one in the group problem yields an answer of 22 miles for the length of the Hilly Trail between Bloomington and Nashville.
3. The distance between the two signs is $8^2/_3$ miles. This can be found by using the method illustrated above.

## Candle

### *Group Problem*

After 4.5 hours the slower burning candle will be exactly twice as long as the faster burning one.

### *Individual Questions*

1. After 2 hours, the slower burning candle has burned 4/18—leaving 14/18 of the candle. This faster burning candle has burned 6/18—leaving 12/18 of the candle. Thus, the slower candle is 2/18 or 1/9 longer than the faster burning candle.
2. Divide the candle into sixths. After 2 hours, the slower burning candle has burned 2/6—leaving 4/6 of the candle. The faster burning candle has burned 4/6—leaving 2/6 of the candle. Thus the slower burning candle is twice as long as the faster burning candle after 2 hours.
3. Divide the red candle into 12/12 and the blue candle into 24/12. After 3.5 hours, the blue candle has burned 21/12—leaving 3/12 of the candle. The red candle has burned 7/12—leaving 5/12 of the candle. Thus the blue candle's length is now 3/5 of the red candle's length.

**Number Game**
Find all two-digit natural numbers, which are three times the sum of their digits.

**Completely "Variable"**
The variables $a$, $b$ and $c$ of the expression $a(c-b)/(b-a)$ are to be substituted with 13, 15 and 20 in such a way that the value of the expression results in a positive whole number.

# Of the Fourth Dimension

*Sandra M. Pulver*

Until the beginning of the 20th century, mathematicians and lay people alike looked on geometries of more than three dimensions with skepticism. It was believed that physical conditions alone precluded the existence of more than three dimensions.

The ancient Greeks devoted much time to geometry and the concept of dimensions, but concentrated solely on one, two and three dimensions. In the fourth century B.C., Aristotle wrote in his book *Heaven* that "the line has magnitude in one way, the plane in two ways, and the solid in three ways and beyond these there is no magnitude because the three are all" (Hess 1977, 1–2). Greeks calculated area and volume in geometric terms. When they studied equations and their solutions, they did so within the framework of geometry. Therefore, they regarded equations higher than cubic as unreal. Girolamo Cardano (1501–1576) stated, "The first power (of a number) refers to a line, the square to a surface, the cube to a solid, and it would be fatuous indeed for us to progress beyond for the reason that it is contrary to nature" (Eves 1969, 212).

Some progress seemed to be made by the middle of the 16th century, as evidenced by Michael Stifel's statement in 1553 that in arithmetic "we set down corporeal lines and surfaces and pass beyond the cube as if there were more than three dimensions, although this is contrary to nature" (Eves 1969, 212). However, it was only with the advent of Einstein's theory of relativity that discussion of four-dimensional space began to be spoken of more realistically. It was realized that physical existence or nonexistence of a four-dimensional body in our universe has nothing to do with its existence as a mathematical entity.

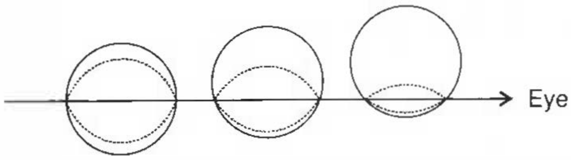
When approaching a new geometry such as four-dimensional geometry, problems that arise are almost always perceptual rather than conceptual in nature. Euclidian geometry can be used as a springboard for the study of four-dimensional space, or $E_4$, as it is sometimes called, to make their conceptual part less complicated.

Just as we had to assume the existence of a point not on a given plane to work with three dimensions, to enlarge our space for the study of four dimensions, we must now assume the existence of a point not belonging to our three-dimensional system. We will thereby be creating a new space in which there will exist many three-dimensional subspaces (similar to the planes in a three-dimensional space). We should logically include a name for these subspaces, and we shall accept the word "prime" as the name of our three-dimensional subspace. And just as we use parallelograms to denote planes in our study of $E_3$ geometry, we will represent primes by parallelepipeds:

**Definition**: The points of a set are said to be coprimal if and only if there is a prime which contains them all. This is not to suggest that a prime has faces like a cube or is any way limited in size. A prime extends infinitely in all directions. At this point many of us encounter a significant perceptual difficulty.

Given the definition, should there not, in effect, be only one prime in existence? Let us try to put in perspective the concept of the existence of more than one prime.

In the late 19th century, Edwin Abbott put forth a way of conceiving of a fourth dimension in his fictional book *Flatland, a Romance of Many Dimensions*. He asked us to imagine "people" in a two-dimensional world. Their "universe" would be a flat plane. Now think of how impossible it would be to visualize through the eyes of those people our three-dimensional world or even to see a three-dimensional object passing through their two-dimensional world. Because they could perceive only in two dimensions, they would see a sphere passing through their world as a circle, gradually increasing in size and then gradually decreasing.




Sphere Passing Through a Plane (Abbott 1952, 73)

The inhabitants of Flatland can see only on their plane. They are not physically capable of looking "upward" or "downward" to see that there are an infinite number of planes or "universes" above and below theirs.

It would be incredibly hard for a two-dimensional being, who lived on a vast plane and who understood "forward, backward, left and right" perfectly well, to understand the concepts of "up and down." The confusion about the third dimension would be similar to the confusion we feel when thinking about the fourth.

We, as three-dimensional beings, can see the sphere and know that there are innumerable plane "universes" and we can pass freely through them all.

Now consider our three-dimensional universe as the plane of the Flatlanders. Through analogy, it stands to reason that there are other planes, or in the case of the fourth dimension, other primes, in addition to ours. We tend to dismiss this idea because we have no physical means of perceiving it or traveling to another prime.

It is true that another prime does not lie "next" to ours or "on top" of ours as planes lie on top of each other. Nevertheless, it follows that other primes, other dimensions, exist somewhere in relation to ours. If a solid can pass through planes, there should be a four-dimensional "hypersolid" that can pass through different primes, and when these hypersolids "disappear," they are actually passing out of our prime.

Because we have concluded that four-dimensional space should consist of more than one prime, we have to introduce new postulates describing their intersection properties:

***Four-Dimensional Euclidean Geometry Postulates***

1. Every line is a set of points and contains at least two points.
2. If X and Y are any two points, there is one and only one line which contains them.
3. Every plane is a set of points and contains at least three noncollinear points.
4. If X, Y, Z are any three noncollinear points, there is one and only one plane which contains them.
5. Every prime (three-space) contains at least four points which are neither collinear nor coplanar.
6. If two points of a line lie in a plane, then every point of the line lies in the plane.

[The remaining postulates hold true only in four dimensions]

7. If W, X, Y, Z are any four noncoplanar points, there is one and only one prime which contains them.
8. Space contains at least five points which are neither collinear, coplanar or coprimal.
9. If three noncollinear points of a plane lie in a prime, then every point of the plane lies in the prime.
10. If a plane and a prime have a point in common, their intersection is a line.

Other postulates that hold equally true in both three- and four-dimensional space include, among others, the parallel postulate and the plane separation postulate.

"The geometry of *n* dimensions is an intellectual journey that takes us through fascinating and purely mental country, and never ends" (Reid 1959, 109)

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# Algorithms and Technology

*Lynn Gordon Calvert*

## Part 1

Scenario 1: I wonder what would work better for this problem a spreadsheet or a computer program?

Scenario 2: The program and instructions are for Logo and the only graphing software I have is Geometer's Sketchpad. What would it take to convert this to a Geometer's Sketchpad script?

Both scenarios point to the ever-increasing variety of technology available to us in our homes and in our classrooms. Modeling a mathematical situation when several technological devices are available requires decision making about ease of use, efficiency of algorithm and the output desired or preferred. To make an informed decision, creating or re-creating an algorithm using different tools requires a global understanding of (1) the problem, (2) the algorithm used to simulate the situation and (3) common features and processes used across technology. As mathematics educators, we must promote the use of specific technological devices and provide opportunities for students to develop the ability to adapt to the technology that is or becomes available to them. We must also decide which tool is best suited for modeling the mathematical situation at hand.

The following radioactive decay simulation illustrates the benefits, limitations and decision making involved when choosing different technological tools to simulate the same event.

### Radioactive Decay

A rich site for mathematics is the simulation of radioactive decay. Exploration into this area may initially be simulated with dice (Lovitt and Lowe 1993). Starting with 50 dice, all active "particles" are rolled. "Six's" are particles that have "decayed" in that year and are removed from the active ones. This process is repeated until all the particles have decayed which usually takes somewhere between 14 to 35 years. Exponential decay, the graphical analysis of half-life, rate of decay, and experimental vs. theoretical probability all come into play. Exploring the phenomenon further—that is, to approximate the average number of years for 50 particles to decay, determine the distribution of this event, compare the half-life and total decay for 50, 100 or 20,000 particles or change the rate of decay—is too time consuming or simply impossible to do with dice . . . but not with technology.

Creating an algorithm with technology to simulate this event requires an awareness of the process or iterative algorithm involved and the application of this process to the technology available. Spreadsheets and programmable calculators are used here to illustrate two possible sources to model the radioactive decay problem.

### Spreadsheet

*Display*

**Figure 1: Spreadsheet Display**

| | A | B | C | ... | AG | AR | AS |
|---|---|---|---|---|---|---|---|
| Atom 1 | 1 | 1 | 0 | | 0 | 3 | 6 |
| Atom 2 | 1 | 1 | 1 | | 0 | 3 | 2 |
| Atom 3 | 1 | 0 | 0 | | 0 | 6 | 1 |
| ... | | | | | | | |
| Remaining | 50 | 43 | 32 | ... | 0 | | |
| Year | 1 | 2 | 3 | ... | 25 | | |

This spreadsheet is set up so that columns A, B, C and so on show individual particles that are active ('1') and particles that have decayed ('0'). Column A represents the beginning of year one. All particles are active. Column B represents the beginning of the second year. Connected to column B is column AR which is a series of randomly generated numbers from 1 to 6. If a '6' was generated, the particle was said to decay in that year.

**Figure 2: Spreadsheet Chart Display**

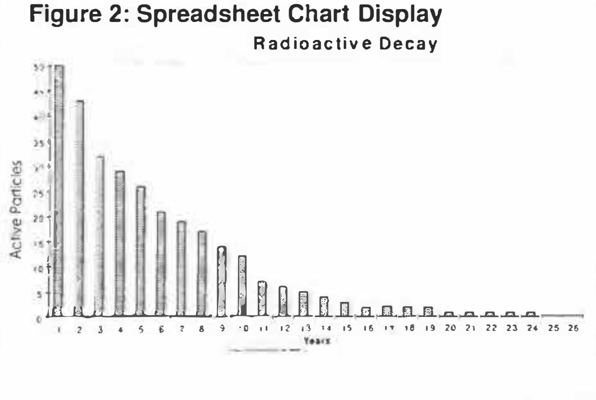


**Figure 3: Spreadsheet Formulas**

| Year 1<br>All active | Year 2<br>Checks random number value | All other years<br>Checks previous year and then random number value | Random Numbers<br>(1–6) |
|---|---|---|---|
| 1 | =IF(AR2<5,0,1) | =IF(B2=0,0, IF(AS2>5,0,1)) | =INT (RAND()*(5)+1) |

## TI-82 Calculator

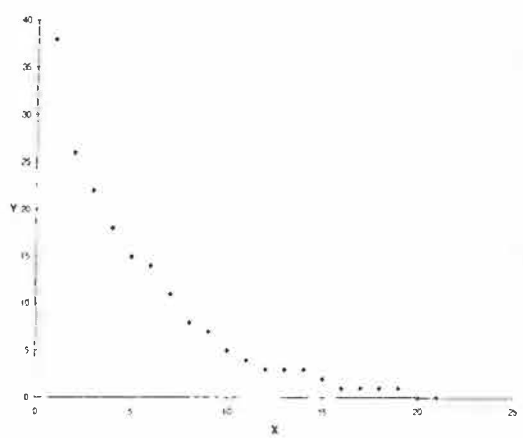
### *Display*

**Figure 4: Calculator Screen Display**

```
YEARS
                                         20
{38   26      22      18      15        ...
                                       Done
```

This program calculates how many years it took for all 50 particles to decay (20 years) and it also contains a list of particles remaining after each reiteration of the program. That is, after year 1 there were 38 particles active, after year 2 there were 26 particles active and so on.

**Figure 5: Calculator Graph Display**



### *Program*

The variables used include the following:

A = the number of active particles remaining
T = the number of decayed particles (a counter for the number of times 6 occurs)
P = the number of years or loops required to get to zero active particles
R = a random number between 1 and 6
L1 = the list of the number of active particles remaining at the end of each year
K = an index counter in the "for" loop

**Figure 6: TI-82 Decay Program**

```
PROGRAM: DECAY
: ClrHome
: ClrDraw
: 0 → a → T
: 0 → P
: 50 → A
: While A>0
: 0 → T
: P + 1 → P
: For (K, 1, A)
: int (rand*6+1) → R
: If R = 6
: Then
: T + 1 → T
: End
: End
: A – T → A
: Pt-On (P, A)
: A → L1 (P)
: End
: Pause
: Disp "YEARS", P
: Disp L1
```

Note: Active particles remaining are totaled at the beginning of each year in the spreadsheet and at the end of the year in the calculator program.

Modeling radioactive decay using different forms of technology and providing an opportunity for students to display and discuss their results develop a richer understanding of the situation and of the iterative algorithm involved. Similar features of the algorithms include generating a random number between 1 and 6 and finding its integer value. Related outcomes include determining whether a particle decays or not and counting the number of active particles remaining each year; however, the procedures used to determine these are significantly different depending on the technology used. For instance, a nested loop is used on the spreadsheet to determine particles' activity individually. The algorithm is to initially reference the previous column to see if the particle decayed in the previous year (If B2 = 0 then 0); then, if it is still active it references the random number generated and decides whether it decays or remains active (If AR2 > 5 then 0, otherwise 1).

The "For" loop and "List 1" in the calculator program does this quite differently. The algorithm here keeps a count of the particles remaining (A), randomly generates a number A times, counts how many 6s occurred and subtracts that from the previous total to determine the number remaining this year.

In discussions, students choosing to use a spreadsheet will find that defining the steps needed to simulate the situation and create a graph are simpler to complete than those choosing a programmable calculator; however, changing the parameters of the problem, such as the number of initial particles and the rate of decay, is much more cumbersome making it more difficult to explore related situations. The discussions surrounding the connections made between the displayed results, about the similarities and differences between the algorithms created and about the possibilities and constraints of the tools chosen become the most interesting and perhaps the most valuable part of this activity.

## Part 2

Scenario 3: Hey, I've written a program on my TI-82 for something like this before. If I change it a bit and add a few lines, I can probably use it for this problem.

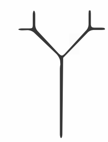
The scenario above approaches algorithms and technology from a perspective different from the previous discussion. Rather than starting from scratch, previously known algorithms can be altered or extended to model related events or to create new ones. Providing students with opportunities to recognize and analyze relationships between different situations and their algorithms promotes the development of mathematical connections. It also provides more creative opportunities in the mathematics classroom by allowing students to explore algorithms they are familiar with to create or pose new problems.

The following activity explores the extension of a basic recursive algorithm to create new mathematical objects. Recursion is an interesting phenomenon found in stories ("For my third wish," said the peasant to the genie, "I want three more wishes."), in art (Escher prints) and in nature (fractals). Its appeal and complexity are inherent in its structure: "a recursive definition is a circular definition that manages to avoid paradox. When something is defined recursively, it is defined in terms of itself" (Poundstone 1985, 123). Recursive programs are usually compact because they reduce the given problem to a simpler one or a subroutine of the same structure. However, this is also the cause of their complexity. After providing instruction on recursion, Harvey (1992, 432) stated that "almost all of the students could understand recursive procedures that [he] presented, but not all could reliably write their own recursive procedures." It may be inappropriate to expect that all secondary students be able to write their own recursive programs, but that should not exclude recursion from the curriculum. Students are quite capable of understanding basic recursive algorithms and making appropriate changes or additions necessary for the task.

Fractal trees can be created using technological devices that support graphics and recursion. For example, Logo and Geometer's Sketchpad can both be used to create binary trees. The basic geometric structure of the binary tree is Y-shaped; that is, each branch divides into two new smaller branches half the size and form a 90° angle symmetric about the previous branch. The Terrapin Logo program and the binary tree (level 3) are shown in Figure 7.

**Figure 7: Binary Tree Logo Program**

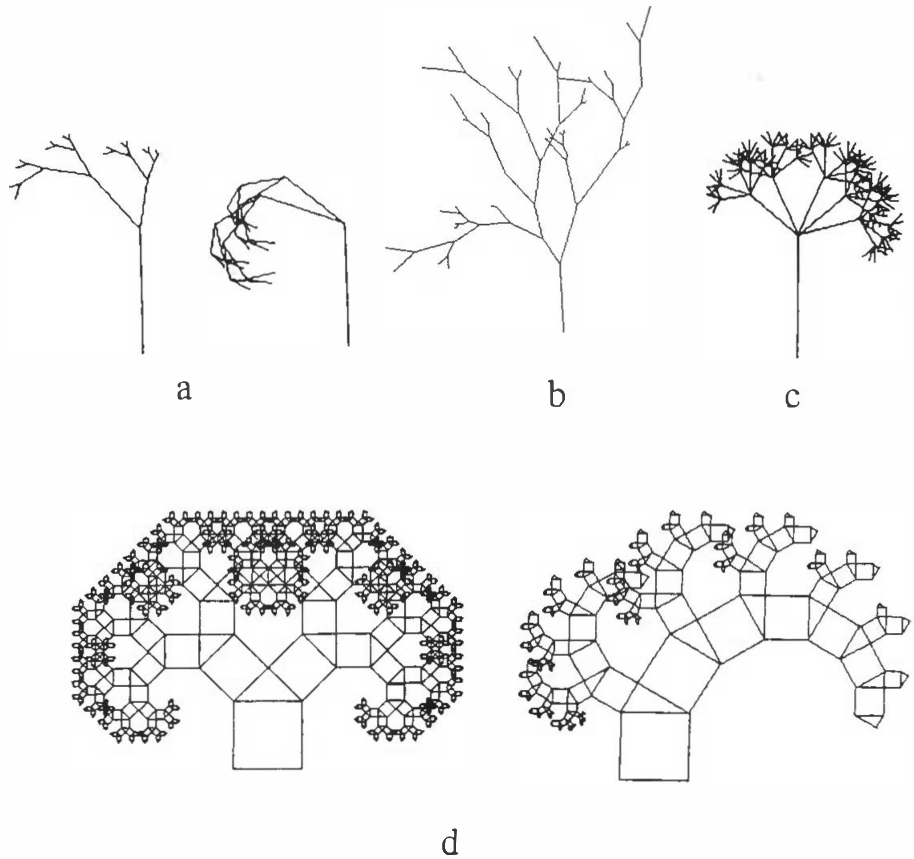
```
TO TREE :SIZE :LEVEL
  IF :LEVEL < 1 [STOP]
  FORWARD :SIZE
  LEFT 45
    TREE :SIZE/2 :LEVEL - 1
  RIGHT 90
    TREE :SIZE/2 :LEVEL - 1
  LEFT 45
  BACK :SIZE
END
```



Binary Tree in Logo and Geometer's Sketchpad

Again, it is of interest to compare the algorithms and products created from different technological tools (as we did in Part 1), but this particular activity also allows a number of creative possibilities by altering and extending the basic algorithm to create new but related objects. Once an algorithm is understood, a student is free to experiment by making changes and additions to it. For example, the Logo program above provided the basic structure of the fractal tree. Simple adjustments were made that altered the length and angle of the branches (Figure 8a); a random number generator available in Logo was used to produce branches of random lengths (Figure 8b); adding another call for recursion in the algorithm created trees or bushes with three or more branches extending from each node (Figure 8c); and numerous other variations are limited only by the imagination (Figure 8d Bosman's Pythagorus tree and a lopsided version).

Figure 8: Alterations and Extensions to Binary Tree



Using a basic algorithm and then adjusting it to fit alternate needs is a technique that computer programmers use frequently. It requires an understanding of the original program and problem solving skills to change it appropriately. Through investigation, experimentation and exploration, it is possible for algorithms to be used toward creative mathematical thinking.

## Conclusion

Although we have broadened our view of the place of algorithms in mathematics, our increased access to a variety of technological tools requires that we push our thinking even further. The recent attention to discrete mathematics promotes the view of algorithms as useful problem solving aids. We presently encourage students "to develop and analyze algorithms" (NCTM 1989, 176) and to compare the efficiency of algorithms (Maurer and Ralston 1991); however, there is a need to include activities that allow students to make decisions as to which technological device best suits their purposes, to alter one algorithm to fit the constraints of a different tool and to alter and extend previously known algorithms to model related events or to create new ones.

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# Reflections on an Extracurricular Parent-Child Mathematics Program

*Elaine Simmt*

*"What do you want me to do?" the man asked his daughter sitting at the table beside him.*

*The young girl paused and then looked up to her father, "We should probably look for a pattern. That might help us with the other ones."*

*"Okay, I'm game. So, what do you think?"*

*"Maybe—I know! Let's make a chart."*

*"Did you do this one?" the eight year old asked her mother as she pointed to a set of dominoes arranged in a path.*

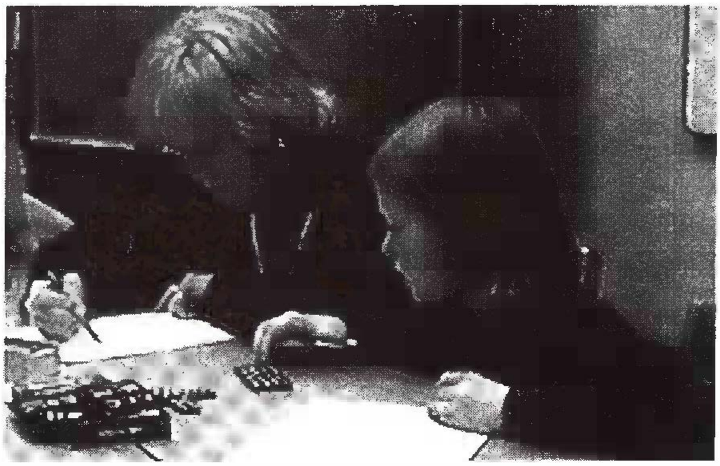
*The mother looked over her records of the tiling patterns she had made. "One in the middle— No! That's good, Casey. Okay. Wait a minute— one, two, three, four, five. No, I didn't get that. Good."*

*The girl continued to scan her mother's records for other missed tilings.*

*"I think, Casey, that I might as well just go home." The mother laughed.*

*"I can't believe you didn't get that one, Mom!"*

*"Well—you know what I never thought of . . ."*



## A Mathematics Program for Parents and Children

These parents and children are participating in an extracurricular mathematics program developed in conjunction with a local school board. The program was set up in response to a need expressed by some parents to supplement their children's experiences with mathematics. The program is designed to encourage and challenge students, aged 8–14, and their parents to engage in mathematical thinking and problem solving and at the same time to educate parents to help their children with school mathematics.

The program consists of ten 1.5-hour sessions. Each session begins with a short opener, such as a number game, a game of strategy or a puzzle, that takes the first 10–15 minutes (Appendix 1). For the rest of the evening, parents and children work together in response to a variable-entry prompt (Appendix 2). This part of the session might be thought of as problem solving, although many participants (adults and children alike) begin to think of themselves as playing with mathematics rather than solving problems. I have come to think of their activity as bringing forth a world of mathematics.

## About the Participants

In general, the participants (parents and children) come to the sessions wanting to learn more about doing mathematics. Parents come because they want to be able to better help their children, and the children come so they can do better at their school mathematics. Some children explained it as follows:

- *I want to find shortcuts and hints to make it easier.*
- *I hope I can bring my grades up and my understanding of what the question is asking.*
- *Let me speed up.*
- *Let me understand so that I won't be frustrated. To get out of my special help classes.*
- *To do math with my dad.*
- *Just help me and for me to like math.*
- *I hope it can give me a better view of math and explain problems clearly.*

Not surprisingly, the motivations for the parents are somewhat different than for the children, although it is not difficult to see how they fit with the children's purposes for participating in the mathematics program:

- *I want to learn ways to make math more enjoyable and easier for my daughter to understand.*
- *Help me to help my son and others to enjoy math and see the importance of it.*
- *I want to learn about the math my daughter is learning in junior high school.*
- *Provide me with the tools to help me coach my daughter in math.*
- *Teach me patience in helping my daughter to learn and enjoy math.*
- *I want to spend quality time with my daughter and this seems like a unique opportunity to do this.*

Participation in the program is often flavored by a person's attitudes toward mathematics. More than once, an adult has hesitated to engage in an activity, suggesting that he or she is not good at mathematics or does not like doing mathematics. The parents said:

- *I do not like doing mathematics. I find it difficult. I find it vague.*
- *I like working with figures.*
- *I don't mind math. It affects everyone everyday. Simple math (+, –, ×, ÷) is enjoyable.*
- *I like doing math. Most of it makes sense. For the most part, I understand it.*
- *I like math. It is straightforward. It's logical and it's not ambiguous.*
- *I like the challenge. I like the feeling of "Oh-yea" when you think of something in a new way or gain a fresh insight.*
- *I am very poor at mathematics. It has always been a fear of mine. My mind just doesn't work that quickly—but hey, give me a computer and I'll soar.*
- *I don't particularly like mathematics. There are not enough grey areas.*

The children's comments were not much different:

- *I don't like doing mathematics. It is confusing and hard.*
- *I like it because I like working with numbers.*
- *I like figuring things out.*
- *I don't like doing mathematics. It's hard and boring.*
- *I don't like mathematics because I find it hard to understand basic things. When I move on to harder things that have basic steps in it I find it hard.*

I intended to provide the parents and children experiences with mathematics that were challenging but also fun, interesting and do-able. I hoped this would foster participants' positive attitudes toward mathematics and beliefs in themselves as quite capable of doing mathematics while enhancing their understanding of mathematics.

## Facilitating the Program

For the most part, my responsibility was to bring to the group different activities. I tried to vary these activities so that the students worked in different areas of mathematics. I also tried to come up with activities that tied into their school work but that did not look like typical school mathematics. To facilitate diversity among participants (in terms of their ages, background knowledge and experiences in mathematics), I used variable-entry prompts. These prompts open a space for mathematical activity at varying levels of mathematical sophistication and occasion a variety of actions. The prompts are such that the participants do not require specific background knowledge or specific mathematical skills; however, the prompts must be intriguing and they should lead to important mathematical ideas, concepts and processes.

Obviously, my role in the math program was different from my role in my previous teaching experiences. I did almost no explaining. I provided no answers to the problems (although I did offer some of my solutions after we had exhausted participants' solutions). Mostly, I paraphrased or repeated what someone else said. I often found myself wandering around looking over shoulders and just listening.

Although participants looked to me to provide the prompts for the weekly activities, they learned to not depend on me to tell them what to do and how to do it. In most cases, both child and parent focused on doing the activity together (and sometimes apart) and, in doing so, learned new mathematics. I did have to provide more guidance to some participants than others. For some pairs, I had to come up with many small tasks within the general prompt.

I did consider it my responsibility to teach parents how they could interact with their children in the context of mathematics. For the most part this meant modeling the kinds of questions parents might ask their children when doing mathematics. I used such questions as "How did you do that?", "Why did you do that?" and "Could you explain that to me?" It took a few classes to convince most parents that they did not need to tell their children "the answer" but rather that they needed to pay attention to their children's thinking. When it came to mathematics the parents had not done or prompts they had never worked on (almost always the case), I suggested that one of the best strategies is simply for the parent to do the mathematics with the child. Finally, I tried to recommend books (Appendix 3) and activities that would be interesting to do at home.

## Parents and Children Doing Mathematics Together

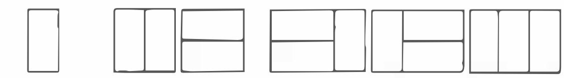
Young learners in classrooms often do much of their mathematical thinking independently of one another and usually have little interaction with an adult who is a fellow learner. In the parent-child program, each child learns alongside an adult learner. When observing the mathematical activity of parents and children working together, I am struck by how the mathematical understanding of both adult and child are enhanced through such interaction.

Let's go back to the two parent-child pairs introduced at the beginning of the article. This vignette is taken from a night when participants worked with the prompt in Figure 3.

Dave and Krista, the father/daughter pair in the first vignette, worked together. Krista arranged the dominoes and called out the tilings to her dad who kept records of the tilings in a table (Figure 4). Together, Dave and Krista examined the number pattern that was being generated and then determined a rule to satisfy the growth pattern they observed. They found that they could determine the number of tilings given a particular number of tiles in a set if they knew the number of tilings for the previous two sets of tiles. It is important to note here that both the parent and the child were involved in looking for a solution to this problem. The father did not know what the sequence would turn out to be in advance. And it is not clear who actually saw the recursive relationship first—Dave or Krista. What is clear is that both Dave and Krista engaged in mathematical thinking and, through their interaction with each other and by manipulating tiles and reflecting on the records they kept, both came to better understand the sequence they created through their mathematical activity.

**Figure 3: Tiling Prompt**

How many paths can you tile with a given number of dominoes (2×1 tiles) if the path must be two units wide?



1 tile 1  2 tiles 2 ways  3 tiles 3 ways

In contrast, Robin and Casey, the mother/daughter pair from the second vignette, were so interested in the geometry of the tilings that they did not bother to look for the relationship between the number of tiles and the number of tilings that could be generated by the set. Instead, they played with the tiles and the images of the tilings as geometrical objects. Their interaction focused on the placement of the dominoes, the symmetries they noticed and the use of mirror images to generate new tilings. Their records captured the tilings not as abstractions noted with hatch marks in a table but as full-page drawings of the domino patterns themselves (Figure 5). For Robin (the mother), the prompt for mathematical actions was shaped by her ability to draw and her interest in shape and orientation of shape in relation to space. Casey, too, was taken by the arrangements the tiles made and was quite interested in knowing if she had found *all* the possible arrangements. When Robin and Casey checked to see if they had all the tilings, it was the geometry of the patterns that provided them with a means of checking their records.

The variable-entry nature of the prompt left room for Robin and Casey to interact with each other and with mathematics in ways quite different from how Dave and Krista interacted. Although the mathematics Robin and Casey did was not the same as Dave and Krista's, in both cases the parent and child interacted in the context of mathematics and, by doing so, enhanced their personal and shared understanding of mathematics. Dave and Krista's understandings of sequences grew based on their work with number patterns. Robin and Casey's understanding of geometry grew based on their work with symmetry, flips and rotations.

**Figure 4: Dave and Krista's Table**

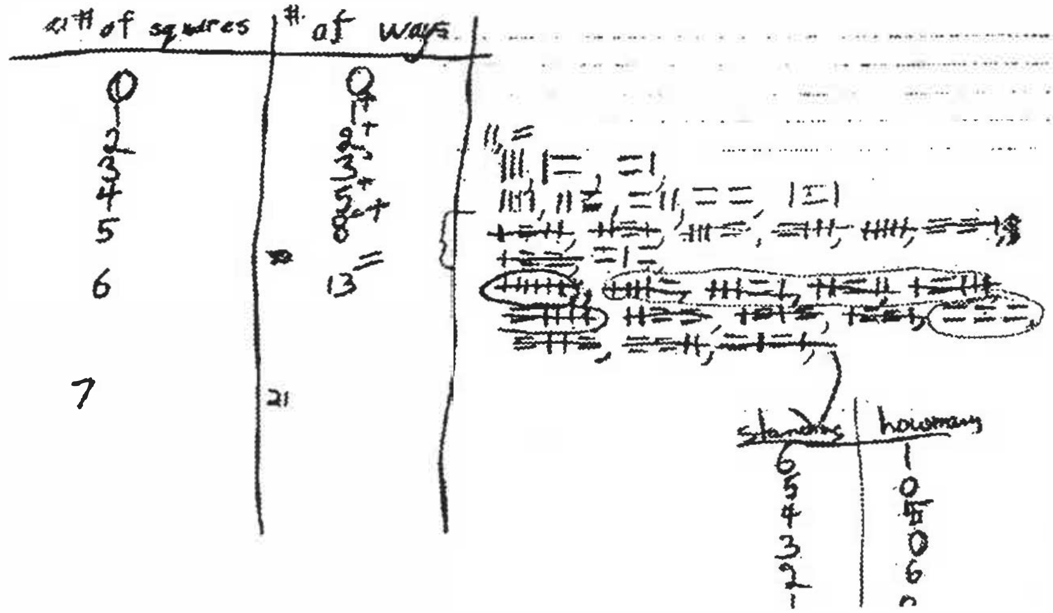
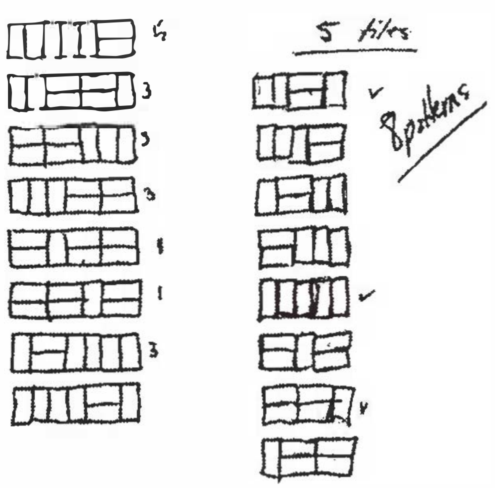


**Figure 5: Casey (left) and Robin's (right) Pictures**



## Involving Parents

At a time when teachers and parents alike are looking for ways to enhance children's mathematical understanding, there is a need to consider alternative roles for parents to play in their children's mathematics education. Traditionally, parents have helped children with their homework. But perhaps what is needed is some work for parents and children to do together at home. Mathematics educators need to encourage parents to do mathematics with their children—to engage in purposeful, meaningful and significant mathematical activity. There is a need for programs and materials that might facilitate such activity. However, it is important that these programs and materials include the parents not as proctors or tutors but as fellow learners; that is, a person whose thinking stimulates the child's and whose thinking in turn is stimulated by the child. The extracurricular mathematics program for parents and children discussed here is one possibility for facilitating such interaction between parents and children.

## Appendix 1
## Warm-Up Activities

***Guess My Number***

The parent and child take turns trying to guess the other's secret number which is between 1 and 20 (for example).

***Xs and Os***

Parent and child play a game of Xs and Os on a 3 × 3 grid.

***Mathematics in the Room***

Have parents and children look around the room and identify mathematical "things" they can see, hear and touch.

### *Doubling*

Ask parents and children to decide which method, A or B, they would prefer to calculate the weekly allowance: Method A: \$1/day, Method B: 1¢ on the first day and double the amount of the previous day every day thereafter.

### *Guess My Rule*

The leader begins the session by putting on the board a set of numbers that share a common property or are generated by the same rule and then has the parents and children work on guessing the property or rule. Then the parents and children make rules for each other to guess.

# Appendix 2

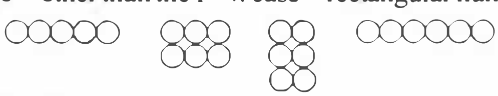
## A Sample of the Variable-Entry Prompts Used in this Program

### *Handshake Problem*

How many handshakes would there be if all of us in this room shook hands with each of the other people just once. If there were 20 of us, how many handshakes would there be?

### *Rectangular Numbers*

Using bingo chips, find the numbers that form rectangles. For example, five chips can form only a 1 × 5 line, whereas six chips can be arranged in a 2 × 3 or 3 × 2 rectangle, as well as the 1 × 6 line. We will call those numbers of chips for which we can form a rectangle—other than the 1 × $n$ case—rectangular numbers.



### *Pentominoes*

Using graph paper, you are to make as many shapes as you can using five squares. The squares must be touching another square on at least one edge.

This is an example. This is not an example.

### *Common Letters*

What do you think is the most common letter used in the English language? Using a book, newspaper or magazine, try to determine the most common letters.

### *Halloween Statistics*

Without showing each other your candy bag, find a way to show the rest of us how much candy you collected on Halloween and the various kinds of candy you collected.

### *Mobius Bands*

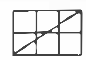
Take a strip of adding machine paper and tape the ends together. Now trace the path an ant would take walking along that tape. How many sides does the band have? Cut the band along the ant's path. How many bands do you have now? Now do the same things but put a twist in the band before you tape it together and then trace and cut the ant's path. Can you predict what will happen. What if the number of twists changes?

### *Square Takeaway*

Cut a rectangle (not a square) from a sheet of graph paper. What is the largest square that can be cut from your rectangle? How many squares can you cut from the leftover piece of paper before you are left with a square? Try this for a number of different rectangles. What do you notice?

### *Diagonals*

Mark off a rectangle on a piece of graph paper. Draw in one of the diagonals. How many squares does the diagonal pass through? Do this for different rectangles. What do you notice?



# Appendix 3

## Resources for Facilitating Parent-Child Mathematical Activity

I have found the following resources suitable for parents and children. Although not all the prompts, questions, problems and concepts are what I would consider to be variable-entry, these books are quite accessible to parents and children (especially if considered by a parent and child acting together).


- *Thinking Mathematically* by J. Mason, L. Burton and K. Stacey. London: Addison-Wesley, 1982.
- *Mathematical Investigations in Your Classrooms: A Pack for Teachers* by S. Pirie. Basingstoke, U.K.: Macmillan, 1987.
- *The Joy of Mathematics: Discovering Mathematics All Around You* (1989) and *More Joy of Mathematics: Exploring Mathematics All Around You* (1991) by T. Pappas. San Carlos, Calif.: World Wide Publishing/Tetra, 1989, 1991.
- *Math for Smarty Pants* (and other titles) by M. Burns. Boston: Little, Brown, 1982.
- *Family Math* by J. K. Stenmark, V. Thompson and R. Cossey. Berkeley, Calif.: University of California, Lawrence Hall of Science, 1986.
- *Exploratory Problems in Mathematics* by F. W. Stevenson. Reston, Va.: National Council of Teachers of Mathematics, 1991. (Look for other resources by the NCTM.)
- *Puzzles, Mazes and Numbers* by C. Snape and H. Scott. Cambridge: Cambridge University Press, 1995.

# MCATA Executive 1997–98

**President**
Florence Glanfield — Res. 489-0084
8215 169 Street NW — Fax 483-7515
Edmonton T5R 2W4 — glanfiel@gpu.srv.ualberta.ca

**Past President and 1998 Conference Cochair**
George Ditto — Res. 289-1709
610 18 Avenue NW — Bus. 777-8927
Calgary T2M 0T8 — Fax 777-7229
gditto@cbe.ab.ca

**Vice President**
Cynthia Ballheim — Res. 278-2991
612 Lake Bonavista Drive SE — Bus. 228-5810
Calgary T2J 0M5 — Fax 229-9280
ballheimcj@aol.com

**Secretary**
Donna Chanasyk — Res. 455-3562
13307 110 Avenue NW — Bus. 459-4405
Edmonton T5M 2M1 — Fax 459-0187
donnac@connect.ab.ca

**Treasurer**
Doug Weisbeck — Res. 459-8464
66 Dorchester Drive — Bus. 434-9406
St. Albert T8N 5T6 — Fax 434-4467
weisguy@v-wave.com

**NCTM Representative**
TBA

**Newsletter Editor**
Art Jorgensen — Res. 723-5370
4411 5 Avenue — Fax 723-4471
Edson T7E 1B7

**Journal Editor**
Klaus Puhlmann — Res. 795-2568
PO Box 6482 — Bus. 723-4471
Edson T7E 1T9 — Fax 723-2414
klaupuhl@gyrd.ab.ca

**Webmaster**
Dick Pawloff — Res. 347-3658
4747 53 Street — Bus. 343-3733
Red Deer T4N 2E6 — Fax 347-8190
dpawloff@tac.net

**Department of Education Representative**
Daryl Chichak — Res. 450-1813
1826 51 Street NW — Bus. 427-0010
Edmonton T6L 1K1 — Fax 469-0414
dchichak@edc.gov.ab.ca

**1997 Conference Chair**
Margaret Marika — Res. 433-0692
9756 73 Avenue NW — Bus. 462-3322
Edmonton T6E 1B4 — Fax 462-7803

**1998 Conference Cochair**
Bob Michie — Res. 246-8597
149 Wimbledon Crescent SW — Bus. 289-9241
Calgary T3C 3J2 — Fax 777-7309
bmichie@cbe.ab.ca

**Mathematics Representative**
Michael Stone — Res. 286-3910
Room 470 Math Sciences Bldg. — Bus. 220-3964
University of Calgary — Fax 282-5150
2500 University Drive NW — mgstone@acs.ucalgary.ca
Calgary T2N 1N4

**Faculty of Education Representative**
Dale Burnett — Res. 381-1281
Faculty of Education — Bus. 329-2416
University of Lethbridge — or 329-2457
Lethbridge T1K 3M4 — Fax 329-2252
dale.burnett@hg.uleth.ca

**Directors**
Daryl Chichak — Res. 450-1813
1826 51 Street NW — Bus. 427-0010
Edmonton T6L 1K1 — Fax 469-0414
dchichak@edc.gov.ab.ca

Rick Johnson — Res. 458-5670
8 Grandin Woods Estates — Bus. 476-7953
St. Albert T8N 2Y4 — Fax 459-0512
rjohnson@freenet.edmonton.ab.ca

Graham Keogh — Res. 347-5113
37568 Rge Rd 275 — Bus. 342-4800
Red Deer County T4S 2B2 — Fax 343-2249
gkeogh@rdcrd.ab.ca

Richard Kopan — Res. 254-9106
72 Sunrise Crescent SE — Bus. 777-7520
Calgary T2X 2Z9 — Fax 777-7529
rjkopan@cbe.ab.ca

Elaine Manzer — Res. 624-3988
9502 79 Avenue — Bus. 624-4221
Peace River T8S 1E6 — Fax 624-4048
manzere@prsdnet.ab.ca

Sandra Unrau — Res. 284-2642
11 Hartford Place NW — Bus. 777-6920
Calgary T2K 2A9 — Fax 777-6922
sunrau@cbe.ab.ca

**PEC Liaison**
Carol D. Henderson — Res. 256-3946
860 Midridge Drive SE Suite 521 — Bus. 938-6666
Calgary T2X 1K1 — Fax 256-3508

**ATA Staff Advisor**
David L. Jeary — Bus. 265-2672
SARO — or 1-800-332-1280
540 12 Avenue SW Suite 200 — Fax 266-6190
Calgary T2R 0H4 — djeary@teachers.ab.ca

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Barnett House
11010 142 Street NW
Edmonton AB  T5N 2R1